Aalto University
School of Science

MS-E2135
Decision Analysis
Lecture 4

- *Risk measures*
- *Value trees*
- *Axioms for preference relations*

Salo, Liesiö, Punkka, Vilkkumaa

Motivation

- **Last time we learned how :**
  - To model the DM's preferences over risk by eliciting her utility function
  - The shape (concave / linear / convex) of the utility function reflects the DM's risk attitude (risk averse / neutral / seeking)
  - Decision recommendations can be inferred from stochastic dominance even if the utility function is not (completely) specified:
    - *If the DM prefers more to less, she should not choose an FSD dominated alternative*
    - *If the DM is also risk averse, she should not choose an SSD dominated alternative*

- **This time ('Part A'):**
  - We cover **risk measures** and examine how they characterize the risks of different alternatives

Risk measures

- **Risk measure** is a real-valued function that maps each decision alternative to a single number which represents the risk level
  - E.g., variance $Var[X] = E[(X - E[X])^2]$
    - *The higher the variance, the higher the risk*

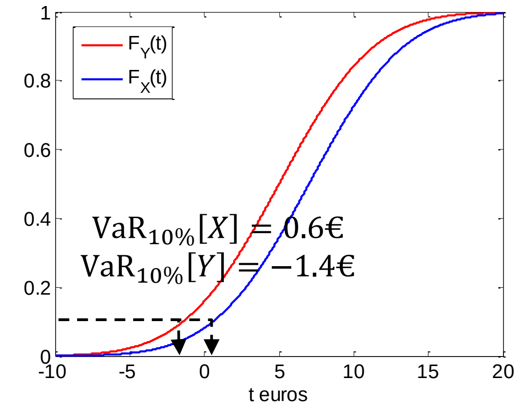
- Risk measures are not based on EUT, but can be used in combination with expected values to generate decision recommendations
  - **Risk constraint**: From the set of alternatives whose risk is below a given threshold, select the one which offers the highest expected value
  - **Risk minimization**: From the set of alternative expected value exceeds a given threshold, select the one with minimum risk
  - **Efficient frontier**: Select an alternative such that these exists no other alternative which would give (i) higher or equal expected value **and** (ii) lower or equal risk, with strict preference for either (i) or (ii)

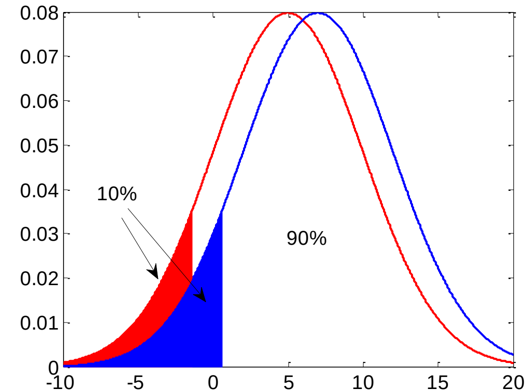
Risk measures: Value-at-Risk (VaR)

- Value-at-Risk ($\mathrm{VaR}_\alpha[X]$) is the outcome such that the probability of getting an outcome which is worse than or equal to this outcome is α:

$$\int_{-\infty}^{\mathrm{VaR}_\alpha[X]} f_X(t)dt = F_X(\mathrm{VaR}_\alpha[X]) = \alpha.$$

