Aalto University
School of Science

MS-E2114 Investment Science
Lecture 4: Applied interest rate analysis

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Overview

Capital budgeting

Portfolio optimization

Dynamic programming

Harmony theorem

Firm valuation

This lecture

- Lectures 2 & 3: We have derived interest rates from fixed income securities and computed the present value of cash flows
- In this lecture we use interest rate analysis to inform different kinds of investment decisions
  - We thus move from descriptive ('what is') to prescriptive ('what should') analysis
- The decisions are informed by interest rates: the weighted average cost of capital (WACC) refers to the rate that a company is expected to pay on average to all its security holders to finance its assets
  - WACC includes both cost of debt (e.g. payments to bond holders) and cost of equity (dividends to stock owners)

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Capital budgeting

- **Task**: Allocate capital C among m investment projects
  - b_i = Benefit of project i
  - c_i = Cost of project i
  - Benefits and costs are aggregated and expressed in monetary terms (€)
- Projects are assumed to be
  - Lumpy "go"/"no-go" investments
    - E.g. bridges - it makes no sense to build half a bridge
    - This is not the case for securities which can be sold and bought in small divisible amounts (relative to the investment size)
  - Projects are independent of each other
    - The benefits and costs of a project do not depend those of other projects (i.e., synergies and cannibalization impacts are here neglected)
  - Not tradeable in established markets
    - E.g., projects for the development of prototypes or new products

Capital budgeting

- The budgeted capital C need not be a hard constraint
  - There is often a tendency to deplete the budget even if the last projects to be funded offer but a small marginal benefit
  - Conversely, it would make sense to increase the capital C if there are excellent projects that cannot be funded

A In theory, one could raise unlimited funding from the markets and fund all projects with positive NPV

B In practice, there are limits
  - Banks limit the amount of credit they provide
  - Heavier credit burden $\Rightarrow$ Greater risk of default $\Rightarrow$ The required interest rate will be higher
  - In large organizations, investment decisions are typically part of divisional budgeting

- Impacts of changing the budget C should be explored
  - 'Soft' (i.e., variable) constraints can be introduced by imposing penalties for budget extensions, for instance

Solving the capital budgeting problem

- Projects denoted by $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \{0, 1\}^m$

$$x_i = \begin{cases} 1, & \text{if project } i \text{ is funded} \\ 0, & \text{otherwise} \end{cases}$$

- The optimum can be solved from

$$\max_{\mathbf{x}} \sum_{i=1}^{m} b_i x_i$$

$$\text{subject to} \sum_{i=1}^{m} c_i x_i \leq C$$

$$x_i \in \{0, 1\}, \quad i = 1, 2, \ldots, n$$

- This is a binary 0-1 optimization problem
- Essentially a knapsack (packing) problem

Solving the capital budgeting problem

- Determining the exact solution may pose computational challenges
  - There are hard-to-solve instances with $m = 50...100$
  - Problems with many similar projects may have multiple optimum solutions
  - Yet many instances with $m = 10\,000$ can be solved in milliseconds with state-of-the-art optimization methods
  - Small and easy instances can be solved with Excel Solver
- An approximate solution can be generated by using benefit-to-cost ratios $r_i = b_i/c_i$
  - Fund projects one by one in decreasing order of ratios r_i (i.e., starting from the project whose ratio is highest) until the budget C has been depleted

Solving the capital budgeting problem, $C = 500$

Project	Cost	Benefit	r_i	Benefit-to-cost solution	Optimal solution
1	100	300	3.00	1	1
2	20	50	2.50	1	0
3	150	350	2.33	1	1
4	50	110	2.20	1	1
5	50	100	2.00	1	1
6	150	250	1.67	0	1
7	150	200	1.33	0	0
Cost				370	500
Benefit				910	1110
NPV				540	610

