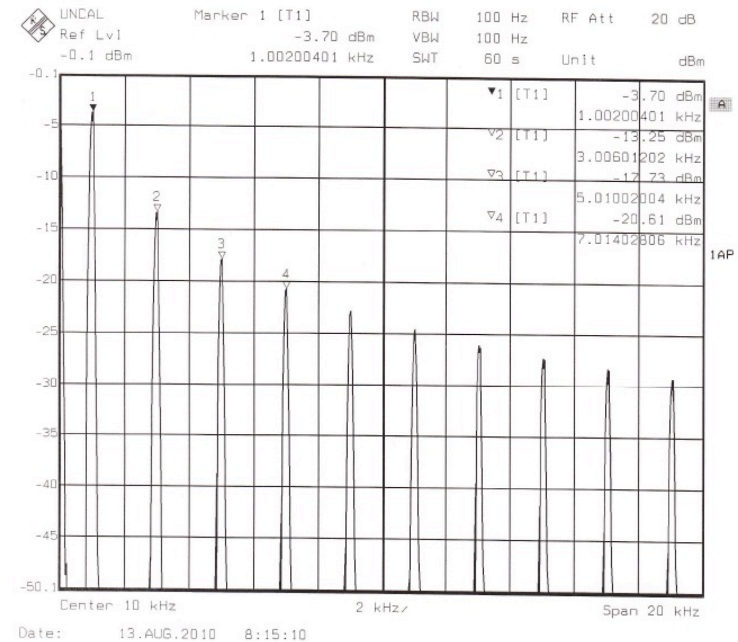
ELEC-A7200

Signals and Systems

Professor Riku Jäntti

Fall 2022

Aalto University
School of Electrical
Engineering



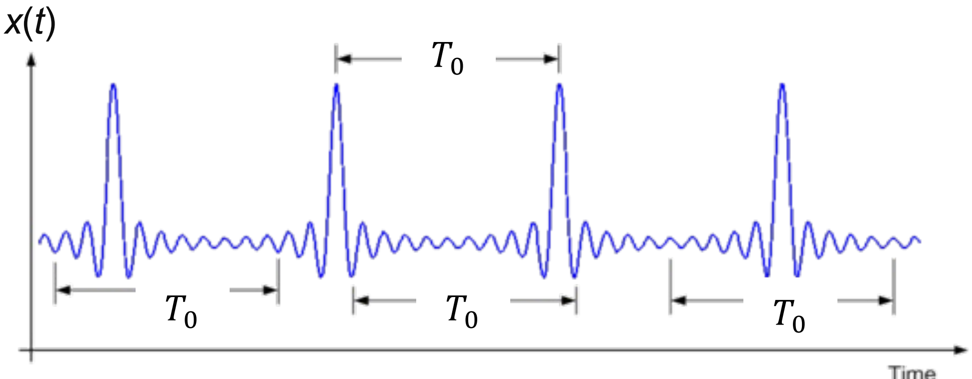
Lecture 4
Fourier series

Contents

- **Periodic signals**
- **Fourier series**
    - Exponential Fourier series
    - Trigonometric Fourier series
- **Amplitude and phase spectrum**
- **Parseval's theorem & Power spectrum**
- **Gibbs phenomena**

Periodic signals

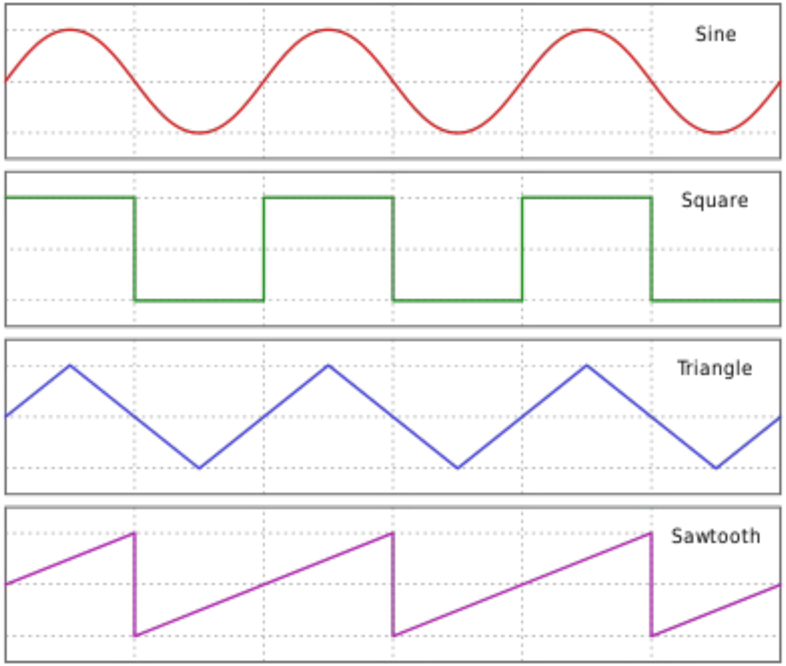
A signal $x(t)$ is periodic with period T_0 if $x(t+T_0) = x(t)$



- The fundamental frequency of the signal is $1/ T_0$
- Periodic signals are power signals

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt \; \forall t_0$$

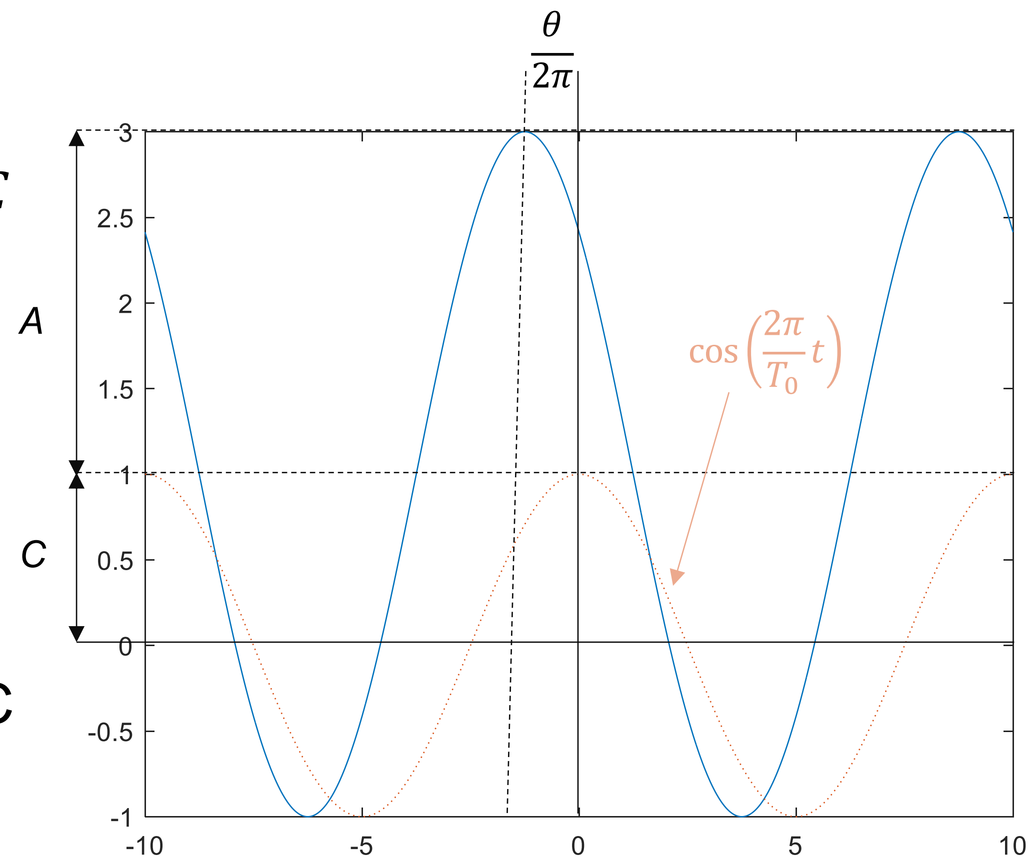
Common periodic signals



Sinusoids

$$x(t) = A\cos\left(\frac{2\pi}{T_0}t + \theta\right) + C$$

- Period T_0
- Frequency $\frac{1}{T_0}$
- Amplitude A
- Phase θ
- Offset (direct current DC component) C