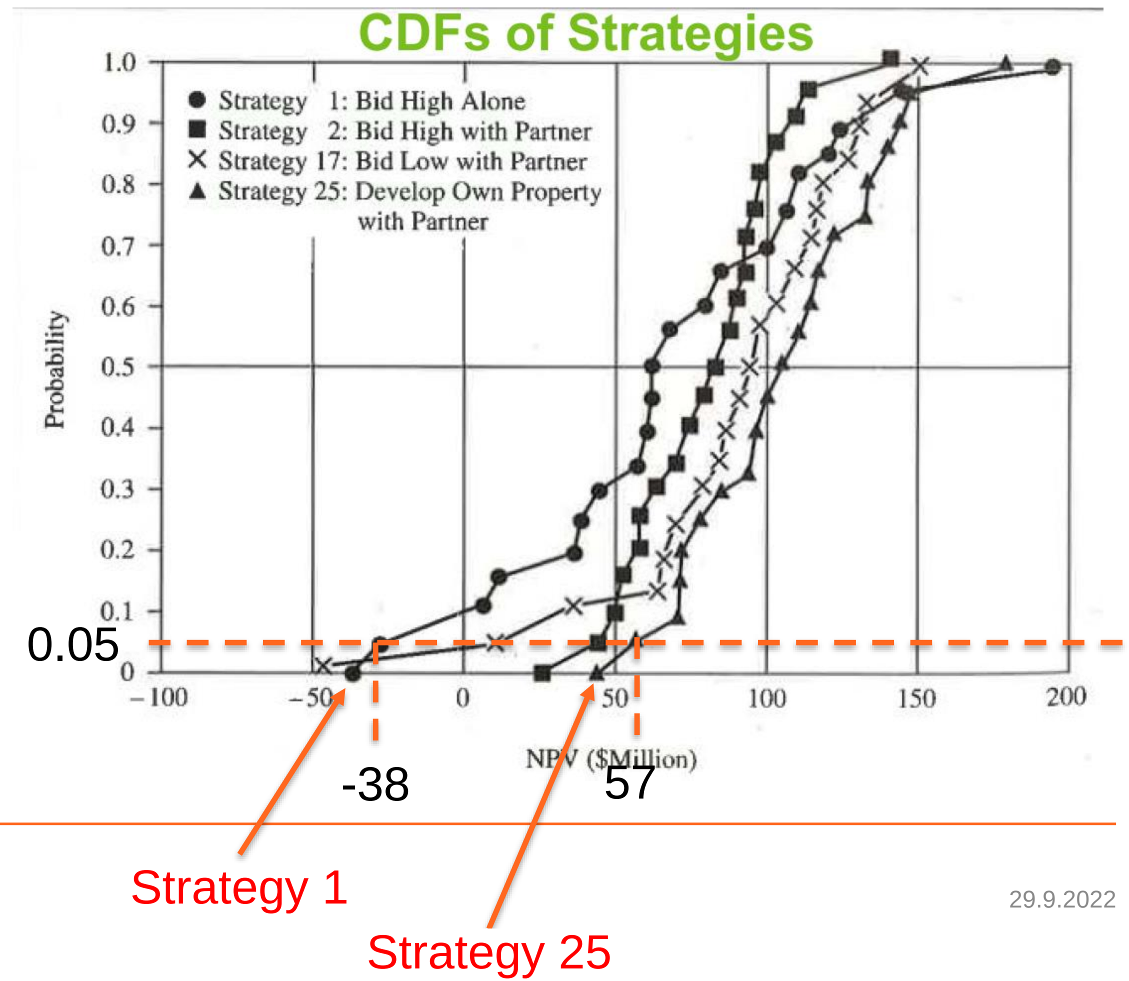
- Higher VaR means smaller risk
  - Unless applied to a loss distribution
- Common values for α: 1%, 5%, and 10%
- Problem: The length/shape of the **tail** is not taken into account





Mining example revisited

- Assess $\mathrm{VaR}_{5\%}$ for strategies 1 and 25

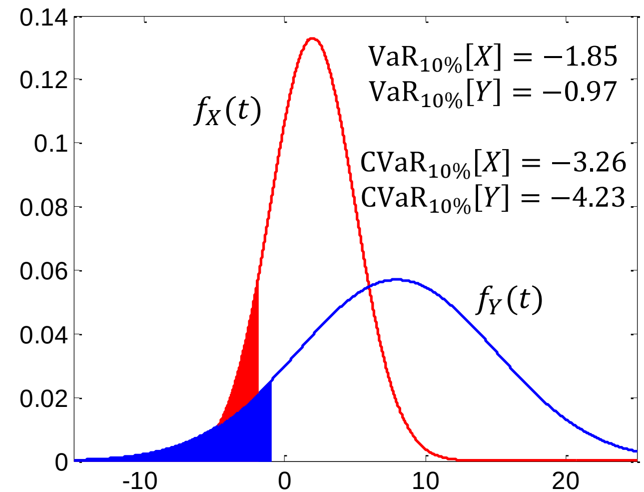


Risk measures: Conditional Value-at-Risk (CVaR)

- Conditional Value-at-Risk ($\mathrm{CVaR}_\alpha[X]$) is the expected outcome given that the outcome is at most VaR_α:

$$\mathrm{CVaR}[X] = E[X|X \leq \mathrm{VaR}_\alpha[X]]$$

- Higher CVaR means smaller risk (unless X represents losses)



- Computation of $\mathrm{CVaR}[X]$ to discrete and continuous X:

$$E[X|X \leq \mathrm{VaR}_\alpha[X]] = \sum_{t \leq \mathrm{VaR}_\alpha[X]} t\,\frac{f_X(t)}{\alpha}, \qquad E[X|X \leq \mathrm{VaR}_\alpha[X]] = \int_{-\infty}^{\mathrm{VaR}_\alpha[X]} t\,\frac{f_X(t)}{\alpha}\,dt.$$

  – Note: $\alpha = P(X \leq \mathrm{VaR}_\alpha[X])$; PMF/PDF $f_X(t)$ is scaled such that it sums/integrates up to 1.

Computation of VaR and CVaR

- If the inverse CDF of X is well-defined, VaR can be obtained from

$$\mathrm{VaR}_\alpha[X] = F_X^{-1}(\alpha)$$

  - In Excel: norm.inv, lognorm.inv, beta.inv, binom.inv etc
  - In Matlab: norminv, logninv, betainv, binoinv etc

- CVaR can then be computed using the formulas on the previous slide
  - Sometimes an analytic solution can be obtained; if, e.g., $X \sim N(\mu, \sigma^2)$ and $\mathrm{VaR}_\alpha[X] = \beta$, then

$$\mathrm{CVaR}_\alpha[X] = \mu - \sigma \frac{\phi\left(\frac{\beta-\mu}{\sigma}\right)}{\Phi\left(\frac{\beta-\mu}{\sigma}\right)},$$

    where ϕ and Φ are the standard normal PDF and CDF, respectively.
  - Sometimes numerical integration is needed

