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Differential and Integral Calculus 1

Exercises, Week 4

Work on Warm-up 1–4 during the exercise sessions of Week 4.
Submit on MyCourses your solutions for Homework 1 and 2 by Sunday, October 2nd.

Warm-up 1: (Not all about new stuff, but instead mostly to revise topics from the previous weeks.) Give an example or explain why an example does not exist:

1. A continuous function $\mathbb{R} \to \mathbb{R}$ with infinitely many local maxima but no global maximum.
2. Two functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ such that
$$\lim_{x\to+\infty} \big(f(x)+g(x)\big) = +\infty \qquad \text{and} \qquad \lim_{x\to+\infty} f(x)g(x) = c,$$
where $c \in \mathbb{R}$ is a given number, not chosen by you.
3. Two continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ whose product is not continuous.
4. Two functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ both not continuous but such that the product fg is continuous.
5. A function $f\colon \mathbb{R} \to \mathbb{R}$ for which $\int_0^{+\infty} f(x)\,dx$ is not a real number but $\int_{-\infty}^{+\infty} f(x)\,dx$ is. For the purposes of this exercise, define
$$\int_0^{+\infty} f(x)\,dx \overset{\text{def}}{=} \lim_{c\to+\infty} \int_0^c f(x)\,dx \qquad \text{and} \qquad \int_{-\infty}^{+\infty} f(x)\,dx \overset{\text{def}}{=} \lim_{c\to+\infty} \int_{-c}^c f(x)\,dx.$$
(The latter is not standard, it's the way Cauchy defined those improper integrals.)

Warm-up 2: Compute first and second derivative for the following functions $\mathbb{R} \to \mathbb{R}$:
$$(a) \quad f(x) = \int_{6x}^{5} e^{-t^2}\,dt \qquad\qquad (b) \quad g(x) = \int_{\pi}^{x} \sin(t)\,dt.$$
In each case, find a value of x that maximizes the corresponding definite integral.

Warm-up 3: Compute the third- and fourth-degree Taylor polynomials for the following functions about 0:
$$(a) \quad f(x) = \ln(1+x) \qquad\qquad (b) \quad g(x) = \int_0^x e^{-t^2}\,dt.$$

Warm-up 4: Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sqrt[3]{x}.$$

1. Find a formula for $f^{(n)}(x)$, for all n. Write the Taylor polynomial $P_n(x, 8)$, for $n = 3$.
2. Find the smallest n for which you are sure that $P_n(9, 8)$ approximates $\sqrt[3]{9}$ with an error smaller than $\frac{1}{1000} = 0,001$.

Submit on MyCourses your solutions **only** for the following two problems. Explain the reasoning behind your solutions, do not just return the final result. If you make use of important results from the lectures, state what they are.

Homework 1: Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \int_0^x e^{-t^2}\, dt.$$

Recall from Warm-up 4 of Sheet 3 that the function e^{-t^2} does not have an elementary antiderivative. Take for granted that $\int_{-\infty}^{+\infty} e^{-t^2}\, dt = \sqrt{\pi}$, which you will see in Calculus 2.

1. What are the limits
$$\lim_{x\to+\infty} f(x) \qquad \text{and} \qquad \lim_{x\to-\infty} f(x),$$
if they exist?
2. Find the points where the graph of f intersects the axes.
3. Compute f' and f'' and study their sign and where they are zero.
4. Draw by hand a sketch of the graph of f, based on all of the above.

Hints: For part 1: The function $g(t) = e^{-t^2}$ is symmetric with respect to the vertical axis, and $\int_{-\infty}^{+\infty} g(t)\, dt = \int_{-\infty}^{0} g(t)\, dt + \int_0^{+\infty} g(t)\, dt$. (It may be helpful to draw the graph of g, also for other parts of the exercise.) [2 points]

Homework 2: Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{2x}.$$

1. Find a formula for $f^{(n)}(x)$, for all n. Use this to write the Maclaurin polynomial $P_n(x, 0)$, for all n.
2. Find the smallest n for which you are sure that $P_n(1, 0)$ approximates e^2 with an error smaller than $\frac{1}{1000} = 0,001$. [2 points]