Modelling project dependencies

- Dependencies can usually be modelled through linear constraints
  - E.g., project i has n_i variants of which but one can be funded:

$$\max_{\mathbf{x}} \sum_{i=1}^{m} \sum_{j=1}^{n_i} b_{ij} x_{ij}$$

$$\text{subject to} \sum_{i=1}^{m} \sum_{j=1}^{n_i} c_{ij} x_{ij} \leq C$$

$$\sum_{j=1}^{n_i} x_{ij} \leq 1, \quad i = 1, 2, \ldots, m$$

$$x_{ij} \in \{0, 1\}, \quad j = 1, 2, \ldots, n_i, i = 1, 2, \ldots, m$$

- b_{ij}, c_{ij} = benefit and cost of variant $j = 1, \ldots, n_i$ of project i
- x_{ij} = decision to fund variant j of project i

Modelling dependencies

- Examples of other extensions
  - Enabler: If project j cannot be started unless the enabling project i is implemented, the constraint $x_j \leq x_i$ must hold
    - E.g. a follow-up feature j that builds on another feature i
  - If at least/at most/exactly k projects from the set $i_1, i_2, \ldots, i_\ell$ must be selected, the constraints are correspondingly

$$x_{i_1} + x_{i_2} + \cdots + x_{i_\ell} \begin{cases} \geq k \\ \leq k \\ = k \end{cases}$$

- These extensions can be solved with integer optimization
  - Small and easy problems can be solved with Excel Solver
  - More challenging ones with CPLEX, Gurobi, LINDO, Matlab, etc.

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Portfolio optimization

- In (financial) portfolio optimization, the objective is to build a portfolio consisting of securities which are traded in the markets
  - Securities are traded in the markets $\Rightarrow$ The price dynamics of securities has to be taken into account
  - Some financial portfolio optimization problems can be solved much in the same way as capital budgeting problems

Cash matching

- **Fixed liabilities**: Obligation to pay y_i euro in period $i = 1, 2, \ldots, n$
- **Task**: Determine the bond portfolio whose cash flow meets or exceeds these liabilities and has the smallest purchasing price
  - c_{ij} = cash flow of bond j in period i
  - p_j = price of bond j

$$\min_{\mathbf{x}} \sum_{j=1}^{m} p_j x_j$$

$$\text{subject to} \sum_{j=1}^{m} c_{ij} x_j \geq y_i, \quad i = 1, 2, \ldots, n$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, m$$

Cash matching

In the above formulation, it is assumed that

- $x_j \geq 0 \Leftrightarrow$ No short selling or issuing new bonds
  - $\Rightarrow$ To allow for shorting, the non-negativity constraint needs to be eliminated
- Excess cash flows are not reinvested
  - $\Rightarrow$ Reinvestments could be modelled e.g. by adding artificial bonds with cash flows $(0, 0, \dots, 0, -1, 1 + r_i, 0, \dots, 0)$ where r_i is the short rate for period i
- The solution may contain fractional (non-integer) bond purchases
  - $\Rightarrow$ Integral solution can be obtained by introducing integer constraints $x_j \in \{0, 1, 2, \dots\}$

Example: Basic cash matching problem

- Use the following 10 bonds to match liabilities

Input

	Bond cash flows c_{ij}										Liability	Portfolio cash flow
$i \setminus j$	1	2	3	4	5	6	7	8	9	10		
1	10	7	8	6	7	5	10	8	7	100	100	171.74
2	10	7	8	6	7	5	10	8	107		200	200
3	10	7	8	6	7	5	110	108			800	800
4	10	7	8	6	7	105					100	119.34
5	10	7	8	106	107						800	800
6	110	107	108								1200	1200
p	109	94.8	99.5	93.1	97.2	92.9	110	104	102	95.2	2381.14 **Cost**	
x	0	11.2	0	6.81	0	0	0	6.30	0.28	0		

Variables

Objective

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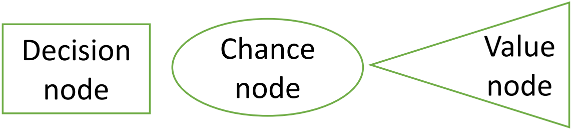
Dynamic cash flows

