Aalto University
School of Electrical
Engineering

ELEC-E8740 — Static Linear Models and Linear Least Squares

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Contents

Intended Learning Outcomes

After this lecture, you will be able to:

- Identify and construct scalar and vector linear models;
- apply and derive (weighted, regularized, and sequential) linear least squares estimators;
- investigate the properties of linear least squares estimators.

Recap

- Sensor fusion involves three components:
  1. Sensor: Measures a variable of interest, directly or indirectly
  2. Model: A mathematical formulation that relates the variables of interest to the measurements
  3. Estimation Algorithm: Combines the measurements and models to estimate the variables of interest
- Multiple measurements and multidimensional measurements can be written in the same vector notation.
- The least squares method is a good way for deriving estimators.
- (Plain) least squares, weighted least squares, and regularized least squares are useful criteria for estimators.

Scalar Model: Model & Cost Function

- Many sensors measure (a scaled) version of a single unknown x

  $$y_n = gx + r_n,$$

  with $\mathrm{E}\{r_n\} = 0$ and $\mathrm{var}\{r_n\} = \sigma_{r,n}^2$
- The error for one measurement is

  $$e_n = y_n - gx$$

  and the least squares cost function is given by

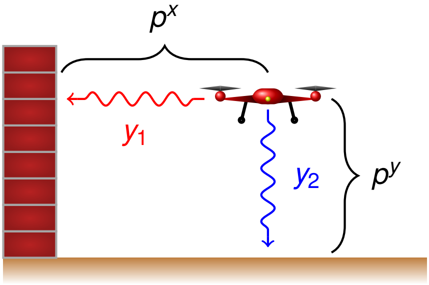
  $$J_{\mathrm{LS}}(x) = \sum_{n=1}^{N} (y_n - gx)^2$$

Scalar Model: Example

Example

- Radar measures the time difference between the sent and reflected signals.
- Double distance divided by the speed of light $\tau = 2p^x / c$, where $c = 299792458 \text{ m/s}$.
- The measurement model is (beware of the notation!)

$$y_n = \frac{2}{c} p^x + r_n, \quad n = 1, \ldots, N.$$



Scalar Model: Minimizing the Cost

- The derivative is given by

$$\frac{\partial J_{\mathrm{LS}}(x)}{\partial x} = -2g \sum_{n=1} y_n + 2Ng^2 x$$

- Setting the derivative to zero and solving for x yields

$$\hat{x}_{\mathrm{LS}} = \frac{1}{Ng} \sum_{n=1}^{N} y_n.$$

- This is the least squares estimator for the model

$$y_n = gx + r_n.$$

Scalar Model: Estimator Properties

- What are the estimator's statistical properties?
- Expected value:

$$\mathrm{E}\{\hat{x}_{LS}\} = x + \sum_{n=1}^{N} \mathrm{E}\{r_n\} = x$$

- Variance:

$$\mathrm{var}\{\hat{x}\} = \frac{1}{N^2 g^2} \sum_{n=1}^{N} \sigma_{r,n}^2$$

and when $\sigma_{r,n}^2 = \sigma^2$ we get

$$\mathrm{var}\{\hat{x}\} = \frac{\sigma^2}{N\, g^2} \quad \text{and} \quad \mathrm{std}\{\hat{x}\} = \frac{1}{\sqrt{N}} \frac{\sigma}{|g|}.$$

- The expectation of $\hat{x}$ is $x \Rightarrow$ estimator is unbiased.

Scalar Model: Example

Example

Let us consider the wall-distance measurement model and assume that we estimate p^x with N measurements:

$$\hat{x}_1 = \frac{c}{2N} \sum_{n=1}^{N} y_n.$$

This estimator is unbiased. Further assume that the standard deviation of the measurement is $\sigma = 10^{-9}$ s (1 nanosecond). Then the standard deviation of the estimator is

$$\mathrm{std}\{\hat{x}_1\} = \frac{1}{\sqrt{N}} \frac{c\,\sigma}{2}.$$

With a single measurement we get the error of 15 cm whereas by averaging 100 measurements the error drops to 1.5 cm.