Computation of VaR and CVaR

- With discrete random variables VaR and CVaR are not always well defined for small values of α
  - Example:

t	-10	-5	1	10	20
$f_X(t)$	0.06	0.02	0.02	0.5	0.4

  - $\mathrm{VaR}_{10\%}[X]=1$
  - $\mathrm{CVaR}_{10\%}[X] = \frac{0.06(-10)+0.02(-5)+0.02(1)}{0.06+0.02+0.02}=-6.8$
  - But what are $\mathrm{VaR}_{5\%}[X]$, $\mathrm{CVaR}_{5\%}[X]$? Formally, $\mathrm{VaR}(X)$ can now be defined as

$$\mathrm{VaR}_\alpha(X) = -\inf\{x \in \mathbb{R} : F_X(x) > \alpha\} = F_Y^{-1}(1-\alpha)$$

VaR and CVaR with Monte Carlo - Excel

=AVERAGE(D12:D211)

	A	B	C	D	E	F
1						
8		Col.mean	Col.mean	CVaR-10%		
9		0.507501	1008.358	147.4443		VaR-10%
10						410.5591
11	Sample	u	x	Below VaR		
12	1	0.691314	1249.789	above		
13	2	0.603076	1130.659	above		
14	3	0.548331	1060.723	above		
15	4	0.058081	214.4534	214.4534		
16	5	0.442469	927.6436	above		
17	6	0.628886	1164.452	above		
18	7	0.157181	496.9445	above		
19	8	0.355657	814.9539	above		
20	9	0.545768	1057.488	above		
21	10	0.416183	894.1666	above		
22	11	0.879097	1585.243	above		
23	12	0.022042	-6.64468	-6.64468		
24	13	0.000927	-556.359	-556.359		
25	14	0.071391	267.2461	267.2461		

=PERCENTILE.INC(C12:C211;0.1)

=IF(C12<=F10;C12;"above")

Note! 200 samples is very little, because only 1/10=20 are used to estimate CVaR

VaR and CVaR with Monte Carlo - Matlab

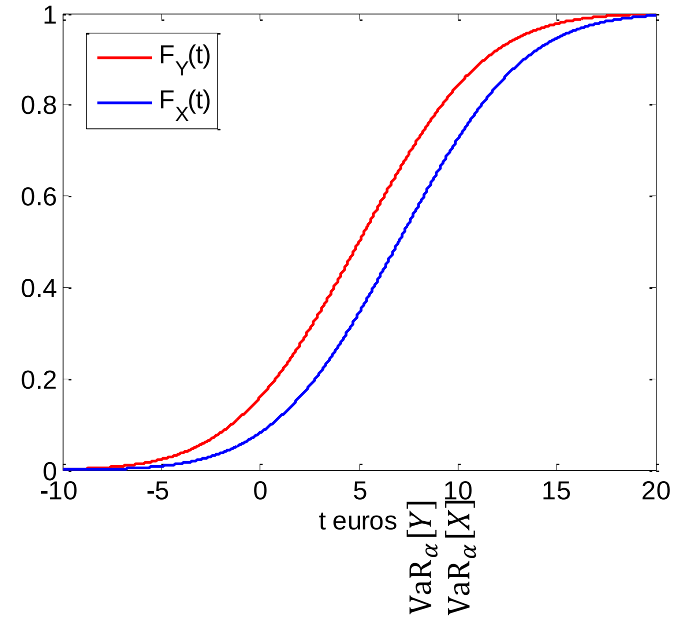
```
S=10^5;                          %Sample size 10,000
mu=1000;
sigma=500;
Sample=normrnd(mu,sigma,S,1);    %Generates 10^5 observations from N(mu,sigma)
VaR=prctile(Sample,10)           %Returns the 10% percentile of the sample
TailIndices=find(Sample<=VaR);   %Returns the indices of those elements
                                 %in the sample below or equal to VaR
CVaR=mean(Sample(TailIndices))   %Computes the arithmetic mean among those
                                 %elements in the sample belor or equal to VaR
```

Risk measures and stochastic dominance

- Recall that FSD and SSD were implied by the cumulative distribution function

- **Theorem:** $X \succcurlyeq_{\text{FSD}} Y$ if and only if $\text{VaR}_\alpha[X] \geq \text{VaR}_\alpha[Y]\ \forall \alpha \in [0,1]$

- **Theorem:** $X \succcurlyeq_{\text{SSD}} Y$ if and only if $\text{CVaR}_\alpha[X] \geq \text{CVaR}_\alpha[Y]\ \forall \alpha \in [0,1]$



EUT vs. Risk measures

- EUT provides a comprehensive way to capture the DM's preferences over uncertain outcomes

- With risk measures, one must answer questions such as
  - Which risk measure should one use?
  - Which α to use in VaR and CVaR?
  - How to combine EV and the value of a risk measure into an overall performance measure?

- Yet, if the answers to these questions are known and incontestable, the use of risk measures can be justified
  - They may be dictated by laws, regulations, industry standards etc.

