Aalto University
School of Science

MS-E2114 Investment Science
Lecture 5: Mean-variance portfolio theory

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Overview

This lecture

- So far, we have analyzed cash flows under certainty
  - Most emphasis has been on fixed income securities
  - Credit risks have not been explicitly addressed
  - Market price volatility (=variability) has not been addressed
- Yet future cash flows and market prices of most investments are uncertain
  - Stock prices, dividends, real estate values, etc.
  - Also the length of the period for which capital is tied can be uncertain
- We cover the Nobel prize framework of Harry Markowitz for portfolio choice under uncertainty
  - Markowitz H. (1952). Portfolio selection, *Journal of Finance* vol. 7, pp. 77-91.
  - Link to Markowitz on the Nobel Prize website
  - Link to Markowitz' Nobel Prize lecture

Overview

Random returns

Portfolio mean and variance

Markowitz model

Two-fund theorem

One-fund theorem

Random returns

- Assume that you invest a fixed amount X_0 now and receive the (random) amount X_1 a year later
- Total return $R = \frac{X_1}{X_0}$
- Rate of return $r = \frac{X_1 - X_0}{X_0}$
  - Thus $R = 1 + r$ and $X_1 = (1 + r)X_0$
- X_1 is random $\Rightarrow r$ is random, too
  - If $X_1 < X_0$, then r will be negative
- The term *return* normally refers to $X_1 - X_0$
  - An absolute sum of money in relevant currency (e.g., €)
  - Sometimes *return* is a shorthand for the rate of return (which is a percentage)

Short selling

- **Short selling** or **shorting** = Selling an asset that one does not own
- To short an asset, one can borrow the asset from someone who owns it (=has a **long position**) (e.g., brokerage firm) and sell it for, say, X_0
- By the end of borrowing period, one has to buy the asset from the market for X_1 to return it (plus the dividends the stock may have paid during the period) to the original owner
- In practice, the borrower has to pay a borrowing cost to the lender
  - A typical borrowing cost for shares of European stocks for an institutional investor is 0.35% (+ dividends paid)
  - Depending on the contract, the lender can call back the asset from the borrower

Short selling

- There are four components in a shorting transaction
  1. Profit or loss from buying the asset back at X_1 at the end of borrowing period: $(X_0 - X_1)$
  2. What happens with X_0 you gained at start?
     - X_0 expands your budget for uses such as investing in other stocks
     - What happens with this extended budget is not here considered a part of profit/loss of shorting, even though one can invest e.g. at the risk-free asset
     - At times, cash X_0 may be used as collateral for the asset loan (interest here belongs to the borrower)
  3. Dividends / coupons paid by the asset during shorting
     - Must be compensated to the lender in shorting
     - Neither the lender nor the borrower receives them, because the asset has been sold
     - Dividends and coupons can be treated as a part of asset return, hence the profit impact will be the same
  4. Margin / fee to compensate the lender

Short selling

- Cost of borrowing equals a margin + dividends / coupons actually paid during the borrowing period
- If the margin and dividends / coupons paid during borrowing are zero, the profit/loss from the transaction is $X_0 - X_1$
- This does not account for what was done with X_0 received in the beginning – it is treated just as an expansion of the budget

Short selling

- If the asset value declines to $X_1 < X_0$, shorting gives a profit $X_0 - X_1 > 0$
- If the asset value increases to $X_1 > X_0$, the difference $X_0 - X_1$ will be negative and shorting leads to a loss of $X_1 - X_0$
- Because prices can increase arbitrarily, losses can become very large $\Rightarrow$ Shorting can be very risky and is therefore prohibited by some institutions

Short selling

- Total return for a short position: Receive $-X_0$ and pay $-X_1$

$$\Rightarrow R = \frac{-X_1}{-X_0} = \frac{X_1}{X_0} = 1 + r$$

  - This is same as for the long position
- Initial position $X_0 < 0$ of asset $\Rightarrow$ profit rX_0
  - Allows one to bet on declining asset values ($r < 0$)
  - Short 100 stock and sell them for $X_0 = 1\ 000$€. If price declines by 10% and you buy the stock back for 900€, and you obtain a profit of 100€