Vector Models

- Scalar observations, several parameters $x_1, x_2, \ldots, x_K$:

$$\begin{aligned} y_n &= g_1 x_1 + g_2 x_2 + \cdots + g_K x_K + r_n \\ &= \mathbf{g} \mathbf{x} + r_n \end{aligned}$$

- Slightly more generally:

$$y_n = \mathbf{g}_n \mathbf{x} + r_n$$

- Stacking measurements together gives:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_N \end{bmatrix} \mathbf{x} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix}$$

- This has the general form

$$\mathbf{y} = \mathbf{G} \mathbf{x} + \mathbf{r}, \text{ with } \operatorname{Cov}\{\mathbf{r}\} = \mathbf{R} = \operatorname{diag}(\sigma_{r,1}^2, \ldots, \sigma_{r,N}^2).$$

Vector Models (cont.)

- Vector observations, several parameters:

$$\mathbf{y}_n = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1K} \\ g_{21} & g_{22} & \cdots & g_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ g_{d_y 1} & g_{d_y 2} & \cdots & g_{d_y K} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_K \end{bmatrix} + \mathbf{r}_n$$
$$= \mathbf{G}_n \mathbf{x} + \mathbf{r}_n,$$

- Batch notation:

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \\ \vdots \\ \mathbf{G}_N \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_N \end{bmatrix}$$

- In compact notation we again get the same general form:

$$\mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{r}, \text{ with } \operatorname{Cov}\{\mathbf{r}\} = \mathbf{R} = \operatorname{diag}(\mathbf{R}_1, \ldots, \mathbf{R}_N).$$

General Linear Model: Definition

- General form of a linear models:

$$\mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{r},$$

  with $\mathrm{E}\{\mathbf{r}\} = 0$ and $\mathrm{Cov}\{\mathbf{r}\} = \mathbf{R}$.
- This is the general linear model, both the scalar and vector cases can be expressed in this way.

Affine Models

- We might also have a constant bias $\mathbf{b}$ in the model:

$$\mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{b} + \mathbf{r}.$$

- We can now compute a modified measurement $\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{b}$ and rewrite this as

$$\tilde{\mathbf{y}} = \mathbf{G}\mathbf{x} + \mathbf{r}.$$

- Thus is again a general linear model.

Example: Localizing a Drone

- Recall the drone model:

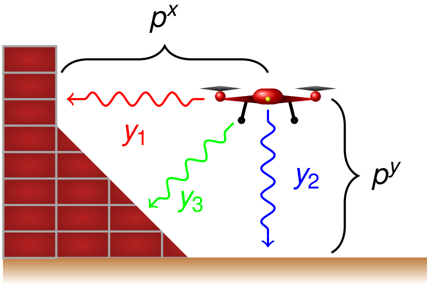
$$y_1 = p^x + r_1,$$
$$y_2 = p^y + r_2,$$
$$y_3 = \frac{1}{\sqrt{2}}(p^x - x_0) + \frac{1}{\sqrt{2}} p^y + r_3.$$

- It has the affine form:

$$\mathbf{y} = \mathbf{G}\,\mathbf{x} + \mathbf{b} + \mathbf{r}.$$

- Can be reduced to linear model by defining

$$\tilde{y}_1 = y_1,$$
$$\tilde{y}_2 = y_2,$$
$$\tilde{y}_3 = y_3 + \frac{1}{\sqrt{2}} x_0.$$



Example: Localizing a Car

- We have:

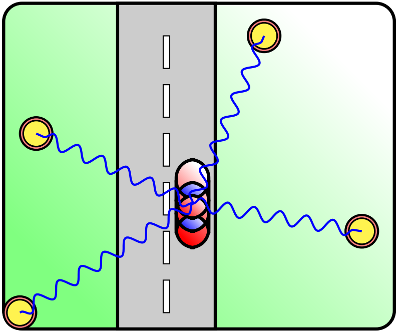
$$
\begin{aligned}
y_1 &= s_1^x - p^x + r_1, \\
y_2 &= s_1^y - p^y + r_2, \\
&\vdots \\
y_{2M} &= s_M^y - p^y + r_{2M}.
\end{aligned}
$$

- Again leads to form

$$\mathbf{y} = \mathbf{G}\,\mathbf{x} + \mathbf{b} + \mathbf{r}.$$

- We can now define

$$
\begin{aligned}
\tilde{y}_1 &= y_1 - s_1^x, \\
&\vdots \\
\tilde{y}_{2M} &= y_{2M} - s_M^y.
\end{aligned}
$$



General Linear Model: Least Squares (1)

- The least squares cost function to minimize:

$$
\begin{aligned}
J_{\text{LS}}(\mathbf{x}) &= (\mathbf{y} - \mathbf{G}\mathbf{x})^{\mathsf{T}}(\mathbf{y} - \mathbf{G}\mathbf{x}) \\
&= \mathbf{y}^{\mathsf{T}}\mathbf{y} - \mathbf{y}^{\mathsf{T}}\mathbf{G}\mathbf{x} - \mathbf{x}^{\mathsf{T}}\mathbf{G}^{\mathsf{T}}\mathbf{y} + \mathbf{x}^{\mathsf{T}}\mathbf{G}^{\mathsf{T}}\mathbf{G}\mathbf{x}
\end{aligned}
$$

- Some vector calculus identities (when $\mathbf{A}$ is symmeric):

$$
\begin{aligned}
\frac{\partial \mathbf{x}^{\mathsf{T}}\mathbf{a}}{\partial \mathbf{x}} &= \frac{\partial \mathbf{a}^{\mathsf{T}}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{a} \\
\frac{\partial \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} &= 2\mathbf{A}\mathbf{x}
\end{aligned}
$$

General Linear Model: Least Squares (2)

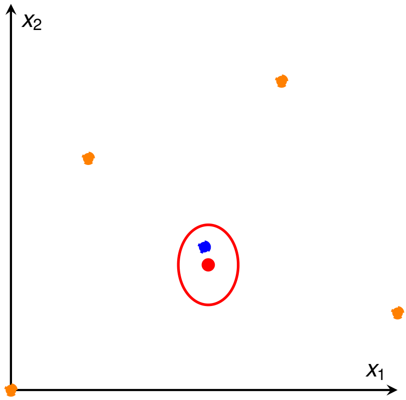
- The least squares estimator for the general linear model is

$$\hat{\mathbf{x}}_{\mathrm{LS}} = (\mathbf{G}^{\mathsf{T}}\mathbf{G})^{-1}\mathbf{G}^{\mathsf{T}}\mathbf{y}$$

- Its statistical properties are

$$\begin{aligned} \mathrm{E}\{\hat{\mathbf{x}}_{\mathrm{LS}}\} &= \mathbf{x} \\ \mathrm{Cov}\{\hat{\mathbf{x}}_{\mathrm{LS}}\} &= (\mathbf{G}^{\mathsf{T}}\mathbf{G})^{-1}\mathbf{G}^{\mathsf{T}}\mathbf{R}\mathbf{G}((\mathbf{G}^{\mathsf{T}}\mathbf{G})^{-1})^{\mathsf{T}}. \end{aligned}$$

Example: Localizing a Car (1)



Weighted Linear Least Squares (1)

- Recall the general linear model:

$$\mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{r}.$$

with $\mathrm{E}\{\mathbf{r}\} = 0$ and $\mathrm{Cov}\{\mathbf{r}\} = \mathbf{R}$.