Motivation

- **So far:**
  - We have considered decision-making situations in which the DM has one objective (e.g., maximize the expected value/utility of a monetary payoff)

- **This time:**
  - We consider decision-making situations in which **the DM has multiple objectives** or, more precisely…
  - **<u>Multiple attributes</u>** with regard to which the achievement of some fundamental objective is measured

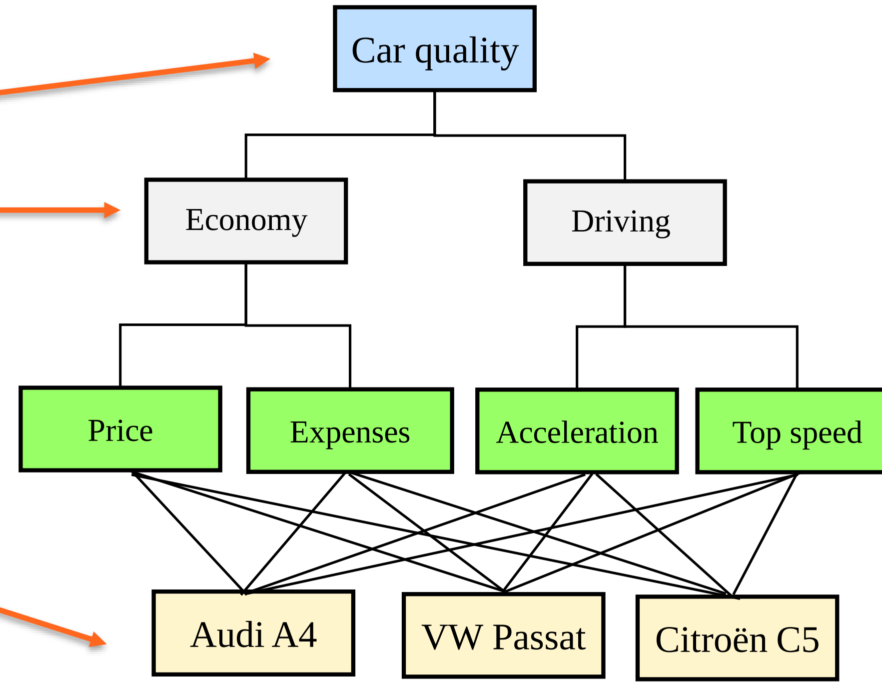
Multiattribute value theory

- Ralph Keeney and Howard Raiffa (1976): *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*, Cambridge University Press.
- James Dyer and Rakesh Sarin (1979): Measurable multiattribute value functions, *Operations Research* Vol. 27, pp. 810-822

- Elements of MAVT
  - A value tree consisting of objectives, attributes, and alternatives
  - Preference relation over the alternatives' *attribute-specific* performances and differences thereof & their representation with an <u>*attribute-specific* value function</u>
  - Preference relation over the alternatives' *overall* performances and differences thereof & their representation with a <u>*multiattribute* value function</u>

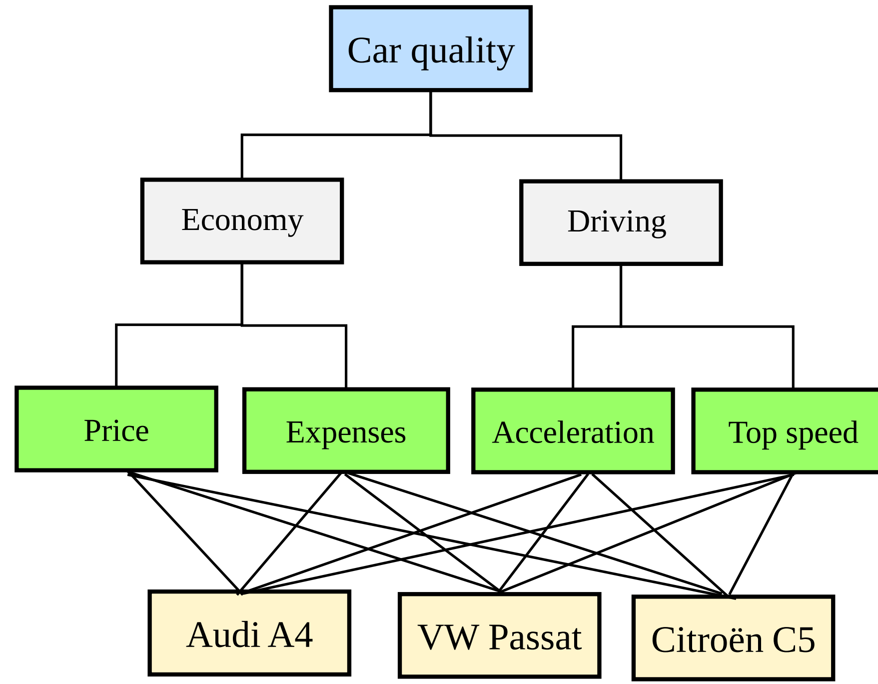
Value tree: objectives, attributes and alternatives

- A value tree consists of
  - A fundamental objective
  - Possible lower-level objectives
  - Attributes that measure the achievement of the objectives
  - Alternatives whose attribute-specific performances are being measured