Phase lead and lag

Phase lag = Delayed sinusoid

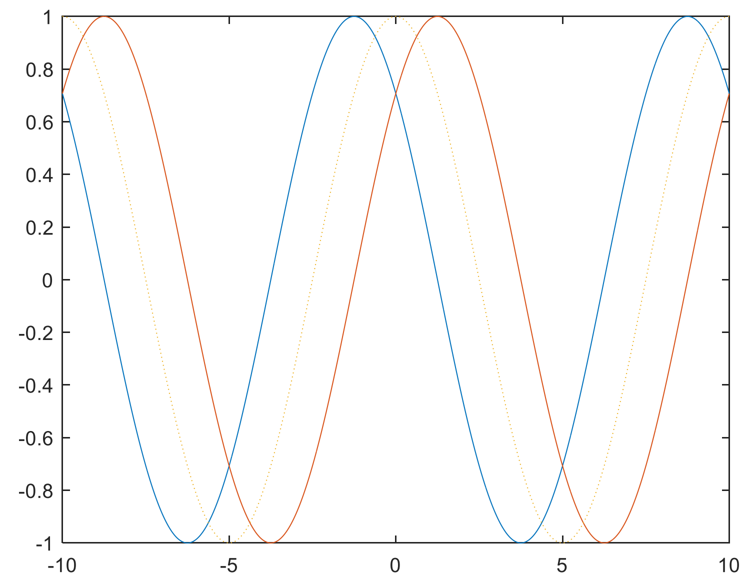
$$x(t) = A\cos\left(\frac{2\pi}{T_0}(t-\tau)\right) = A\cos\left(\frac{2\pi}{T_0}t+\theta\right)$$

$$\Rightarrow\ \theta = -\frac{2\pi}{T_0}\tau$$ Phase lag

Phase lead = Early signal

$$x(t) = A\cos\left(\frac{2\pi}{T_0}(t+\tau)\right) = A\cos\left(\frac{2\pi}{T_0}t+\theta\right)$$

$$\Rightarrow\ \theta = +\frac{2\pi}{T_0}\tau$$ Phase lead



Jean Baptiste Joseph Fourier

Jean Baptiste Joseph Fourier (1768-1830)

French mathematician, Egyptologist and administrator, who exerted strong influence on mathematical physics through his Théorie analytique de la chaleur (1822; The Analytical Theory of Heat).

Fourier was the first person to study the Earth's temperature from a mathematical perspective. In that study, he discovered the *greenhouse effect*.

https://www.britannica.com/biography/Joseph-Baron-Fourier

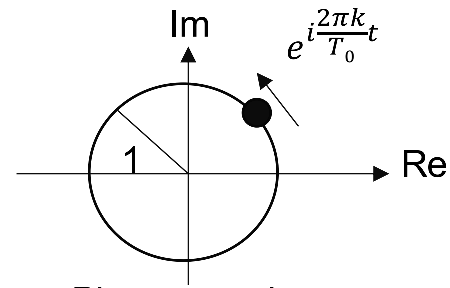
Signal representation in orthonormal basis: Fourier-series

Periodic signal $x(t) = x(t + T_0)$

Orthonormal basis

$$\phi_k(t) = \frac{1}{\sqrt{T_0}} e^{i\frac{2\pi k}{T_0}t}$$ Complex signal

k=…,-2,-1,0,1,2,..



Phasor rotating counter clockwise with frequency $\frac{k}{T_0}$

Signal represented in the orthonormal basis

$$x(t) = \sum_{k=-\infty}^{\infty} \langle x(t), \phi_k(t) \rangle \, \phi_k(t) = \sum_{k=-\infty}^{\infty} \int_{T_0} x(t) \frac{1}{\sqrt{T_0}} e^{-i\frac{2\pi k}{T_0}t} dt \frac{1}{\sqrt{T_0}} e^{i\frac{2\pi k}{T_0}t} = \sum_{k=-\infty}^{\infty} X(k) \, e^{i\frac{2\pi k}{T_0}t}$$

$$X(k) = \frac{1}{T_0} \int_{T_0} x(t) e^{-i\frac{2\pi k}{T_0}t} dt$$

Coefficients of Exponential Fourier Series

Fourier basis

Basis functions of the exponential Fourier-series: $\phi_k(t) = \frac{1}{\sqrt{T_0}} e^{i\frac{2\pi k}{T_0}t}$

Proof that the basis is orthonormal:

$$\langle \phi_k(t), \phi_l(t) \rangle = \frac{1}{T_0} \int_{T_0} e^{i\frac{2\pi(k-l)}{T_0}t}\, dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} e^{i\frac{2\pi(k-l)}{T_0}t}\, dt = \frac{1}{i2\pi(k-l)} \left(e^{i\pi(k-l)} - e^{-i\pi(k-l)}\right)$$

$$= \frac{1}{\pi(k-l)} \frac{1}{i2} \left(e^{i\pi(k-l)} - e^{-i\pi(k-l)}\right) = \frac{1}{\pi(k-l)} \sin(k-l) = \mathrm{sinc}(k-l) = \begin{cases} 0, k \neq l \\ 1, k = l \end{cases}$$

$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Sinc function

Sinc function

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

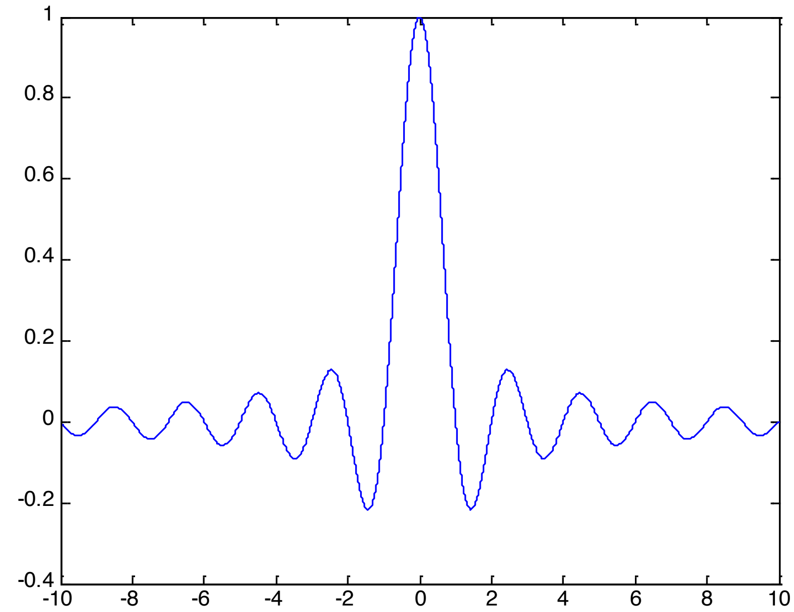
Zeros

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} = 0$$

$$x = \pm 1, \pm 2, \pm, 3, \ldots$$

Limit

$$\lim_{x \to 0} \frac{\sin(\pi x)}{\pi x} = \lim_{x \to 0} \frac{\frac{\partial}{\partial x} \sin(\pi x)}{\frac{\partial}{\partial x} \pi x} = \lim_{x \to 0} \frac{\pi \cos(\pi x)}{\pi} = \frac{\pi}{\pi} = 1$$



Aalto University
School of Electrical
Engineering

Signal representation in orthonormal basis: Fourier series

Exponential Fourier series representation of a periodic signal $x(t) = x(t + T_0)$

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{i\frac{2\pi k}{T_0}t}$$

Fourier series coefficients

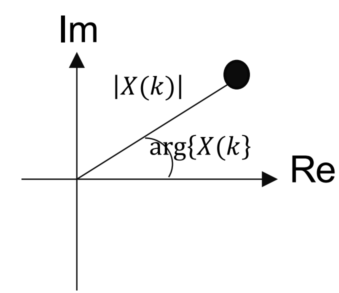
$$X(k) = \frac{1}{T_0}\int_{T_0} x(t)e^{-i\frac{2\pi k}{T_0}t}dt$$

Fourier series coefficients are complex valued:

$$X(k) = |X(k)|e^{-i\arg\{X(k\}} \in \mathbb{C}$$

$$|X(k)| = \sqrt{\mathrm{Re}^2\{X(k)\} + \mathrm{Im}^2\{X(k)\}}$$ Absolute value a.k.a modulus

$$\arg\{X(k\} = \tan^{-1}\frac{\mathrm{Im}\{X(k)\}}{\mathrm{Re}\{X(k)\}}$$ Argument angle



Aalto University
School of Electrical
Engineering

Fourier series properties

Let x(t) **and** y(t) **be periodic signals with the same period** T_0

- **Linearity** z(t)=x(t)+y(t) ⇒ Z(k) = X(k)+Y(k)
- **Komplex konjugation** $y(t) = x^*(t) \Rightarrow Y(k) = X^*(-k)$
- **Time shifting** $y(t)$=x(t-t_0) => $Y(t) \Rightarrow X(k)e^{-i\frac{2\pi k}{T_0}t_0}$
- **Frequency shifting** $y(t) = x(t)e^{i\frac{2\pi l}{T_0}t} \Rightarrow Y(k) = X(k-l)$
- **Differentation** $y(t) = \frac{d}{dt}x(t) \Rightarrow Y(k) = i\frac{2\pi k}{T_0}X(k)$
- **Integration** $y(t) = \int_{-\infty}^{t} x(\tau)d\tau \Rightarrow \frac{1}{i\frac{2\pi k}{T_0}}X(k)$

Exponential Fourier-series representation of a sinusoidal signal

Sinusoidal:

$$x(t) = C + A\cos\left(\frac{2\pi}{T_0}t + \theta\right) = \frac{A}{2}e^{-i\theta}e^{-i\frac{2\pi}{T_0}t} + C + \frac{A}{2}e^{i\theta}e^{i\frac{2\pi}{T_0}t}$$