- Weighted least squares cost function:

$$J_{\mathrm{WLS}}(\mathbf{x}) = (\mathbf{y} - \mathbf{G}\mathbf{x})^{\mathsf{T}} \mathbf{R}^{-1} (\mathbf{y} - \mathbf{G}\mathbf{x})$$

- We can now derive the estimator in the same way as for (plain) least squares.

Weighted Linear Least Squares (2)

- Weighted linear least squares estimator:

$$\hat{\mathbf{x}}_{\mathrm{WLS}} = (\mathbf{G}^\mathsf{T}\mathbf{R}^{-1}\mathbf{G})^{-1}\mathbf{G}^\mathsf{T}\mathbf{R}^{-1}\mathbf{y}.$$

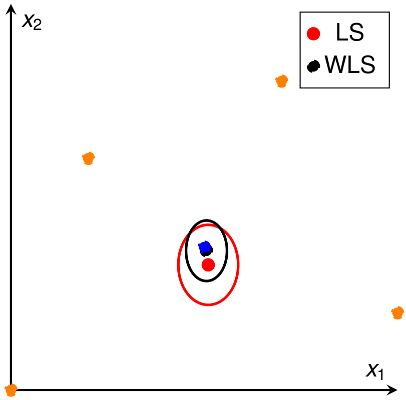
- Properties:

$$\begin{aligned}\mathrm{E}\{\hat{\mathbf{x}}_{\mathrm{WLS}}\} &= \mathbf{x} \\ \mathrm{Cov}\{\hat{\mathbf{x}}_{\mathrm{WLS}}\} &= (\mathbf{G}^\mathsf{T}\mathbf{R}^{-1}\mathbf{G})^{-1}\end{aligned}$$

- It can be shown that $\mathbf{W} = \mathbf{R}^{-1}$ minimizes $\mathrm{Cov}\{\hat{\mathbf{x}}_{\mathrm{WLS}}\}$ over all choices for $\mathbf{W}$ and in this case

$$\mathrm{Cov}\{\hat{\mathbf{x}}_{\mathrm{WLS}}\} \le \mathrm{Cov}\{\hat{\mathbf{x}}_{\mathrm{LS}}\}$$

Example: Localizing a Car (2)



Regularized Linear Least Squares (1/2)

- The regularized least squares criterion

$$J_{\mathrm{ReLS}}(\mathbf{x}) = (\mathbf{y} - \mathbf{G}\mathbf{x})^{\mathsf{T}} \mathbf{R}^{-1} (\mathbf{y} - \mathbf{G}\mathbf{x}) + (\mathbf{x} - \mathbf{m})^{\mathsf{T}} \mathbf{P}^{-1} (\mathbf{x} - \mathbf{m}).$$

- The regularized linear least squares estimator

$$\hat{\mathbf{x}}_{\mathrm{ReLS}} = (\mathbf{G}^{\mathsf{T}} \mathbf{R}^{-1} \mathbf{G} + \mathbf{P}^{-1})^{-1} (\mathbf{G}^{\mathsf{T}} \mathbf{R}^{-1} \mathbf{y} + \mathbf{P}^{-1} \mathbf{m}).$$

- The expectation is not $\mathbf{x}$!
- The covariance of the estimator is

$$\mathrm{Cov}\{\hat{\mathbf{x}}_{\mathrm{ReLS}}\} = (\mathbf{G}^{\mathsf{T}} \mathbf{R}^{-1} \mathbf{G} + \mathbf{P}^{-1})^{-1}.$$

- The covariance is always smaller (or equal) to the WLS estimator.