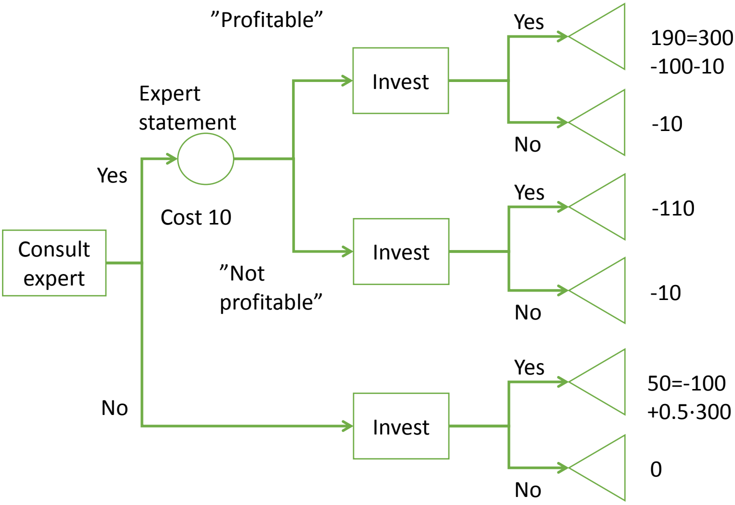
- In dynamic problems, decisions and the information that supports them depend on previous decisions
- **Note**: Many dynamic programming problems involve risk and uncertainties which are resolved over time
- The examples in this section of Luenberger do not consider the partial resolution of uncertainties $\Rightarrow$ Bayesian updating
- Example: Oil well as an investment
  - Oil well can be drilled only if the site has been acquired
  - Information about profitability can be obtained through testing
  - How much is the test worth?
  - Based on the results of the test, should one buy or not?
- Dynamic decisions can be structured as decision trees or lattices
  - Lattice = Tree with recombining branches

Dynamic cash flows

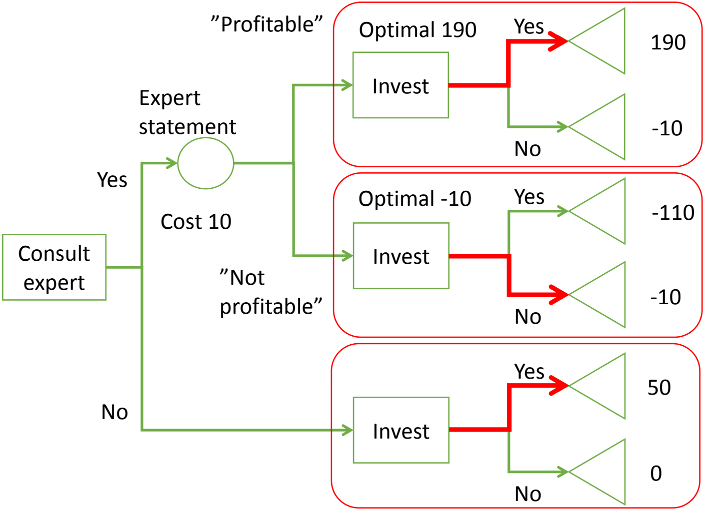
- Consider a dynamic investment problem such that:
  - Initial investment cost is 100 €
  - The investment yields cash flows of 300 € or 0 € with the same probability 0.50
  - With a cost of 10 € an expert can be consulted to assess whether the investment is profitable (300 €) or not (0 €)
  - The expert knows for sure the status of the well which is thus the only uncertainty (otherwise, a larger decision tree based on Bayesian updating would be built; cf. MS-E2135 Decision Analysis)
- This problem can be represented as a decision tree with three kinds of nodes