$X(-1)$ $X(0)$ $X(1)$

The modulus of the Fourier series coefficient is amplitude divided by 2

The argument angle of the Fourier series coefficient gives the phase of the sinusoidal

$$\cos(x) = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}$$

Trigonometrical Fourier Series

For real periodic signal $x(t) = x(t + T_0) \in \mathbb{R}$ we have

$$X(-k) = \frac{1}{T_0}\int_{T_0} x(t)e^{i\frac{2\pi k}{T_0}t}dt = X^*(k)$$ Complex conjugate

By noting that $\cos(x) = \frac{1}{2}\left(e^{-ix} + e^{ix}\right)$, we can write

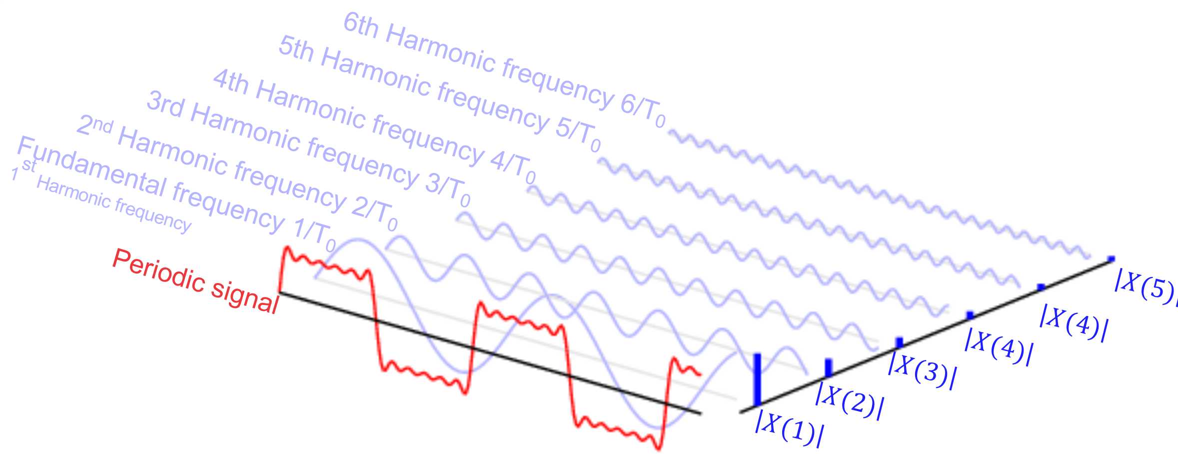
$$x(t) = X(0) + \sum_{\boldsymbol{k}=1}^{\infty} 2|X(k)|\cos\left(2\pi\frac{k}{T_0}t + \arg\{X(k)\}\right)$$

Trigonometrical Fourier Series

Periodic signal can be expressed as a sum of sinusoids with different frequencies and phases

$$x(t) = X(0) + \sum_{k=1}^{\infty} 2|X(k)| \cos\left(2\pi \frac{k}{T_0} t + \arg\{X(k)\}\right)$$

Harmonic frequency is an integer multiple of the fundamental frequency



https://pgfplots.net/fourier-transform/

Fourier sine and cosine series

Fourier series coefficients

$$X(k) = \frac{1}{T_0}\int_{T_0} x(t) e^{-i\frac{2\pi k}{T_0}t} dt = \underbrace{\frac{1}{T_0}\int_{T_0} x(t)\cos\left(\frac{2\pi k}{T_0}t\right)dt}_{a_k} - i\,\underbrace{\frac{1}{T_0}\int_{T_0} x(t)\sin\left(\frac{2\pi k}{T_0}t\right)dt}_{b_k}$$

If the periodic signal is real, then *X(k)=X*(-k)*

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{i\frac{2\pi k}{T_0}t} = \sum_{k=-\infty}^{\infty} (a_k - ib_k)\, e^{i\frac{2\pi k}{T_0}t} = a_0 + \sum_{k=1}^{\infty}\Big[\underbrace{(a_k + ib_k)}_{X(-k)} e^{-i\frac{2\pi k}{T_0}t} + \underbrace{(a_k - ib_k)}_{X(k)} e^{+i\frac{2\pi k}{T_0}t}\Big]$$

$$= a_0 + \sum_{k=1}^{\infty} 2a_k \cos\left(\frac{2\pi k}{T_0}\right) + \sum_{k=1}^{\infty} 2b_k \sin\left(\frac{2\pi k}{T_0}\right)$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \qquad \frac{1}{i} = -i$$

Fourier sine and cosine series

Fourier sine and cosine series for a real signal

$$x(t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cos\left(\frac{2\pi k}{T_0}\right) + \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{2\pi k}{T_0}\right)$$

- **Cosine series coefficients**

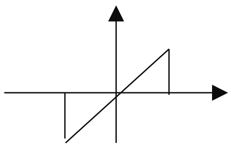
$$\alpha_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

$$\alpha_k = \frac{2}{T_0}\int_{T_0} x(t)\cos\left(\frac{2\pi k}{T_0}t\right)dt$$

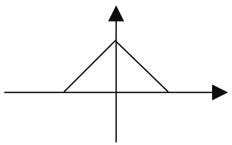
- **Sine series coefficients**

$$\beta_k = \frac{2}{T_0}\int_{T_0} x(t)\sin\left(\frac{2\pi k}{T_0}t\right)dt$$

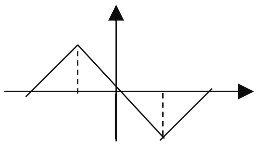
If x(t) is odd i.e. x(-t)=-x(t), then cosine series disappear and α_k=0 for k=0,1,2,…



If x(t) is even i.e. x(-t)=x(t), then sine series disappear and β_k =0 for k=0,1,2,…

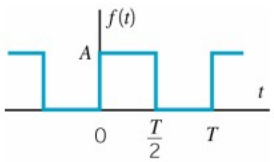


If x(t) is half-wave symmetric x(t)=-x(t+T_0/2), then even idex terms dissappear i.e. $\alpha_{2k} = \beta_{2k}$ =0 for k=0,1,2,…



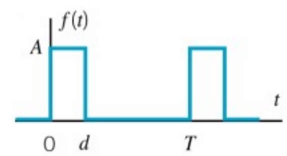
Fourier series repesentations of periodic signals

f(t)-A/2 is odd => Sine series



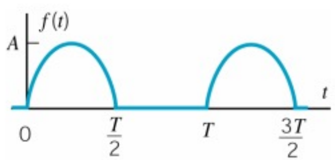
Square wave: $\omega_0 = \frac{2\pi}{T}$

$$f(t) = \frac{A}{2} + \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left((2n-1)\omega_0 t\right)}{2n-1}$$



Pulse wave: $\omega_0 = \frac{2\pi}{T}$

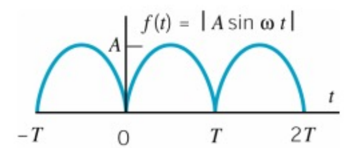
$$f(t) = \frac{Ad}{2} + \frac{2Ad}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\frac{n\pi d}{T}}\cos(n\omega_0 t)$$



Half wave rectified sine wave: $\omega_0 = \frac{2\pi}{T}$

$$f(t) = \frac{A}{\pi} + \frac{A}{2}\sin\omega_0 t - \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n\omega_0 t)}{4n^2-1}$$

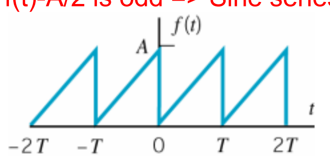
f(t) is even => Cosine series



Full wave rectified sine wave: $\omega_0 = \frac{2\pi}{T}$

$$f(t) = \frac{2A}{\pi} - \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\omega_0 t)}{4n^2-1}$$