Regularized Linear Least Squares (2/2)

- By using the matrix inversion formula we can write

$$(\mathbf{G}^\mathsf{T}\mathbf{R}^{-1}\mathbf{G} + \mathbf{P}^{-1})^{-1} = \mathbf{P} - \mathbf{P}\,\mathbf{G}^\mathsf{T}\,(\mathbf{G}\,\mathbf{P}\,\mathbf{G}^\mathsf{T} + \mathbf{R})^{-1}\,\mathbf{G}\,\mathbf{P}.$$

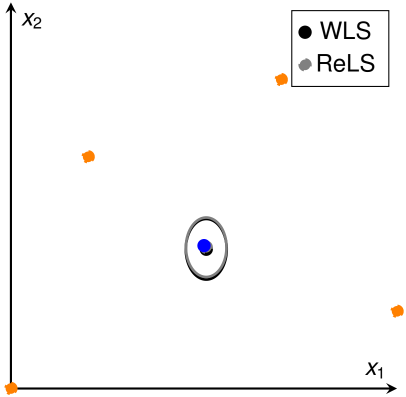
- This gives

$$\begin{aligned}
\mathbf{K} &= \mathbf{P}\mathbf{G}^\mathsf{T}(\mathbf{G}\mathbf{P}\mathbf{G}^\mathsf{T} + \mathbf{R})^{-1}, \\
\hat{\mathbf{x}}_{\mathrm{ReLS}} &= \mathbf{m} + \mathbf{K}(\mathbf{y} - \mathbf{G}\mathbf{m}), \\
\mathrm{Cov}\{\hat{\mathbf{x}}_{\mathrm{ReLS}}\} &= \mathbf{P} - \mathbf{K}(\mathbf{G}\mathbf{P}\mathbf{G}^\mathsf{T} + \mathbf{R})\mathbf{K}^\mathsf{T}.
\end{aligned}$$

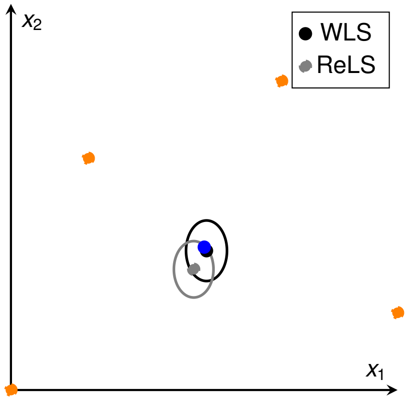
- Finally, we can always rewrite regularized least squares as weighted least squares:

$$\begin{aligned}
J_{\mathrm{ReLS}}(\mathbf{x}) &= (\mathbf{y} - \mathbf{G}\mathbf{x})^\mathsf{T}\mathbf{R}^{-1}(\mathbf{y} - \mathbf{G}\mathbf{x}) + (\mathbf{m} - \mathbf{x})^\mathsf{T}\mathbf{P}^{-1}(\mathbf{m} - \mathbf{x}) \\
&= \left(\begin{bmatrix}\mathbf{y} \\ \mathbf{m}\end{bmatrix} - \begin{bmatrix}\mathbf{G} \\ \mathbf{I}\end{bmatrix}\mathbf{x}\right)^\mathsf{T} \begin{bmatrix}\mathbf{R}^{-1} & 0 \\ 0 & \mathbf{P}^{-1}\end{bmatrix} \left(\begin{bmatrix}\mathbf{y} \\ \mathbf{m}\end{bmatrix} - \begin{bmatrix}\mathbf{G} \\ \mathbf{I}\end{bmatrix}\mathbf{x}\right).
\end{aligned}$$

Example: Localizing a Car (3)



Example: Localizing a Car (4)



Sequential Linear Least Squares (1/2)

- In many cases, the sensor data arrives sequentially at the estimator.
- Assume that we have calculated the weighted least squares (WLS) estimate using $\mathbf{y}_{1:n-1} = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{n-1}\}$:

$$\hat{\mathbf{x}}_{n-1} = (\mathbf{G}_{1:n-1}^\mathsf{T} \mathbf{R}_{1:n-1}^{-1} \mathbf{G}_{1:n-1})^{-1} \mathbf{G}_{1:n-1}^\mathsf{T} \mathbf{R}_{1:n-1}^{-1} \mathbf{y}_{1:n-1},$$
$$\mathrm{Cov}\{\hat{\mathbf{x}}_{k-1}\} = (\mathbf{G}_{1:n-1}^\mathsf{T} \mathbf{R}_{1:n-1}^{-1} \mathbf{G}_{1:n-1})^{-1} = \mathbf{P}_{n-1}.$$

