Statistical Mechanics
E0415

Fall 2022, lecture 2
Random walks

Take-home 1

Take-home task 1

Study the material for the week: Sethna Chapter 1. Then, answer the following question(s) and return your answer to MyCourses **deadline on the day before the lecture: September 17 at noon.**

1.t

Have a look at `https://royalsocietypublishing.org/doi/10.1098/rsos.200307` and write a short (1 paragraph) essay commenting on two points: which issues of the book do you recognize in the study, what do you think you would have done in the study?

To the first part of the question:

”The issues I discuss here involve largely the data, while touching on the rendering of crowds and the selection of participants.”

“In doing so I would like to try to answer what types of noise
subjects are most sensitive to which could lead to more convincing simulated crowds.”

“I'd expect the real crowd to have some distinct characters:
Some people would be in a hurry; walking quickly from A to B, avoiding any
big groups or slipping through them. Others would be groups of friends walking more slowly.”

More answers

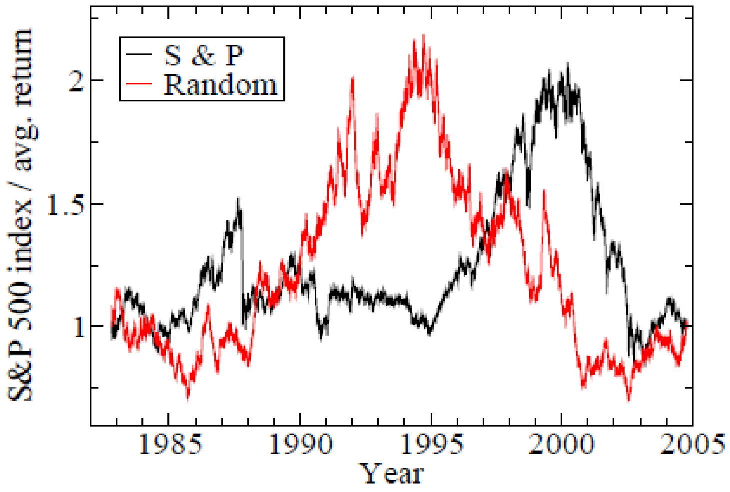
Second part:

”… many of the statistical facets mentioned in chapter 1 of Sethna have applications in crowd behaviour such as abrupt phase transitions and phases. A phase transition in crowd behaviour will be observed during an intense event, like a fire, where a crowd may go from a loosely packed ”gas-like” state to a ”solid-like” state as everyone rushes to the exits. A property of the crowds, polarization, reminded me of how magnetic phases are

Universality, scale invariance

- **Three main points with random walks**: scaling (scale-free) behavior, universality (small details do not matter), probability distributions (and the governing equation(s)).
- Example: coin flips/tosses – do heads or tails win? Square-root law with N.
- Example II: drunkard’s walk (on a 2D plane).
- Universality – compare with polymers (self-avoidance). “Entropic repulsion”, walk exponent becomes ϒ = ¾ (2D), 0.59 (3) – exact and numerical values. “Universal critical exponent” (Self-Avoiding Walks, SAW).