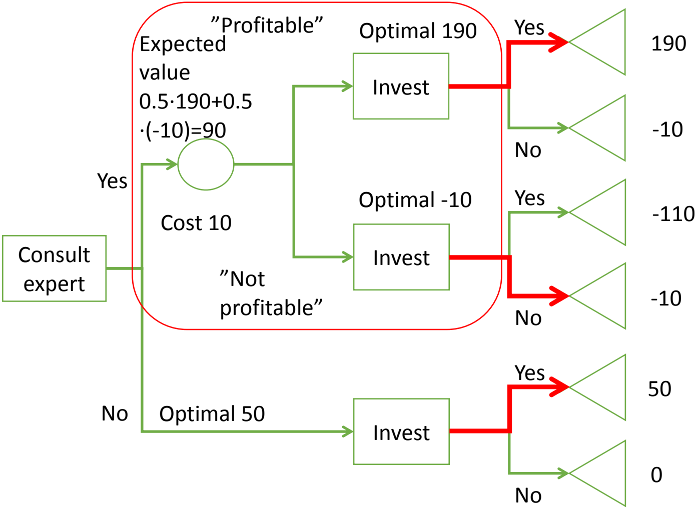
Dynamic cash flows



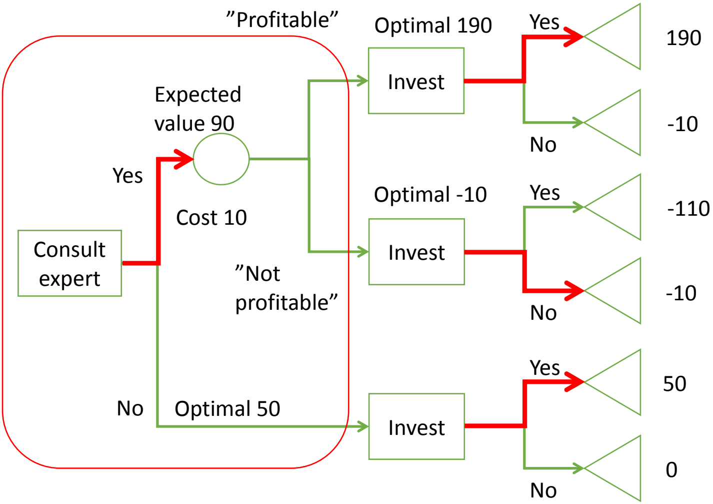
Example: Dynamic programming



Example: Dynamic programming



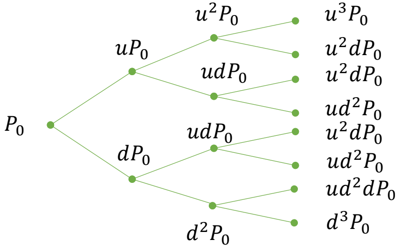
Example: Dynamic programming



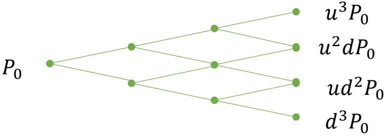
Lattices: Dynamic choice and uncertainties

- Can be used to model dynamic choice and dynamic uncertainties (e.g., evolution of a stock price over time)
- Consider *n* periods such that price goes up by factor *u* or down by factor *d* in each period, starting from price P_0

Binary tree (3-periods)



Binary lattice (equivalent to the tree)

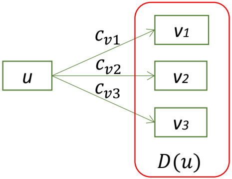


Dynamic choice models

- Helps solve complex decision trees and simple lattices
  - A binomial tree with n periods has 2^n paths
  - Traversing all paths becomes impossible with large n
- Solution principle (for dynamic choice):
  - What is optimal decision at node u?
  - Decision leads to node $v \in D(u)$
  - Cash flow of c_v
  - Optimal cash flow

$$V(u) = \max_{v \in D(u)} \{c_v + V(v)\}$$

  - Solve recursively from the end to the beginning

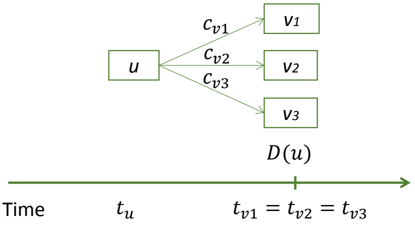


Running present value

- The preceding slide had no discounting
- Discounting can be included through

$$V(u) = \max_{v \in D(u)} \{c_v + d_{t_u,t_v} V(v)\}$$

- Discount factor $d_{t_u,t_v} = 1/(1 + f_{t_u,t_v})^{t_v - t_u}$
  - f_{t_u,t_v} = forward rate from time t_u to time t_v
  - t_u, t_v times of decisions u, v
- Decisions in every period $\Rightarrow$ short rate $r_k = f_{k,k+1}$