$$r = \frac{X_1 - X_0}{X_0} = \frac{-900 - (-1\ 000)}{-1\ 000} = -0.1$$

$$X_1 - X_0 = rX_0 = -0.1 \cdot (-1\ 000) = 100$$

Portfolio return

- Portfolio of n assets
  - X_{0i} = investment in the i-th asset (negative when shorting)
  - $X_0 = \sum_{i=1}^n X_{0i}$ = total investment
  - Weight of the i-th asset i
    $w_i = \frac{X_{0i}}{X_0} \Rightarrow \sum_{i=1}^n w_i = \sum_{i=1}^n \frac{X_{0i}}{\sum_{j=1}^n X_{0j}} = 1$
  - X_{1i} = cash flow from investment at the end of the period
  - Total return of i-th asset $R_i = 1 + r_i$
- Portfolio return

$$R = \frac{\sum_{i=1}^n X_{1i}}{X_0} = \frac{\sum_{i=1}^n R_i X_{0i}}{X_0} = \frac{\sum_{i=1}^n R_i w_i X_0}{X_0} = \sum_{i=1}^n w_i R_i$$

$$\Rightarrow 1 + r = \sum_{i=1}^n w_i(1 + r_i) = 1 + \sum_{i=1}^n w_i r_i$$

$$\Rightarrow r = \sum_{i=1}^n w_i r_i$$

Random variables

- Expected value $\mathbb{E}[x]$ is the mean ('average') outcome of a random variable
  - For a finite number of realizations x_i with probabilities $p_i, i = 1, 2, \ldots, n$,

$$\mathbb{E}[x] = \sum_{i=1}^{n} p_i x_i = \bar{x}$$

- Variance $\mathrm{Var}[x]$ is the expected value of the squared deviation from the mean $\bar{x}$

$$\begin{aligned} \sigma^2 = \mathrm{Var}[x] &= \mathbb{E}[(x - \bar{x})^2] \\ &= \mathbb{E}[x^2 - 2x\bar{x} + \bar{x}^2] \\ &= \mathbb{E}[x^2] - 2\,\mathbb{E}[x]\bar{x} + \bar{x}^2 \\ \Rightarrow \sigma^2 = \mathrm{Var}[x] &= \mathbb{E}[x^2] - \mathbb{E}[x]^2 \end{aligned}$$

Random variables

- Standard deviation is the expected deviation from the mean

$$\sigma = \mathrm{Std}[x] = \sqrt{\mathrm{Var}[x]}$$

- Covariance $\mathrm{Cov}[x_1, x_2]$ is the expected product of deviations from the respective means of two random variables x_1, x_2

$$\begin{aligned}
\sigma_{12} = \mathrm{Cov}[x_1, x_2] &= \mathbb{E}[(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)] \\
&= \mathbb{E}[x_1 x_2 - x_1 \bar{x}_2 - \bar{x}_1 x_2 + \bar{x}_1 \bar{x}_2] \\
&= \mathbb{E}[x_1 x_2] - \mathbb{E}[x_1]\bar{x}_2 - \bar{x}_1 \mathbb{E}[x_2] + \bar{x}_1 \bar{x}_2 \\
\Rightarrow \sigma_{12} = \mathrm{Cov}[x_1, x_2] &= \mathbb{E}[x_1 x_2] - \mathbb{E}[x_1]\,\mathbb{E}[x_2]
\end{aligned}$$

- Covariance and variance closely related

$$\sigma_1^2 = \mathrm{Var}[x_1] = \mathrm{Cov}[x_1, x_1] = \sigma_{11}$$

Random variables

- Correlation coefficient $\text{Corr}[x_1, x_2]$ measures the strength of the linear relationship of two random variables

$$\rho_{12} = \text{Corr}[x_1, x_2] = \frac{\sigma_{12}}{\sigma_1 \sigma_2} \frac{\text{Cov}[x_1, x_2]}{\sqrt{\text{Var}[x_1]}\sqrt{\text{Var}[x_2]}}$$

  - No correlation $\Leftrightarrow \rho_{12} = 0 \Leftrightarrow \sigma_{12} = 0$
  - Positive correlation $\Leftrightarrow \rho_{12} > 0$
  - Negative correlation $\Leftrightarrow \rho_{12} < 0$
  - Perfect correlation $\Leftrightarrow \rho_{12} = \pm 1$
- We have

$$|\rho_{12}| \leq 1 \Leftrightarrow |\sigma_{12}| \leq \sigma_1 \sigma_2$$

Random variables

- Variance of a linear combination of two random variables

$$\begin{aligned}\sigma^2_{a_1x_1+a_2x_2} &= \mathrm{Var}[a_1x_1 + a_2x_2] \\ &= a_1^2\,\mathrm{Var}[x_1] + a_2^2\,\mathrm{Var}[x_2] + 2a_1a_2\,\mathrm{Cov}[x_1, x_2]\end{aligned}$$