Sorts of RWs



Stock Exchange Index vs.
a simple RW

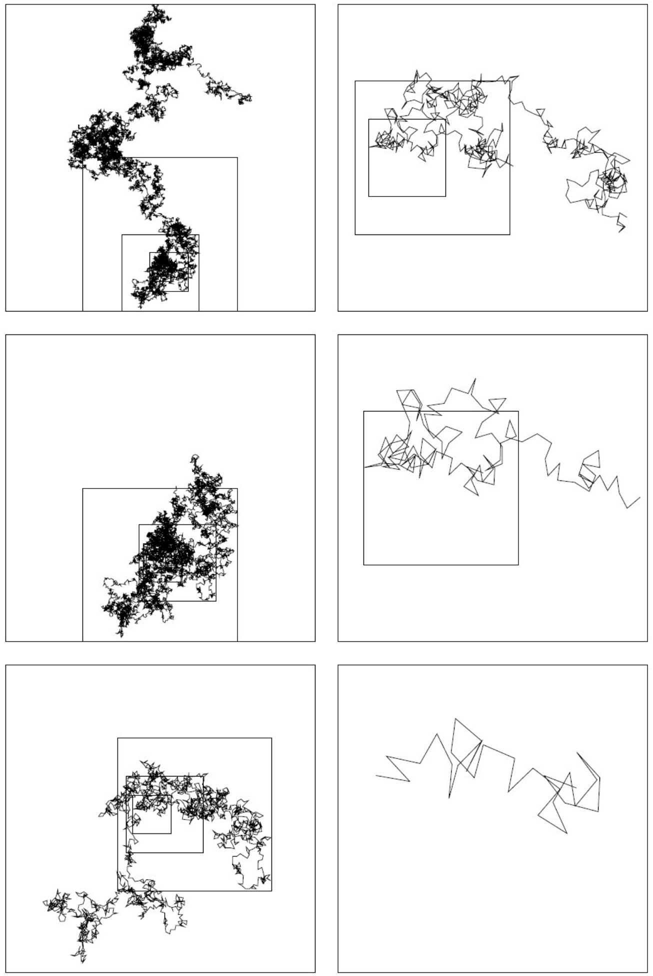


Fig. 2.2 Random walk: scale invariance. Random walks form a jagged, fractal pattern which looks the same when rescaled. Here each succeeding walk is the first quarter of the previous walk, magnified by a factor of two; the shortest random walk is of length 31, the longest of length 32 000 steps. The left side of Fig. 1.1 is the further evolution of this walk to 128 000 steps.

The Diffusion Equation

An equation for ρ: two interpretations – density of a cloud, pdf of a single RW.

Derivation of DE: separation of scales (RW step against the gradient of ρ).

Relation of the diffusion constant D > 0 to the microscopic RW details:

Step size a, timescale Δt.

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho = D\frac{\partial^2 \rho}{\partial x^2}.$$

$$D = a^2/2\Delta t.$$

Currents & external forces

Remember: DE conserves particles, thus the density – "conservation law".

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x}. \qquad J_{\text{diffusion}} = -D\frac{\partial \rho}{\partial x},$$

In the presence of external forces the particles have a deterministic drift.

$$x(t + \Delta t) = x(t) + F\gamma\Delta t + \ell(t).$$

This shows up in the current, and in the equation for ρ.

$$J = \gamma F \rho - D\frac{\partial \rho}{\partial x}.$$

$$\frac{\partial \rho}{\partial t} = -\gamma F\frac{\partial \rho}{\partial x} + D\frac{\partial^2 \rho}{\partial x^2},$$

Case study: density profile with gravity ("atmosphere").

$$\rho^*(x) = A\exp\left(-\frac{\gamma}{D}mgx\right).$$

Solving the diffusion equation

Example: Fourier method. FT in space, substitute: reveals the diffusive timescale and the role of D.

General solution as superposition of the FT of the initial profile or condition .

$$\frac{\partial \rho}{\partial t} = \frac{\mathrm{d}\widetilde{\rho}_k}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}kx} = D\frac{\partial^2 \rho}{\partial x^2} = -Dk^2\widetilde{\rho}_k\mathrm{e}^{\mathrm{i}kx},$$

$$\frac{\mathrm{d}\widetilde{\rho}_k}{\mathrm{d}t} = -Dk^2\widetilde{\rho}_k,$$

$$\widetilde{\rho}_k(t) = \widetilde{\rho}_k(0)\mathrm{e}^{-Dk^2t}.$$

$$\rho(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\rho}_k(0)\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-Dk^2t}\,\mathrm{d}k.$$

$$\widetilde{\rho}_k(0) = \int_{-\infty}^{\infty}\rho(x,0)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x,$$

Homework

1.2 Waiting times (Sethna 1.3 p. 6) HOMEWORK (5 points)

Study Sethna Ch2, answer the following in the same spirit as the first take-home. Let us turn this into an exercise in gambling. You play heads and tails (toss a coin, and guess the outcome: win or lose the coin).

Three questions: you start with 10 coins. Give an argument how the distribution of times it takes for you to lose all your coins looks like.

What happens if you play till you have zero, or until you won all the 10 coins of your friend?

Let us now consider the case where the coin is not fair: the fractional Brownian motion, where the subsequent outcomes are correlated (positively or negatively). How does that influence qualitatively those outcomes?

Other scheduling

Paper presentations and groups and project groups: DL tomorrow, we summarize the situation and make the groups and inform you (MC) early next week. First paper presentation probably 7th of October – so the group HAS to be present. If you can not be present at your slot: contact us ASAP.

Friday lectures: we usually do 13.15