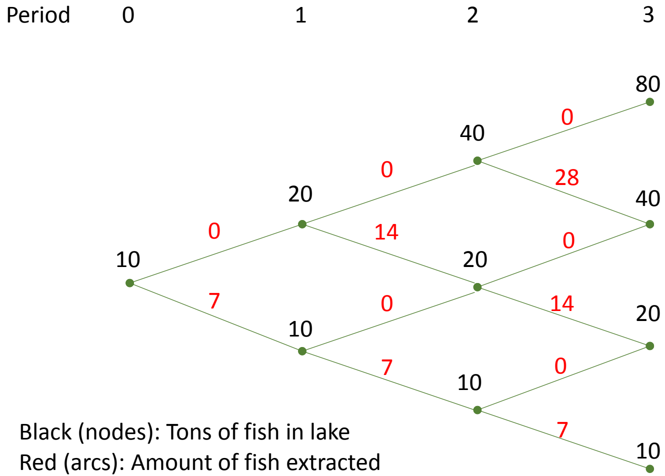


Example: Fishing problem

- You have the exclusive right to harvest fish for 3 years
  - The initial fish stock in the lake is 10 tons
  - If you do harvest (max once per year), you extract 70% of all the fish that are in the lake
    - Profit 1000 € /ton
  - Each year, the fish population grows
    - If you harvest, the fish stock will be restored to the level at which it was before harvesting
    - If you do not harvest, the size of the fish stock will double
- What is optimal harvesting policy when using a 25% discounting rate?
  - Discount factor $d = 1/1.25 = 0.8$

Example: Fishing problem

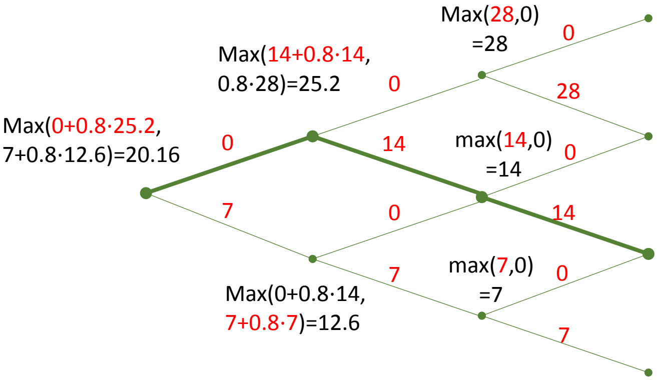
- Build the binary lattice (for dynamic choice)



Black (nodes): Tons of fish in lake
Red (arcs): Amount of fish extracted
Up: No fishing, Down: fish

Example: Fishing problem

- Use $V(u) = \max_{v \in D(u)} \{c_v + dV(v)\}$
  - Discounting yields optimum 20.16 k€ with policy (N, Y, Y)
  - Without discounting, two solutions (N, Y, Y) and (N, N, Y) both yield 28 k€



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Harmony theorem

- If you invest in a venture as a new owner, what decision principle should you adopt to steer investment decisions in the venture to get the best possible return for <u>your investment</u> in the venture?
- IRR may lead to different decision than NPV $\Rightarrow$ Do current and new owners have conflicting objectives?