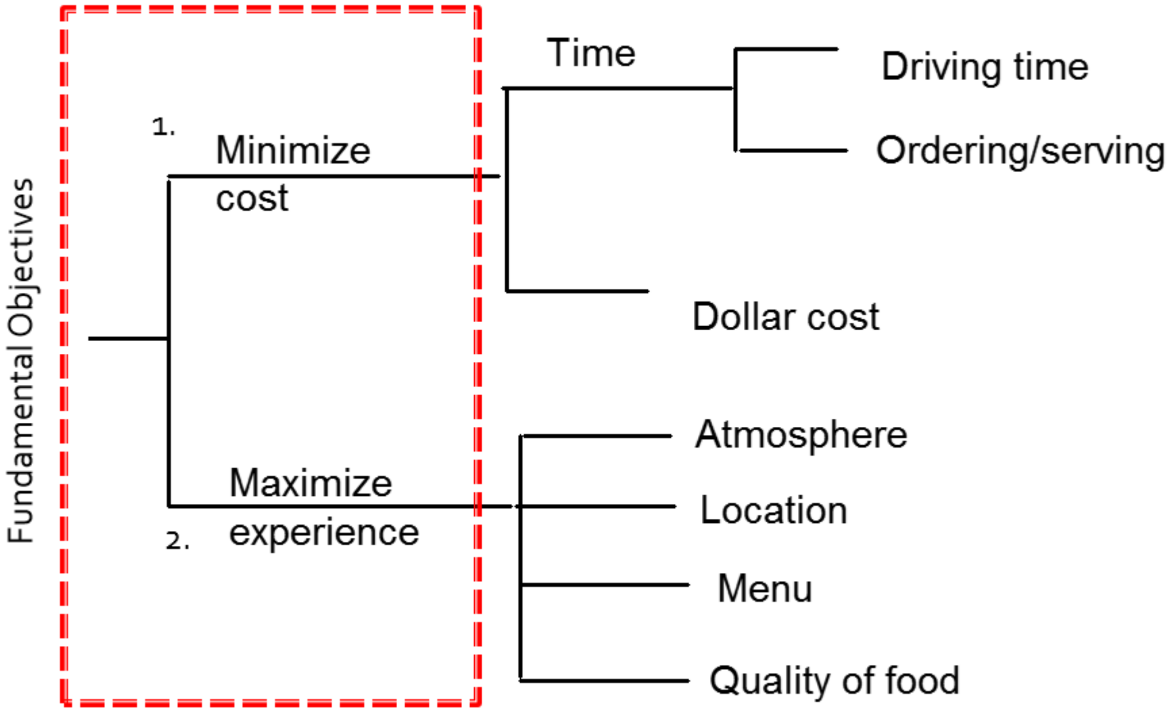
Relations between objectives

- **Fundamental** objectives are the essential reasons you care about a decision
- **Means** objectives help generate alternatives and deepen your understanding of the decision problem