- We can now rewrite the WLS the cost function in sequential form

$$\begin{aligned} J_{\mathrm{SLS}}(\mathbf{x}) = {} & (\mathbf{y}_{1:n-1} - \mathbf{G}_{1:n-1}\mathbf{x})^\mathsf{T} \mathbf{R}_{1:n-1}^{-1} (\mathbf{y}_{1:n-1} - \mathbf{G}_{1:n-1}\mathbf{x}) \\ & + (\mathbf{y}_n - \mathbf{G}_n\mathbf{x})^\mathsf{T} \mathbf{R}_n^{-1} (\mathbf{y}_n - \mathbf{G}_n\mathbf{x}). \end{aligned}$$

Sequential Linear Least Squares (2/2)

- Setting gradient to zero and substituting the already computed result gives:

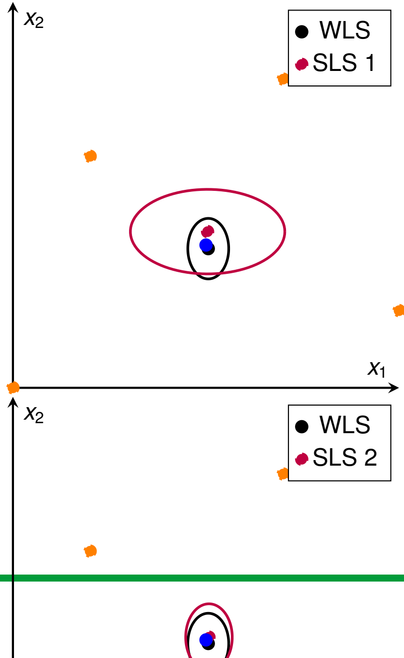
$$\begin{aligned}\hat{\mathbf{x}}_n &= (\mathbf{G}_{1:n-1}^\mathsf{T}\mathbf{R}_{1:n-1}^{-1}\mathbf{G}_{1:n-1} + \mathbf{G}_n^\mathsf{T}\mathbf{R}_n^{-1}\mathbf{G}_n)^{-1} \\ &\qquad \times (\mathbf{G}_{1:n-1}^\mathsf{T}\mathbf{R}_{1:n-1}^{-1}\mathbf{y}_{1:n-1} + \mathbf{G}_n^\mathsf{T}\mathbf{R}_n^{-1}\mathbf{y}_n) \\ &= (\mathbf{P}_{n-1}^{-1} + \mathbf{G}_n^\mathsf{T}\mathbf{R}_n^{-1}\mathbf{G}_n)^{-1}(\mathbf{G}_{1:n-1}^\mathsf{T}\mathbf{R}_{1:n-1}^{-1}\mathbf{y}_{1:n-1} + \mathbf{G}_n^\mathsf{T}\mathbf{R}_n^{-1}\mathbf{y}_n).\end{aligned}$$

- Using matrix inversion formula gives

$$\begin{aligned}\mathbf{K}_n &= \mathbf{P}_{n-1}\mathbf{G}_n^\mathsf{T}(\mathbf{G}_n\mathbf{P}_{n-1}\mathbf{G}_n^\mathsf{T} + \mathbf{R}_n)^{-1}, \\ \hat{\mathbf{x}}_n &= \hat{\mathbf{x}}_{n-1} + \mathbf{K}_n(\mathbf{y}_n - \mathbf{G}_n\hat{\mathbf{x}}_{n-1}), \\ \mathrm{Cov}\{\hat{\mathbf{x}}_n\} &= \mathbf{P}_{n-1} - \mathbf{K}_n(\mathbf{G}_n\mathbf{P}_{n-1}\mathbf{G}_n^\mathsf{T} + \mathbf{R}_n)\mathbf{K}_n^\mathsf{T} = \mathbf{P}_n.\end{aligned}$$

- This is now a recursion for the estimates.
- Regularized least squares results from $\hat{\mathbf{x}}_0 = \mathbf{m}$ and $\mathbf{P}_0 = \mathbf{P}$.