Harmony theorem

- Your friend has invented a new gizmo, for which he holds the patent rights
- There are $i = 1, 2, \ldots, n$ commercialization options
- Present value of commercialization option i is

$$P(i) = -c_i + \frac{1}{1+r} b_i$$

  where b_i, c_i are benefit and cost of option i, respectively
- If your friend seeks to maximize the present value of his venture, he selects the alternative i^* such that

$$P(i^*) = \max_{i \in \{1,2,\ldots,n\}} P(i)$$

Harmony theorem

- If you think that your investment will be used for covering the expenses, you may (wrongly) conclude that you would like to select the alternative that yields

$$\max_{i \in \{1,2,\ldots,n\}} \left\{ \frac{b_i}{c_i} \right\}$$

- However, to get your share of benefits, you must buy share α of the gizmo (the company) at a cost $\alpha P(i^*)$
- Thus, your cost-benefit decision criterion is

$$\max_{i \in \{1,2,\ldots,n\}} \left\{ \frac{\alpha b_i}{\alpha P(i^*) + \alpha c_i} \right\}$$

Harmony theorem

- **Theorem:** Your cost-benefit decision criterion is maximized for $i = i^*$, meaning that your and your friend's preferred options coincide.
- **Proof**: Suppose there is an alternative i such that

$$\frac{\alpha b_i}{\alpha P(i^*) + \alpha c_i} > \frac{\alpha b_{i^*}}{\alpha P(i^*) + \alpha c_{i^*}} = 1 + r.$$

Now, solving this for the definition of $P(i)$ we get

$$-c_i + \frac{b_i}{1+r} > P(i^*) \iff P(i) > P(i^*),$$

which contradicts with the definition of i^*. $\square$

Harmony theorem

Theorem

*(**Harmony theorem**) Current owners of a venture should want to operate the venture to maximize the present value of its cash flow stream.*
Potential new owners, who must pay the full value of their prospective share of the venture, will want the company to operate in the same way, in order to maximize the return on their investment.

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Valuation of a firm

- The valuation of a firm can be based on its cash flows
- Different aspects, such as
  1. dividends to stock holders,
  2. net earning of the firm
  3. cash flow that could be realized by selling the firm's assets

  lead to different valuations of firms
- We do not consider cash flow uncertainties explicitly

Dividend discount models

- Dividend cash flows D_k in year $k = 1, 2, \ldots$
- Present value of this cash flow

$$V_0 = \frac{D_1}{1+r} + \frac{D_2}{(1+r)^2} + \cdots + \frac{D_k}{(1+r)^k} + \cdots$$

- In the **constant-growth dividend model** dividends grow at a constant rate g which leads to the **Gordon formula**

$$V_0 = \frac{D_1}{1+r} + \frac{(1+g)D_1}{(1+r)^2} + \cdots + \frac{(1+g)^{k-1}D_1}{(1+r)^k} + \cdots$$

$$\Rightarrow V_0 = \frac{D_1}{r-g} = \frac{1+g}{r-g}D_0, \quad r > g,$$

which is obtained from the geometric sum $\sum_{i=0}^{\infty} at^i = \frac{a}{1-t}$ for any real a and $|t| < 1$ (here, $a = \frac{D_1}{1+r}$ and $t = \frac{1+g}{1+r}$).

Example: Dividend discount models

- **Discounted growth (Gordon) formula**:

$$V_0 = \frac{1+g}{r-g} D_0, \quad r > g,$$

- If the firm has paid 1.5 M€ of dividends, grows 8% each year and is discounted using with 20% interest rate, then

$$V_0 = \frac{1+0.08}{0.2-0.08} \cdot 1.5\ \text{M€} = 13.5\ \text{M€}$$

Free cash flow models

- Startups pay little or no dividends in order to retain more capital for growth
- Another possibility is analysis based on the **free cash flow** that the firm can pay without compromising growth
- Other possibilities would include analysis of earnings of the firm rather than dividends and pricing based on balance sheet value (assets)