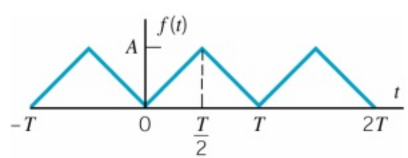
f(t)-A/2 is odd => Sine series



Sawtooth wave: $\omega_0 = \frac{2\pi}{T}$

$$f(t) = \frac{A}{2} + \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\omega_0 t)}{n}$$

f(t) is even => Cosine series



Triangle wave: $\omega_0 = \frac{2\pi}{T}$

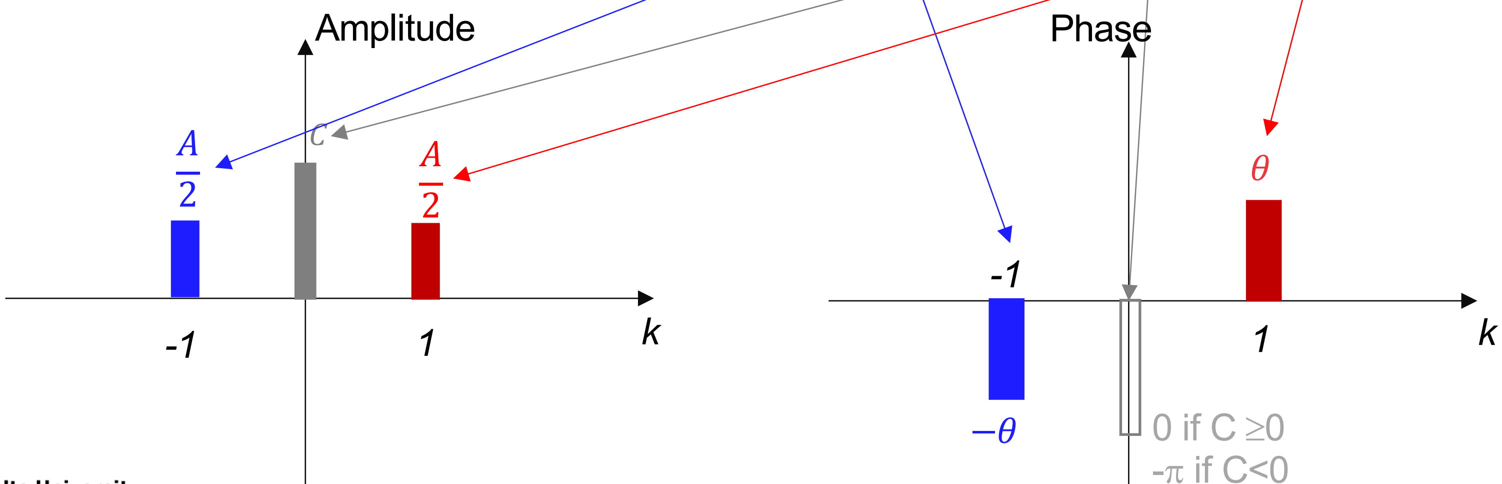
$$f(t) = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left((2n-1)\omega_0 t\right)}{(2n-1)^2}$$

Amplitude and phase spectrum

$$\cos(x) = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}$$

Two-sided spectrum for a sinusoidal:

$$x(t) = C + A\cos\left(\frac{2\pi}{T_0}t + \theta\right) = \frac{A}{2}e^{-i\theta}e^{-i\frac{2\pi}{T_0}t} + C + \frac{A}{2}e^{i\theta}e^{i\frac{2\pi}{T_0}t}$$

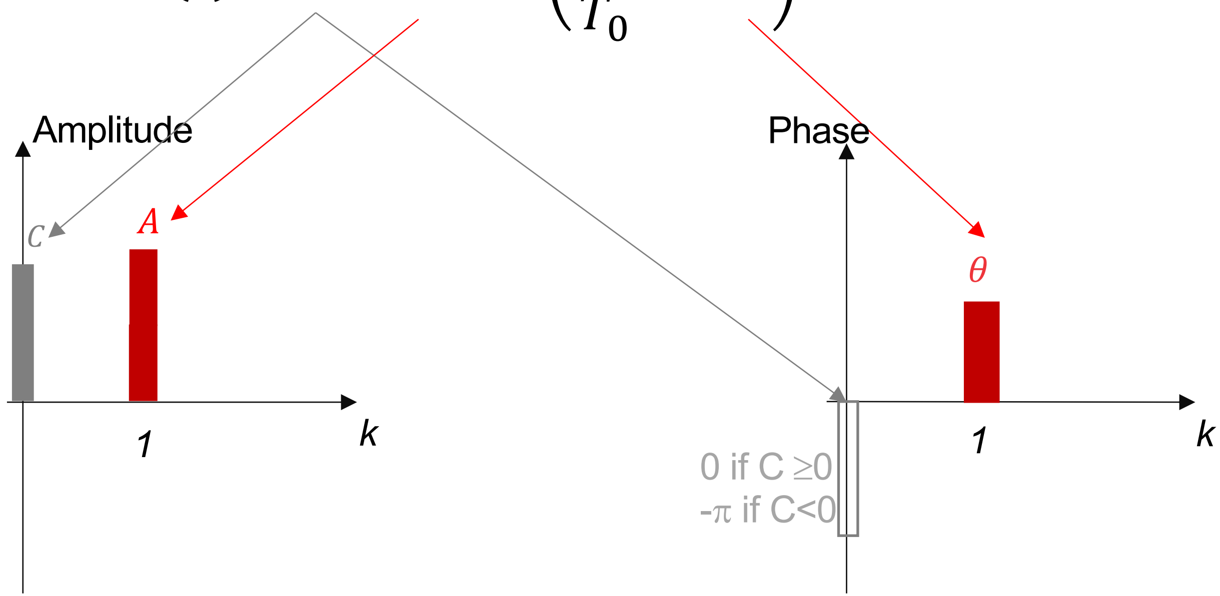


Aalto University
School of Electrical
Engineering

Amplitude and phase spectrum

One-sided spectrum for a sinusoidal:

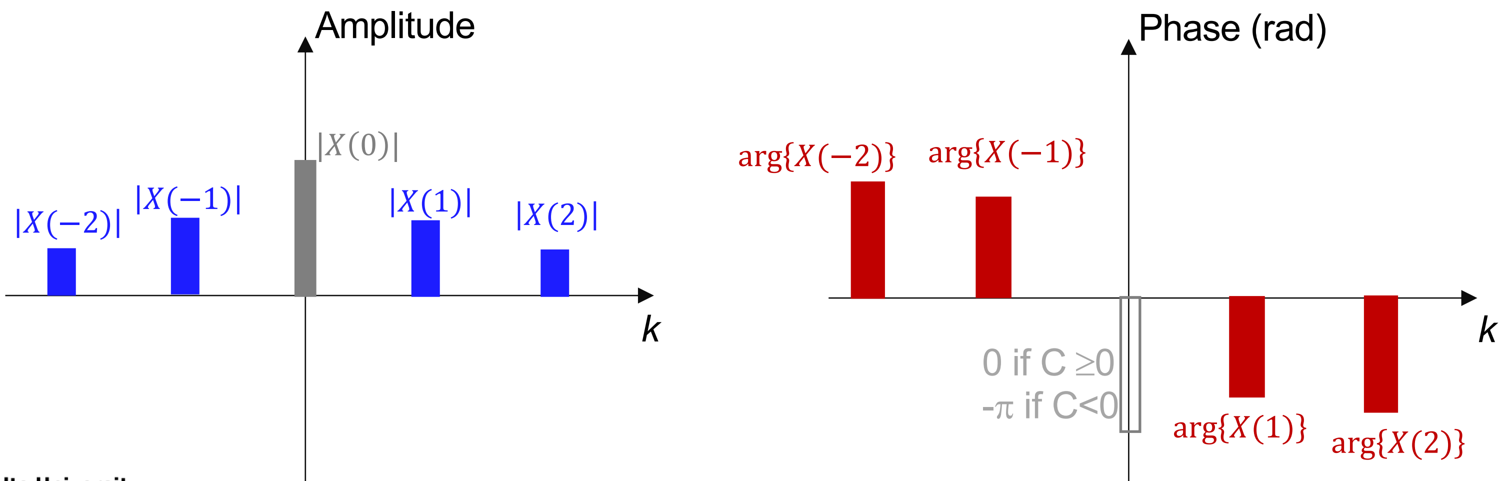
$$x(t) = C + A\cos\left(\frac{2\pi}{T_0}t + \theta\right)$$



Amplitude and phase spectrum

Two-sided amplitude and phase spectrum

$$x(t) = \sum_{k=-\infty}^{\infty} |X(k)| e^{-i\arg\{X(k)\}} e^{i\frac{2\pi k}{T_0}t}$$

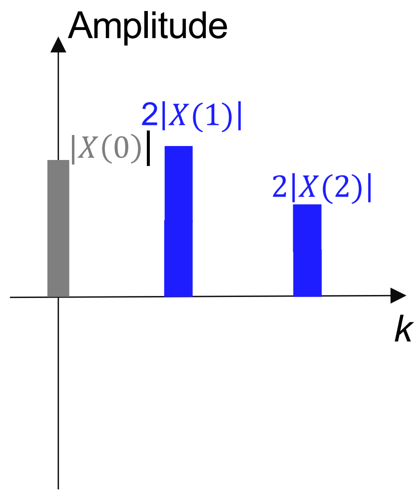


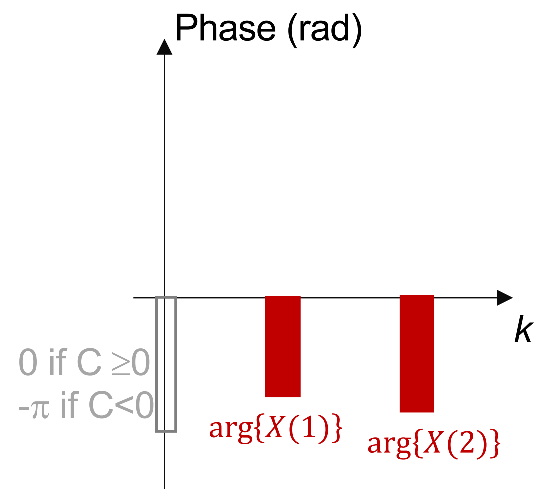
Aalto University
School of Electrical
Engineering

Amplitude and phase spectrum

One-sided amplitude and phase spectrum (real signal)

$$x(t) = X(0) + \sum_{k=1}^{\infty} 2|X(k)| \cos\left(2\pi \frac{k}{T_0} t + \arg\{X(k)\}\right)$$





Parseval’s theorem

Parsevals’ theorem:

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X(k)|^2$$

Squared modulus of the Fourier series coefficients $|X(k)|^2$ describe how the signal power is distributed among the harmonic frequencies.