Example: Localizing a Car (5)



Summary (1)

- The general linear model is given by

$$\mathbf{y} = \mathbf{G}\mathbf{x} + \mathbf{r}, \; \mathrm{E}\{\mathbf{r}\} = 0, \; \mathrm{Cov}\{\mathbf{r}\} = \mathbf{R}$$

- Affine models can be tackled by rewriting

$$\begin{aligned} \mathbf{y} &= \mathbf{G}\mathbf{x} + \mathbf{b} + \mathbf{r}, \\ \underbrace{\mathbf{y} - \mathbf{b}}_{\tilde{\mathbf{y}}} &= \mathbf{G}\mathbf{x} + \mathbf{r}. \end{aligned}$$

- Different least squares estimators:

$$\begin{aligned} \hat{\mathbf{x}}_{\mathrm{LS}} &= (\mathbf{G}^\mathsf{T}\mathbf{G})^{-1}\mathbf{G}^\mathsf{T}\mathbf{y}, \\ \hat{\mathbf{x}}_{WLS} &= (\mathbf{G}^\mathsf{T}\mathbf{R}^{-1}\mathbf{G})^{-1}\mathbf{G}^\mathsf{T}\mathbf{R}^{-1}\mathbf{y}, \\ \hat{\mathbf{x}}_{\mathrm{ReLS}} &= (\mathbf{G}^\mathsf{T}\mathbf{R}^{-1}\mathbf{G} + \mathbf{P}^{-1})^{-1}(\mathbf{G}^\mathsf{T}\mathbf{R}^{-1}\mathbf{y} + \mathbf{P}^{-1}\mathbf{m}). \end{aligned}$$

- We also computed their expectations and covariances.

Summary (2)

- Alternative form of regularized least squares estimator:

$$
\begin{aligned}
\mathbf{K} &= \mathbf{P}\mathbf{G}^\mathsf{T}(\mathbf{G}\mathbf{P}\mathbf{G}^\mathsf{T} + \mathbf{R})^{-1}, \\
\hat{\mathbf{x}}_{\mathrm{ReLS}} &= \mathbf{m} + \mathbf{K}(\mathbf{y} - \mathbf{G}\mathbf{m}), \\
\mathrm{Cov}\{\hat{\mathbf{x}}_{\mathrm{ReLS}}\} &= \mathbf{P} - \mathbf{K}(\mathbf{G}\mathbf{P}\mathbf{G}^\mathsf{T} + \mathbf{R})\mathbf{K}^\mathsf{T}.
\end{aligned}
$$

- Sequential (weighted/regularized) least squares estimator:

$$
\begin{aligned}
\mathbf{K}_n &= \mathbf{P}_{n-1}\mathbf{G}_n^\mathsf{T}(\mathbf{G}_n\mathbf{P}_{n-1}\mathbf{G}_n^\mathsf{T} + \mathbf{R}_n)^{-1}, \\
\hat{\mathbf{x}}_n &= \hat{\mathbf{x}}_{n-1} + \mathbf{K}_n(\mathbf{y}_n - \mathbf{G}_n\hat{\mathbf{x}}_{n-1}), \\
\mathbf{P}_n &= \mathbf{P}_{n-1} - \mathbf{K}_n(\mathbf{G}_n\mathbf{P}_{n-1}\mathbf{G}_n^\mathsf{T} + \mathbf{R}_n)\mathbf{K}_n^\mathsf{T}.
\end{aligned}
$$