Example: Free cash flow models

- Consider a firm which wishes to determine how large a share of its cash flow it should invest in its capital (e.g., machines) to maximize the present value of its dividends / free cash flow
- Profit in year n is Y_n, out of which the share $u \in [0, 1]$ is invested
- Annual growth rate modelled as factor $g(u)$
- Profit of the firm in period n is

$$Y_{n+1} = [1 + g(u)]\, Y_n$$
$$\Rightarrow Y_n = [1 + g(u)]^n Y_0$$

- Capital depreciates by a factor α annually but increases by the investment

$$C_{n+1} = (1 - \alpha) C_n + u Y_n$$

Example: Free cash flow models

- By recursion, the capital in year $n > 0$ is

$$C_n = (1-\alpha)^n C_0 + uY_0 \sum_{i=1}^{n} (1-\alpha)^{n-i} (1+g(u))^{i-1}$$

- By using the identity

$$x^n - y^n = (x-y) \sum_{i=1}^{n} x^{n-i} y^{i-1}$$

with $x = 1-\alpha$ and $y = 1+g(u)$, we get

$$C_n = (1-\alpha)^n C_0 + uY_0 \frac{(1+g(u))^n - (1-\alpha)^n}{g(u)+\alpha}$$

Note that $x - y = 1 - \alpha - (1+g(u)) = -\alpha - g(u)) < 0$, hence the fraction $\frac{x^n - y^n}{x-y}$ is written as $\frac{y^n - x^n}{y-x}$

Example: Free cash flow models

- By combining the terms containing $(1-\alpha)^n$, this formula becomes

$$C_n = (1-\alpha)^n \left(C_0 - \frac{uY_0}{g(u)+\alpha} \right) + uY_0 \frac{(1+g(u))^n}{g(u)+\alpha}$$

- Due to depreciation $\alpha > 0$, the first term with $(1-\alpha)^n$ will tend to zero over time and thus we get

$$C_n \approx uY_0 \frac{(1+g(u))^n}{g(u)+\alpha} = \frac{u}{g(u)+\alpha} Y_n$$

- For other combinations of parameters, the numerical solution can be found e.g. with Excel

Example: Free cash flow models

Income statement	
Before-tax cash flow	Y_n
Depreciation	αC_n
Taxable income	$Y_n - \alpha C_n$
Taxes (34%)	$0.34(Y_n - \alpha C_n)$
After-tax income	$0.66(Y_n - \alpha C_n)$
After-tax income + depreciation	$0.66(Y_n - \alpha C_n) + \alpha C_n$
Sustaining investment	uY_n
Free cash flow	$0.66(Y_n - \alpha C_n) + \alpha C_n - uY_n$

Note: From the last equation on slide 41, the sustaining investment $C_{n+1} - C_n = uY_n - \alpha C_n$ needs to be substracted from the after-tax income to obtain the free cash flow

Example: Free cash flow models

- By using $C_n = \dfrac{u}{g(u)+\alpha} Y_n$, we obtain an analytic formula for the free cash flow as a function of period n:

$$FCF_n = 0.66(Y_n - \alpha C_n) + \alpha C_n - uY_n$$

$$\Rightarrow FCF_n = \left[0.66 + 0.34\frac{\alpha u}{g(u)+\alpha} - u\right](1+g(u))^n Y_0,$$

- Because this cash flow grows by a constant factor, we can use Gordon's formula to calculate its present value:

$$PV = \left[0.66 + 0.34\frac{\alpha u}{g(u)+\alpha} - u\right]\frac{1}{r-g(u)} Y_0$$

Example: Free cash flow

- How much to invest in order to maximize NPV?
  - Turnover 10M€
  - Capital 19.8M€
  - Depreciation $\alpha = 0.1$
  - Discount rate $r = 0.15$
  - Growth $g(u) = 0.12(1 - e^{5(\alpha - u)})$
- The optimal solution $u = 0.3776$ can be obtained e.g. with Excel Solver

u	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
g(u)	-0.08	-0.03	0.00	0.03	0.05	0.06	0.08	0.09	0.09	0.10	0.10
PV	29.0	34.5	39.6	44.6	49.3	53.3	56.4	58.1	58.2	56.4	52.6

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