- The division may not be very clear, though

Choosing a restaurant for dining

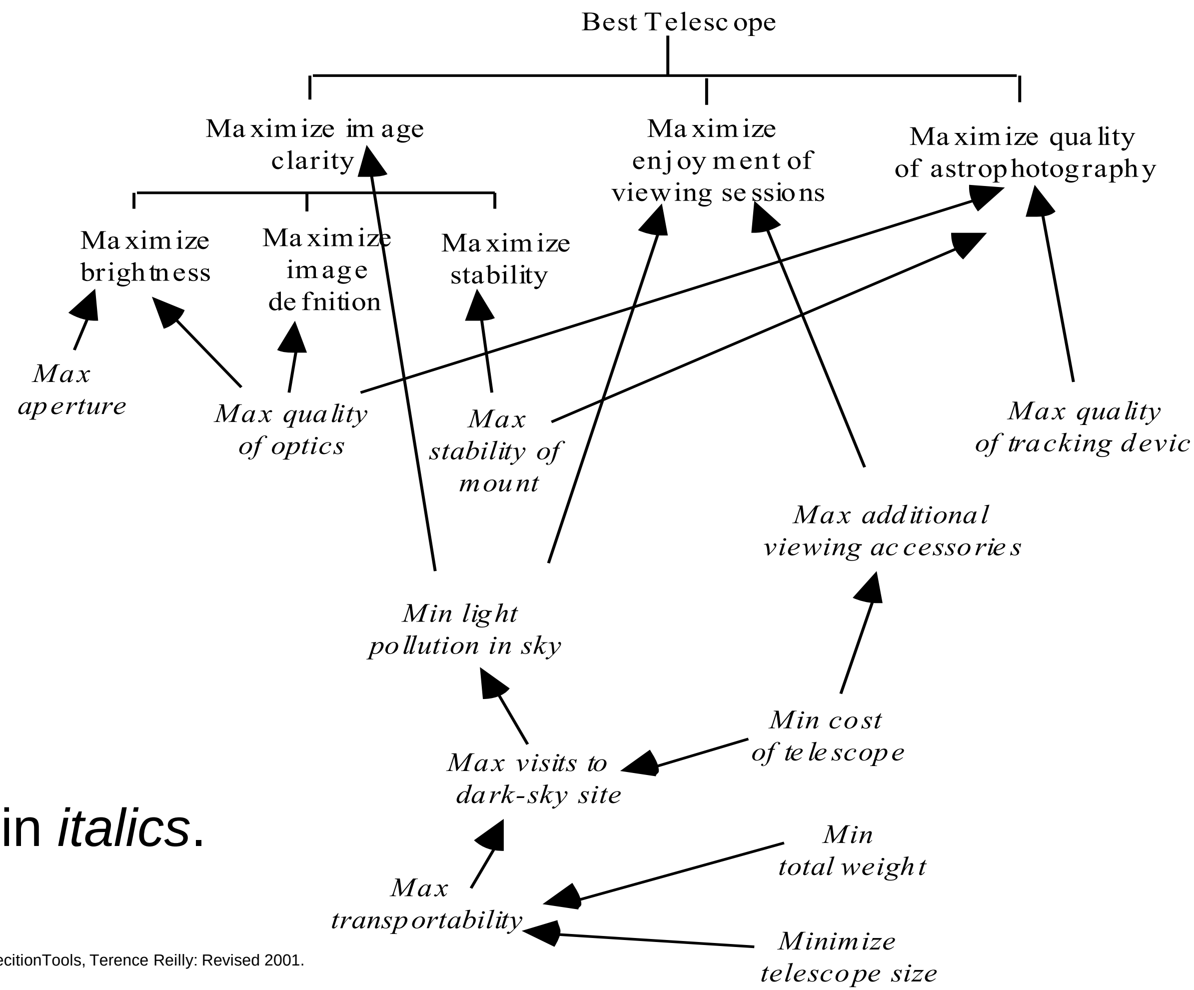


Source: Clemen, Dillon, Reilly, Making Hard Decisions with DecitionTools, Terence Reilly: Revised 2001.

Structuring objectives

	Fundamental Objectives	Means Objectives
To Move: Ask:	*Downward in the Hierarchy:* "What do you mean by that?"	*Away from Fundamental Objectives:* "How could you achieve this?"
To Move: Ask:	*Upward in the Hierarchy:* "Of what more general objective is this an aspect?"	*Toward Fundamental Objectives:* "Why is that important?"

Objectives for choosing a telescope



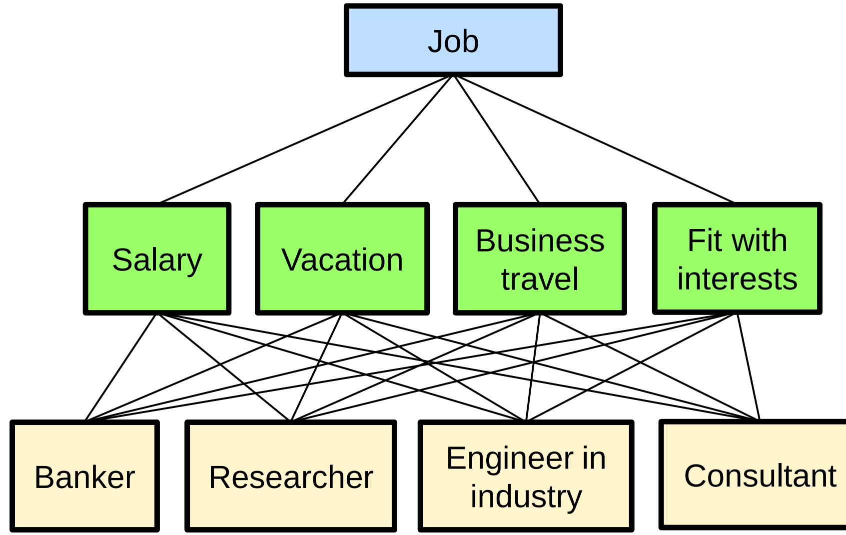
Means objectives in *italics*.

Source: Clemen, Dillon, Reilly, Making Hard Decisions with DecitionTools, Terence Reilly: Revised 2001.

Value tree: objectives, attributes and alternatives

- The attributes $a_1,\ldots, a_n$ have *measurement scales* X_i, i=1,…,n; e.g.,
  - X_1=[1000€/month, 6000€/month]
  - X_2 =[2 weeks/year, 8 weeks/year]
  - X_3 =[0 days/year, 200 days/year]
  - X_4 ={poor, fair, good, excellent}

- Alternatives $x = (x_1, x_2, \ldots x_n)$ are characterized by their performance w.r.t. the attributes; e.g.,
  - Banker=(6000€/month, 5 weeks/year, 40 days/year, fair) $\in X_1 \times X_2 \times X_3 \times X_4$.



