Problem Set 3
Due date: Friday 7.10 at 12.15

Exercise 1

(a) Let $f : \mathbb{R}^3_+ \to \mathbb{R}$ be such that $f(x_1, x_2, x_3) = 10x_1^{\frac{1}{3}} x_2^{\frac{1}{2}} x_3^{\frac{1}{6}}$. We want to approximate the value of this function around the point $x = (27, 16, 64)$.

1. Use differentials to approximate the value of f when x_1 increases to 27.1, x_2 decreases to 15.7, and x_3 remains the same.
2. Compare the approximated value you obtained in the previous answer with the actual value of the function at the new point.
3. Answer again to questions 1. and 2. for $dx_1 = dx_2 = 0.2$ and $dx_3 = -0.4$.

(b) Find the first and second order approximations of the function $f(x, y) = e^{x+2y}$ at $(x, y) = (0, 0)$.

Exercise 2

Consider the set S in $\mathbb{R}^2$ defined as $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$. Show that S is an open set. Is $\hat{S} = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 \leq y \leq 1\}$ open or closed?

Exercise 3

Check if the following equations define z implicitly as a function $g(x, y)$ in a neighbourhood of the given point (x_0, y_0, z_0). If so, calculate $\frac{\partial g}{\partial x}(x_0, y_0)$ and $\frac{\partial g}{\partial y}(x_0, y_0)$.

1. $x^3 + y^3 + z^3 - xyz - 1 = 0$, $(x_0, y_0, z_0) = (0, 0, 1)$;
2. $e^z - z^2 - x^2 - y^2 = 0$, $(x_0, y_0, z_0) = (1, 0, 0)$.

Exercise 4

a) At a given moment of time, the marginal product of labor is 2.5 and the marginal product of capital is 3, the amount of capital is increasing by 2 each unit of time and the rate of change of labor is 0.5. What is the rate of change of output?

b) Assume that $f : \mathbb{R}^n \mapsto \mathbb{R}$ is differentiable, fix $\mathbf{x}^*, \mathbf{d} \in \mathbb{R}^n$, $\mathbf{d} \neq 0$, and form a function $g(t)$, $t \in \mathbb{R}$, by setting $g(t) = f(\mathbf{x}^* + t\mathbf{d})$. Find $g'(0)$.

Exercise 5

Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ be a C^1 utility function, and let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $f'(x) > 0$ for every $x \in \mathbb{R}$ (i.e. f is a strictly increasing function). Define the composite function $v := f \circ u$. Recall that the Marginal Rate of Substitution of u at a point (x_0, y_0) is

$$MRS^u(x_0, y_0) = -\frac{\frac{\partial u}{\partial x}(x_0, y_0)}{\frac{\partial u}{\partial y}(x_0, y_0)}.$$

(a) Write the expression of the MRS at (x_0, y_0) for the composite function v.

(b) Use the chain rule to show that the MRS of u and v at (x_0, y_0) is the same.

(c) Now assume that u is also homogeneous of degree k. Show that the MRS of u is a homogeneous function of degree zero.

Exercise 6

For each of the following production functions, determine whether the corresponding returns to scale are *decreasing*, *increasing*, or *constant*. Throughout the exercise, assume that the parameters a, b, and c are all strictly positive.

(a) $f(x_1, x_2) = ax_1^c + bx_2^c$.

(b) $f(x_1, x_2) = \min\{ax_1, bx_2\}$.

(c) $f(x_1, x_2) = x_1^a x_2^b$.

(d) $f(x_1, x_2) = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2}}$.