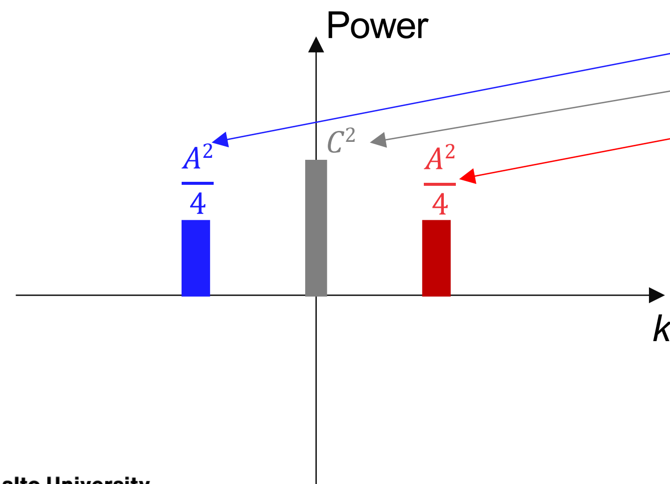
Power spectrum

$$\cos(x) = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}$$

Two-sided power spectrum of a sinusoidal

$$x(t) = C + A\cos\left(\frac{2\pi}{T_0}t + \theta\right) = \frac{A}{2}e^{-i\theta}e^{-i\frac{2\pi}{T_0}t} + C + \frac{A}{2}e^{i\theta}e^{i\frac{2\pi}{T_0}t}$$

$X(-1)$ $X(0)$ $X(1)$

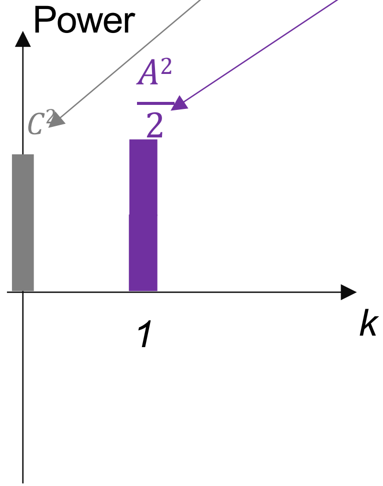


$$P = \sum_{k=-1}^{1} |X(k)|^2 = \frac{A^2}{4} + C^2 + \frac{A^2}{4} = \frac{A^2}{2} + C^2$$

Power Spectrum

One-sided spectrum for a sinusoidal:

$$x(t) = C + A\cos\left(\frac{2\pi}{T_0}t + \theta\right)$$



Power of a sinusoidal

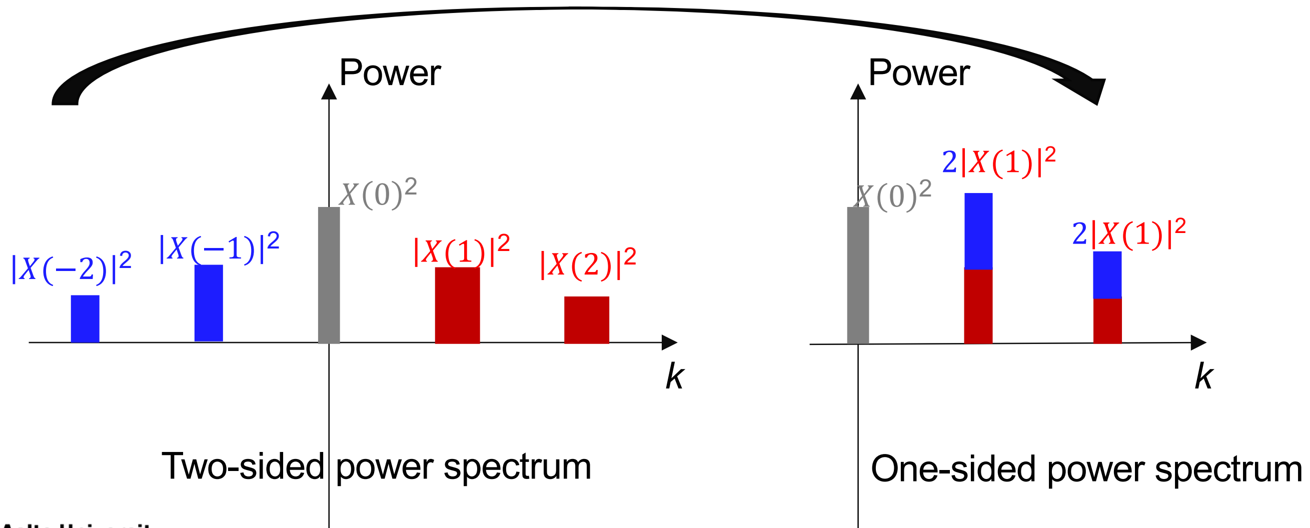
$$|X(-1)|^2 + |X(1)|^2 = \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

$$P = C^2 + \frac{A^2}{2}$$

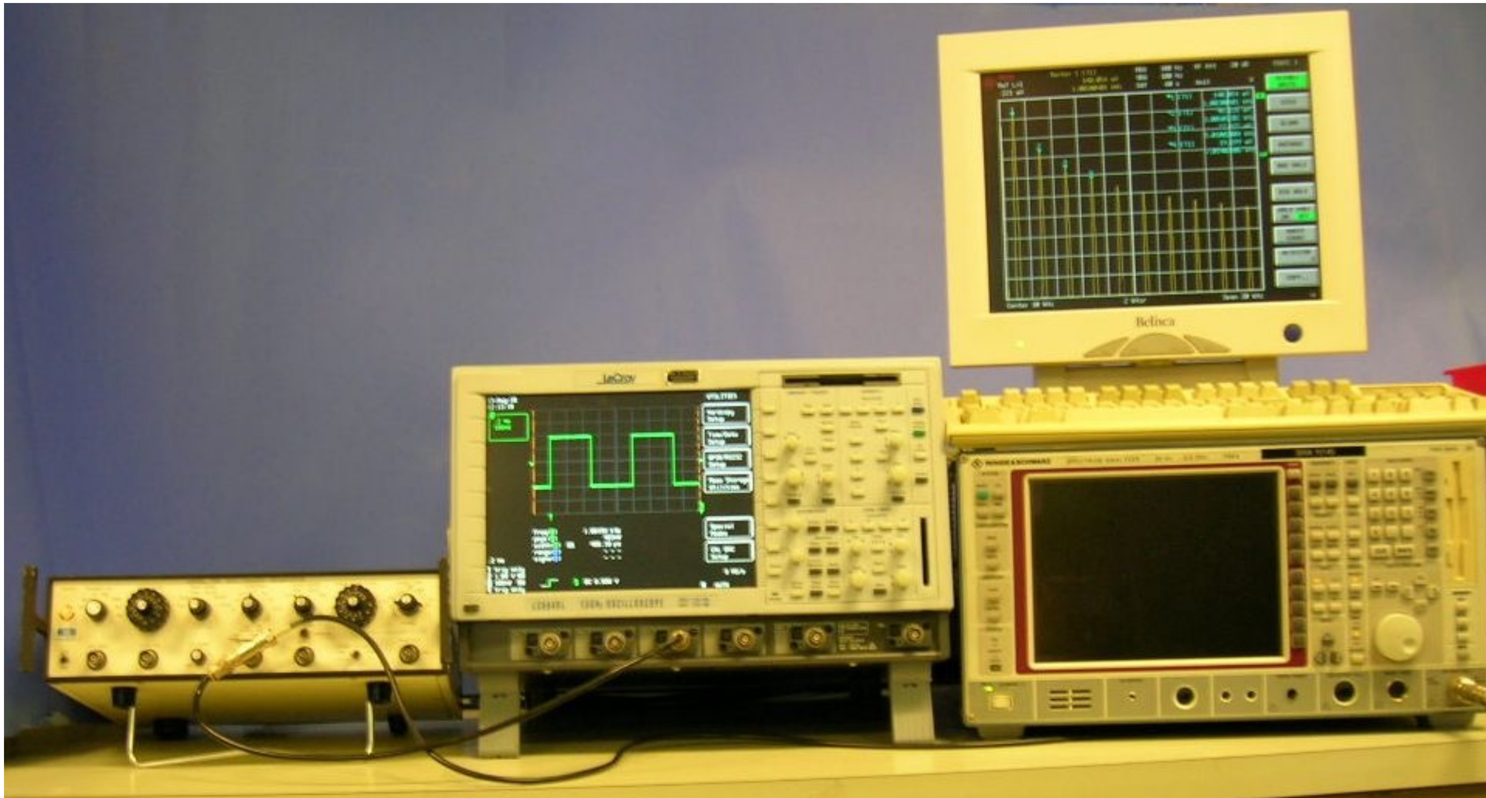
Power Spectrum

Fourier series of a real periodic signal

$$x(t) = \sum_{k=-\infty}^{\infty} |X(k)| e^{-i\arg\{X(k)\}} e^{i\frac{2\pi k}{T_0}t} = X(0) + \sum_{k=1}^{\infty} 2|X(k)| \cos\left(2\pi \frac{k}{T_0} t + \arg\{X(k)\}\right)$$