Types of attributes

1. Natural attributes

- Shared interpretation by all
  - *“Cost measured in euros”*
  - *“Distance to the closest ski lift in kilometers”*

2. Constructed attributes

- No natural measurement scale
- Developed for the decision context
- Often define/clarify what is meant by the objectives
- E.g., scales for measuring pain

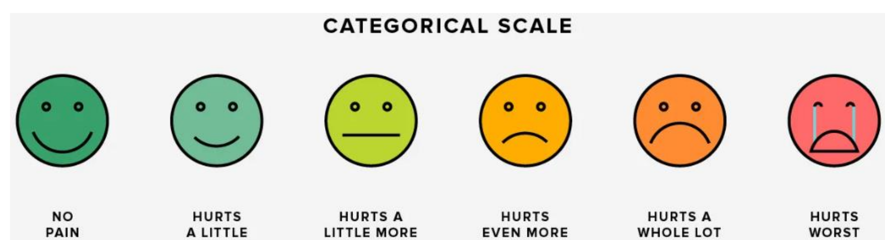
3. Proxy attributes

- Indirect measures to achievement of the objective
  - *“Accommodation quality measured in average review rating”*

From Keeney (1992): *Value-Focused Thinking*, Harvard University Press

Table 4.3. A constructed attribute *Y* of site biological impact

Attribute level	Description of attribute level
0	No loss of productive wetlands or rare species habitat.
1	Loss of 320 acres of productive wetlands and no loss of rare species habitat.
2	Loss of 640 acres of productive wetlands and no loss of rare species habitat or loss of 30 acres of rare species habitat and no loss of productive wetlands.
3	No loss of productive wetlands and loss of 50 acres of rare species habitat.
4	Loss of 640 acres of productive wetlands and loss of 40 acres of rare species habitat.
5	Loss of 640 acres of productive wetlands and loss of 50 acres of rare species habitat.



Preference relation: attribute-specific performance

- Let $\succcurlyeq$ be a preference relation among performance levels *a* and *b* on a given attribute

Preference $a \succcurlyeq b$: a **at least as preferred as** b

Strict preference $a \succ b$ **defined as** $\neg(b \succcurlyeq a)$

Indifference $a \sim b$ **defined as** $a \succcurlyeq b \ \wedge b \succcurlyeq a$

Axioms for preference relation

- **A1:** $\succcurlyeq$ is complete
  - For any $a, b \in X$, either $a \succcurlyeq b$ or $b \succcurlyeq a$ or both
- **A2:** $\succcurlyeq$ is transitive
  - If $a \succcurlyeq b$ and $b \succcurlyeq c$, then $a \succcurlyeq c$

Ordinal value function

Theorem: Let axioms **A1-A2** hold. Then, there exists an <u>ordinal</u> value function $v_i(\cdot)\colon X_i \to \mathbb{R}$ that represents preference relation $\succcurlyeq$ in the sense that

$$v_i(a) \geq v_i(b) \Leftrightarrow a \succcurlyeq b$$

- An ***ordinal*** value function does not capture strength of preference, i.e., it does not indicate <u>by how much</u> more an object is preferred to another

Ordinal value function

- Assume you have two different mopeds A and B with top speeds of 30 and 35km/h, respectively
- You have two (mutually exclusive) *alternatives* for upgrade
  - Increase top speed of moped A to 40
  - Increase top speed of moped B to 45
- You prefer a higher top speed to a lower one
  - 45>40>35>30
  - v(45)=1, v(40)=0.8, v(35)=0.5, v(30)=0.4
  - w(45)=0.9, w(40)=0.8, w(35)=0.6, w(30)=0.4
- Both v and w are *ordinal value functions* representing your preferences but they do not describe your preferences between the two upgrade alternatives
  - v(45)-v(35)=0.5 > v(40)-v(30)=0.4, but w(45)-w(35)=0.3 < w(40)-w(30) =0.4

Ordinal value function

Theorem: Ordinal value functions $v_i(\cdot)$ and $w_i(\cdot)$ represent the same preference relation $\succcurlyeq$ **if and only if** there exists a strictly increasing function $\phi: \mathbb{R} \to \mathbb{R}$ such that $w_i(a) = \phi[v_i(\cdot)] \;\; \forall a \in A$**.**

Example: Let *consultant* $\succ$ *professor* $\succ$ *janitor* be Jim's preferences over these jobs and v(consultant) $= 10 > v$(professor) $= 8 > v$(janitor) $=$ 7.

Then v' and v'' **both represent the same preferences as ordinal value function** v

	consultant	professor	janitor
v	10	8	7
v'	20	16	14
v''	20	16	8

For cardinal measurement, a preference relation over differences is needed

- Let $\succcurlyeq_d$ be preference relation for <u>differences</u> in performance levels on a given attribute

  - Preference $(a \leftarrow b) \succcurlyeq_d (c \leftarrow d)$: a change from b to a is at least as preferred as a change from d to c
  - Strict preference $(a \leftarrow b) \succ_d \ (c \leftarrow d)$ defined as
  $$\neg((c \leftarrow d) \succcurlyeq_d (a \leftarrow b))$$
  - Indifference $(a \leftarrow b) \sim_d (c \leftarrow d)$ defined as
  $$(a \leftarrow b) \succcurlyeq_d (c \leftarrow d) \wedge (c \leftarrow d) \succcurlyeq_d (a \leftarrow b)$$

Axioms for preference relation (cont'd)

- **A3:** $\forall a, b, c \in X_i$: $a \succcurlyeq b \Leftrightarrow (a \leftarrow b) \succcurlyeq_d (c \leftarrow c)$
  - If a is preferred to b, then a change from b to a is preferred to no change
- **A4:** $\forall a, b, c, d \in X_i$: $(a \leftarrow b) \succcurlyeq_d (c \leftarrow d) \Leftrightarrow (d \leftarrow c) \succcurlyeq_d (b \leftarrow a)$
  - E.g., if an increase in salary from 1500€ to 2000€ is preferred to an increase from 2000€ to 2500€, then a decrease from 2500€ to 2000€ is preferred to a decrease from 2000€ to 1500€
- **A5:** $\forall a, b, c, d, e, f \in X_i$: $(a \leftarrow b) \succcurlyeq_d (d \leftarrow e) \wedge (b \leftarrow c) \succcurlyeq_d (e \leftarrow f) \Rightarrow (a \leftarrow c) \succcurlyeq_d (d \leftarrow f)$
  - If two incremental changes are both preferred to some other two, then the overall change resulting from the first two increments is also preferred.
- **A6:** $\forall b, c, d \in X_i\ \exists a \in X_i$ such that $(a \leftarrow b) \sim_d (c \leftarrow d)$ and $\forall b, c \in X_i\ \exists a \in X_i$ such that $(b \leftarrow a) \sim_d (a \leftarrow c)$
  - Equally preferred differences between attribute levels can always be constructed
  - There is always an attribute level a between b and c such that a change from c to a is equally preferred to a change from a to b.
- **A7:** The set (or sequence) $\{a_n | b \succ a_n\ where (a_n \leftarrow a_{n-1}) \sim_d (a_1 \leftarrow a_0)\}$ is finite for any b in X_i
  - The sequence of equally preferred differences over a fixed interval is finite
  - "No b can be infinitely better than other performance levels"

This defines preferred changes that represent preferences

Cardinal value function

- **Theorem:** Let axioms **A1-A7** hold. Then, there exists a <u>cardinal</u> value function $v_i(\cdot)\colon X_i \to \mathbb{R}$ that represents preference relations $\succcurlyeq$ and $\succcurlyeq_d$ in the sense that

$$v_i(a) \geq v_i(b) \Leftrightarrow a \succcurlyeq b$$
$$v_i(a) - v_i(b) \geq v_i(c) - v_i(d) \Leftrightarrow (a \leftarrow b) \succcurlyeq_d (c \leftarrow d).$$

Note: A cardinal value function is unique up to positive affine transformations, i.e., $v_i(x)$ and $v_i'(x) = \alpha v_i(x) + \beta, \alpha > 0$ and represent the same preferences

Cardinal value function: positive affine transformations

Example: Let *consultant* ≻ *professor* ≻ *janitor* and $(consultant \leftarrow professor) \succcurlyeq_d (professor \leftarrow janitor)$ be Jim's preferences and v(consultant) $= 10 > v$(professor) $= 8 > v$(janitor) $= 7$.

Then v' and v'' **both represent same preferences as cardinal value function** v

	consultant	professor	janitor
v	10	8	7
$v' = 2v$	20	16	14
$v'' = v' - 10$	10	6	4

Attribute-specific value functions

- A value function maps the attribute-specific measurement scale onto a numerical scale in accordance with the DM's preferences

- Value and utility:
  - Value is a measure of preference <u>under certainty</u>
  - Utility is a measure of preference <u>under uncertainty</u>

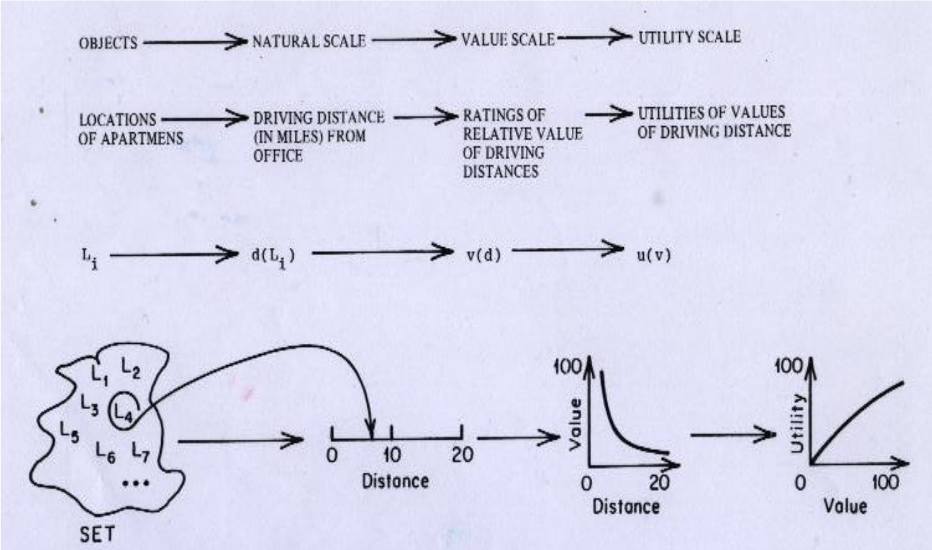


Figure 7.2. The four steps needed to construct value and utility functions.

Summary

- Value trees are used model decisions with multiple objectives

- Under the stated axioms, the DM's preferences for changes on a measurement scale can be captured by a cardinal (measurable) value function
  - "I prefer a change from 0 euros to 10 euros to a change from 10 euros to 22 euros"