Example: Square wave

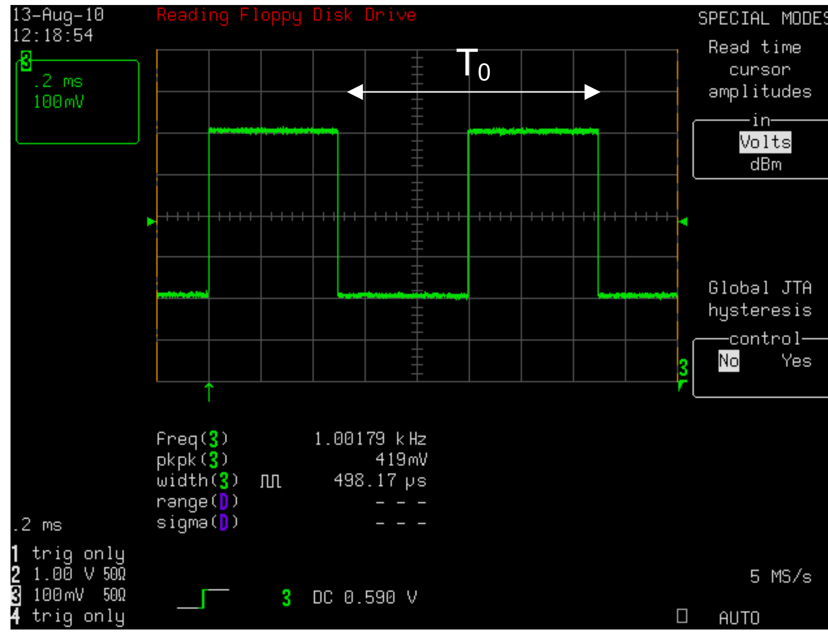


Signal generator Scope Spectrum analyzer

Example: Square wave

Square wave signal

$$x(t) = A\,\mathrm{sgn}\left(\sin\left(\frac{2\pi k}{T_0}t\right)\right)$$



Fourier series coefficients

$$X(k) = \frac{1}{T_0}\int_{T_0} x(t)e^{-i\frac{2\pi k}{T_0}t}dt$$

$$= -\frac{A}{T_0}\int_{-\frac{1}{2}T_0}^{0} e^{-i\frac{2\pi k}{T_0}t}dt + \frac{A}{T_0}\int_{0}^{\frac{1}{2}T_0} e^{-i\frac{2\pi k}{T_0}t}dt$$

$$= -\frac{A}{-i2\pi k}\left(1 - e^{i\pi k}\right) + \frac{A}{-i2\pi k}\left(e^{-i\pi k} - 1\right)$$

$$= \frac{A}{i2\pi k}\left(1 - e^{i\pi k} + 1 - e^{-i\pi k}\right) = -i\frac{A}{\pi k}2\left(1 - \cos(\pi k)\right)$$

$$= -i\frac{2A}{\pi k}\sin^2\left(\frac{\pi k}{2}\right)$$

$$= \begin{cases} -i\dfrac{2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

Example: Square wave (again)

Square wave signal

$$x(t) = A\,\mathrm{sgn}\left(\sin\left(\frac{2\pi k}{T_0}t\right)\right)$$

is odd => cosine series vanish.

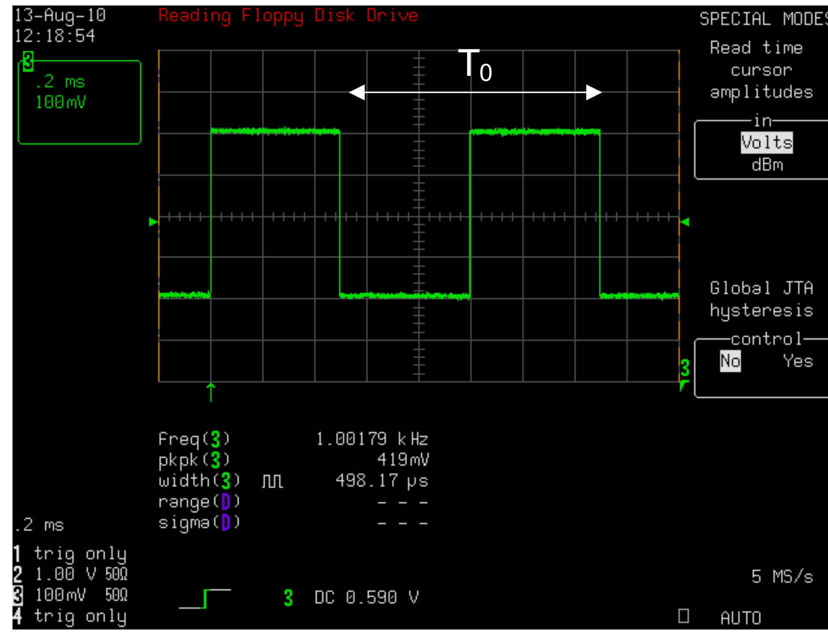
Fourier series coefficients

$$\beta_k = \frac{2}{T_0}\int_{T_0} x(t)\sin\left(\frac{2\pi k}{T_0}t\right)dt$$

$$-\frac{2A}{T_0}\int_{-\frac{T_0}{2}}^{0}\sin\left(\frac{2\pi k}{T_0}t\right)dt + \frac{2A}{T_0}\int_{0}^{\frac{T_0}{2}}\sin\left(\frac{2\pi k}{T_0}t\right)dt$$

$$= \frac{4A}{T_0}\int_{0}^{\frac{T_0}{2}}\sin\left(\frac{2\pi k}{T_0}t\right)dt = \frac{2A}{\pi k}(1-\cos(\pi k)) = \frac{4A}{\pi k}\sin^2\left(\frac{\pi k}{2}\right)$$

$$= \begin{cases} \frac{4A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$



$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

Aalto University
School of Electrical
Engineering

Example: Square wave (again)

Fourier cosine series coefficients

$$\alpha_k = \mathbf{0}$$

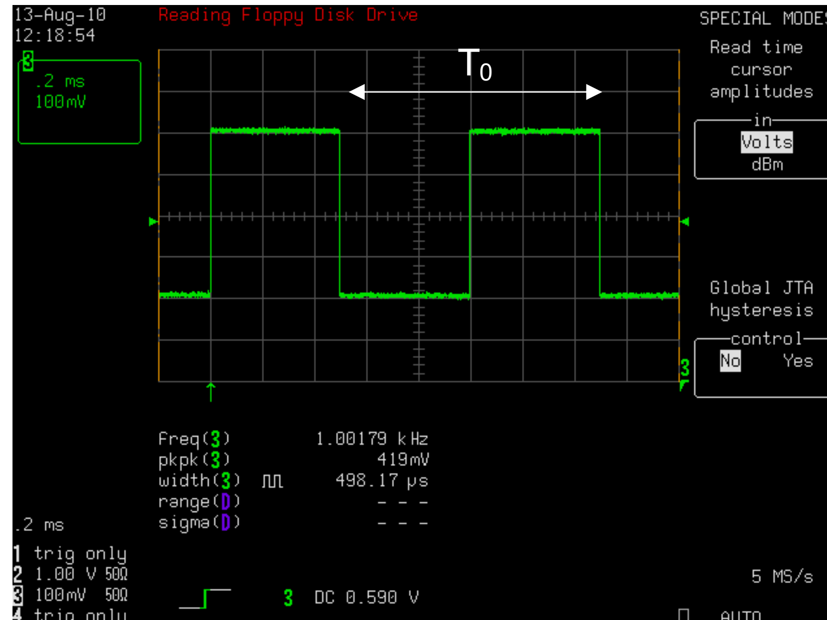
Fourier sine series coefficients

$$\beta_k = \frac{4A}{\pi k} \sin^2\left(\frac{\pi k}{2}\right)$$

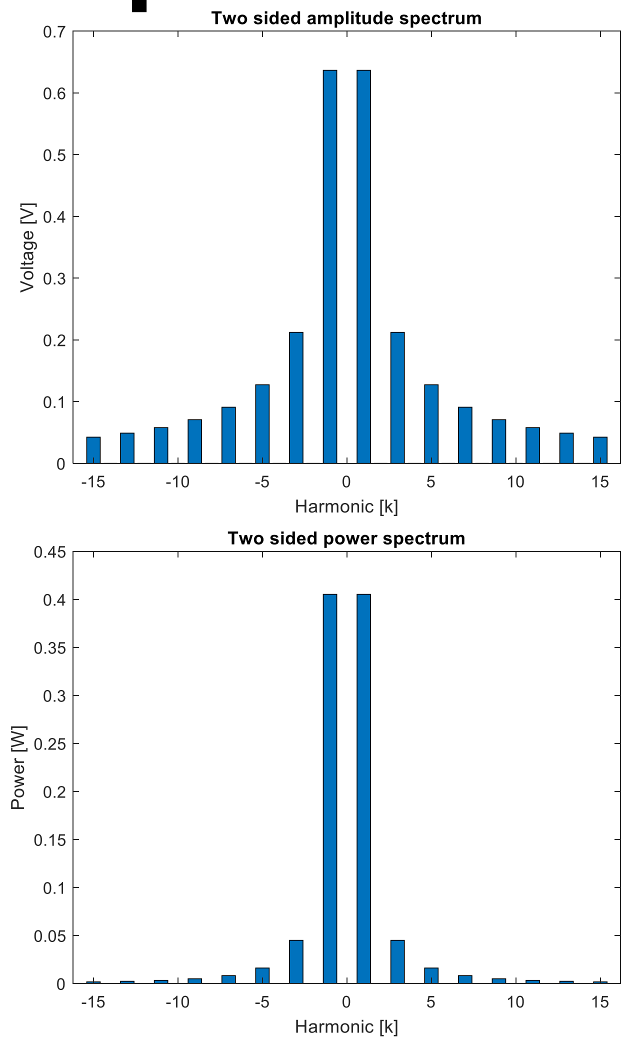
Fourier series coefficients

$$X(k) = \boldsymbol{a}_k - \mathrm{i}\boldsymbol{b}_k = 0 - \mathrm{i}\frac{\beta_k}{2}$$

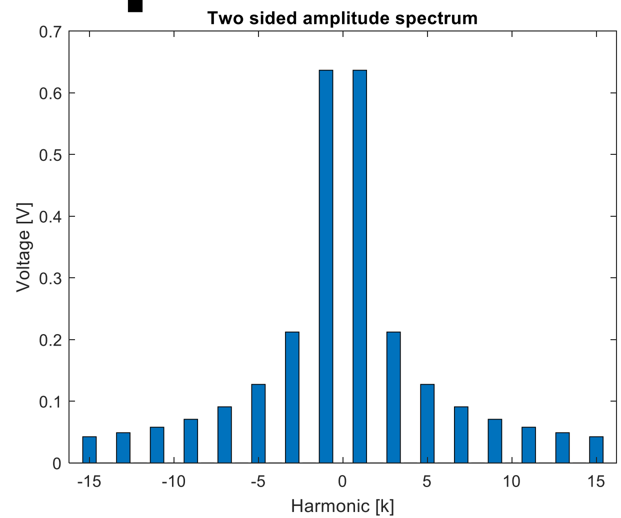
$$= -i\frac{4A}{\pi k} \sin^2\left(\frac{\pi k}{2}\right)$$

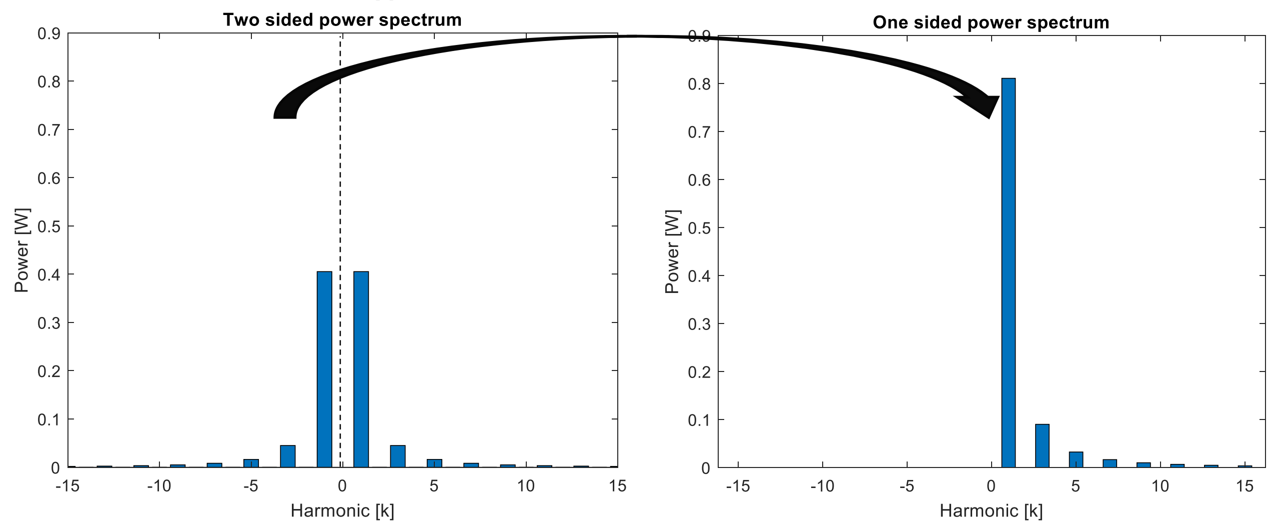


Example: Square wave



Example: Square wave

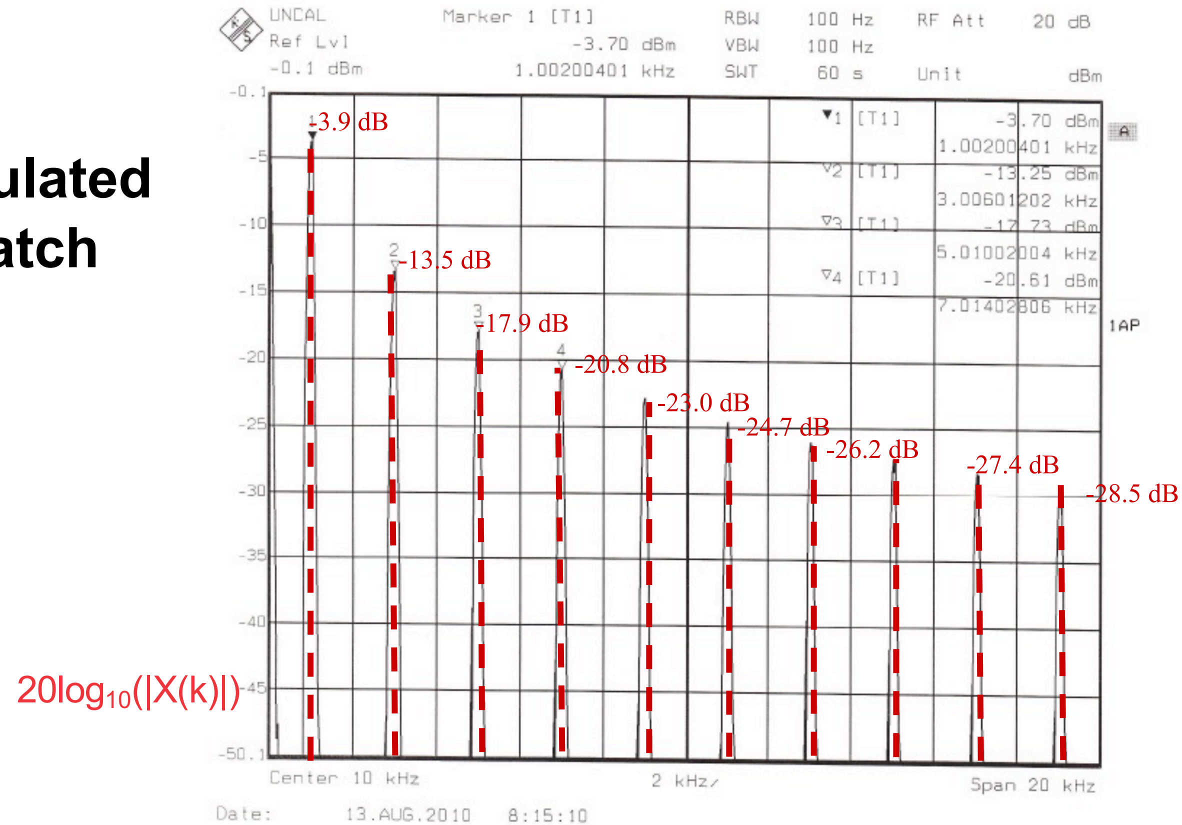




Aalto University
School of Electrical
Engineering

Example: Square wave

- **Measured and calculated power spectrum match well.**



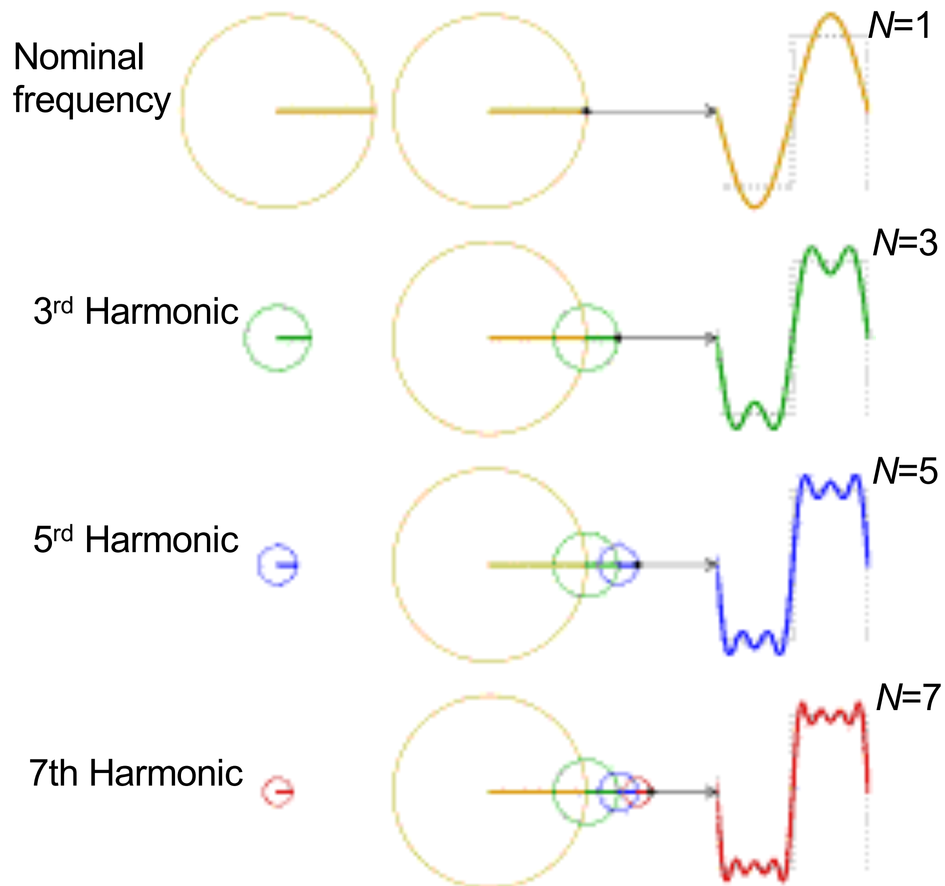
Approximating a periodic with limited number of coefficients

Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{i\frac{2\pi k}{T_0}t}$$

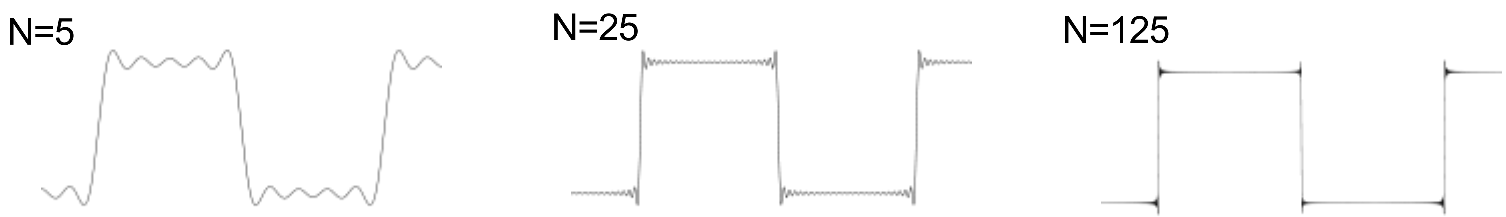
Approximation

$$x_N(t) = \sum_{k=-N}^{N} X(k)\, e^{i\frac{2\pi k}{T_0}t}$$



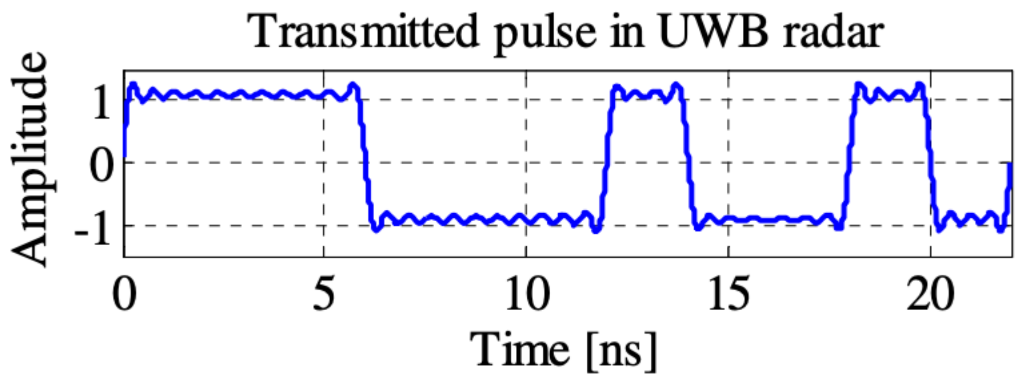
Gibbs phenomena

The Gibbs phenomenon involves both the fact that Fourier sums overshoot at a jump discontinuity, and that this overshoot does not die out as more terms are added to the sum.



Approximating a periodic with limited number of coefficients

Example: Generation of ultrawideband radar signal



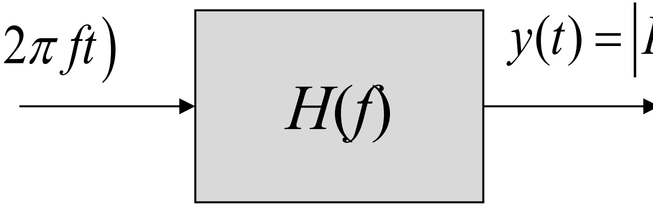
A UWB pulse signal is generated by Fourier series model using 40 harmonies. The pulse is produced by the Barker code with the length of 11, chip width of 2 ns and the total pulse duration 22 of ns

Ali Tayaranian Hosseini, S.M., Amindavar, H. and Ritcey, J.A., 2016. Robust detection in ultra-wideband impulse radar using DPSS-MMSE estimator. *EURASIP Journal on Advances in Signal Processing*, *2016*(1), pp.1-15.

Filtering periodic signals

Given LTI system frequency transfer function H(f), we have

$$x(t) = \cos\left(2\pi f t\right) \qquad y(t) = \left|H\left(f\right)\right| \cos\left(2\pi f t + \arg\left\{H\left(f\right)\right\}\right)$$



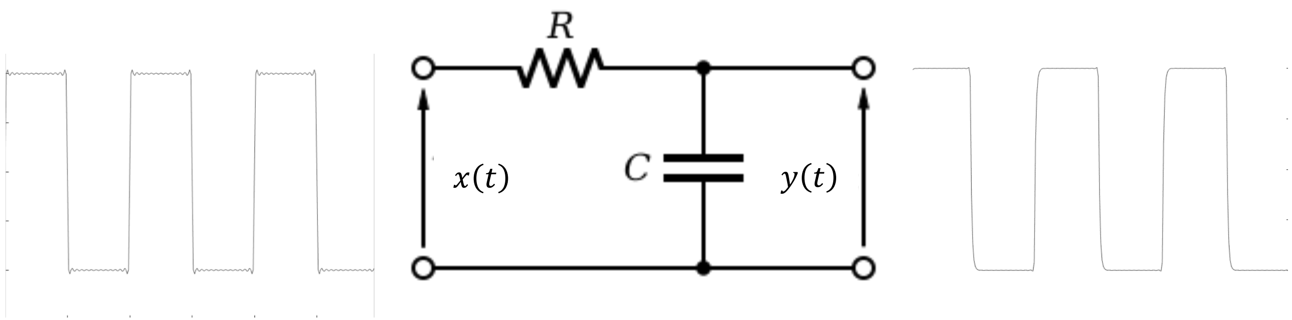
Hence for input $x(t) = X(0) + \sum_{\boldsymbol{k}=1}^{\infty} 2|X(k)| \cos\left(2\pi \frac{k}{T_0} t + \arg\{X(k)\}\right)$

the output becomes

$$y(t) = H(0)X(0) + \sum_{\boldsymbol{k}=1}^{\infty} 2\left|H\left(\frac{k}{T_0}\right)\right| |X(k)| \cos\left(2\pi \frac{k}{T_0} t + \arg\{X(k)\} + \arg\left\{H\left(\frac{k}{T_0}\right)\right\}\right)$$

Filtering periodic signals

- **Example: Square wave passes through a low-pass RC-filter**



aalto.fi

Aalto University
School of Electrical
Engineering