- More generally, the variance of a linear combination of random variables $x_1, x_2, \dots, x_n$ is

$$\begin{aligned}\sigma^2_{\sum_{i=1}^n a_ix_i} &= \mathrm{Var}\left[\sum_{i=1}^n a_ix_i\right] \\ &= \sum_{i=1}^n\sum_{j=1}^n a_ia_j\,\mathrm{Cov}[x_i, x_j] = \sum_{i=1}^n\sum_{j=1}^n a_ia_j\sigma_{ij}\end{aligned}$$

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Two-fund theorem

One-fund theorem

Portfolio mean and variance

- Consider n assets with random returns $r_i, i = 1, 2, \ldots, n$, such that $\mathbb{E}[r_i] = \bar{r}_i$
- Expected return of the portfolio

$$r = \sum_{i=1}^{n} w_i r_i$$

$$\Rightarrow \mathbb{E}[r] = \sum_{i=1}^{n} w_i \, \mathbb{E}[r_i] = \sum_{i=1}^{n} w_i \bar{r}_i$$

- Portfolio variance

$$\sigma^2 = \mathrm{Var}\left[\sum_{i=1}^{n} w_i r_i\right]$$

$$\Rightarrow \sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \, \mathrm{Cov}[r_i, r_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

Diversification

- Investing in several assets tends to lower portfolio variance
  - Deviations from the means tend to average out
  - "Divide your portion to seven, or even to eight, for you do not know what misfortune may occur on the earth." *The Bible, Ecclesiastes 11:2*
- Invest equal amounts in n assets with expected return m, variance σ^2, and uncorrelated returns ($\sigma_{ij} = 0, i \neq j$)

$$\bar{r} = \sum_{i=1}^{n} w_i \bar{r}_i = \sum_{i=1}^{n} \frac{1}{n} m = m$$

$$\mathrm{Var}[r] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n} w_i^2 \sigma_{ii} = \sum_{i=1}^{n} \frac{1}{n^2} \sigma^2 = \frac{1}{n} \sigma^2$$

$$\Rightarrow \lim_{n \to \infty} \mathrm{Var}[r] = \lim_{n \to \infty} \frac{1}{n} \sigma^2 = 0$$

- No variation, yet the expected return is the same!

Diversification

- Uncorrelated assets are ideal for diversification
- If the returns are correlated, say, $\sigma_{ij} = 0.3\sigma^2, i \neq j$, we have

$$\mathrm{Var}[r] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n} \frac{1}{n^2}\sigma_{ii} + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{n^2}\sigma_{ij}$$

$$= \frac{1}{n}\sigma^2 + n(n-1)\frac{1}{n^2}0.3\sigma^2 = \left(0.7\frac{1}{n} + 0.3\right)\sigma^2$$

$$\Rightarrow \lim_{n\to\infty} \mathrm{Var}[r] = \lim_{n\to\infty} \left(0.7\frac{1}{n} + 0.3\right)\sigma^2 = 0.3\sigma^2$$

- Hence, variance <u>cannot</u> be reduced to zero

Mean-standard deviation diagram

- Variance (or standard deviation) of returns is widely used as a measure of risk
  - If two portfolios have the same expected return, then the one with smaller variance is preferred
- Consider 3 assets

Asset i	1	2	3
$\mathbb{E}[r_i]$	10%	12%	14%
σ_i	8%	10%	12%

$$\rho = \begin{bmatrix} 1 & 0.2 & 0.2 \\ 0.2 & 1 & 0.3 \\ 0.2 & 0.3 & 1 \end{bmatrix}$$

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$

$$\Rightarrow \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 0.64\% & 0.16\% & 0.19\% \\ 0.16\% & 1\% & 0.36\% \\ 0.19\% & 0.36\% & 1.44\% \end{bmatrix}$$

Mean-standard deviation diagram

- What are the possible combinations of returns and variances?
  - Pairs $(\sigma, \mathbb{E}[r])$ such that

$$\begin{cases} \mathbb{E}[r] & = \sum_{i=1}^{n} w_i \, \mathbb{E}[r_i] \\ \sigma^2 & = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \end{cases}$$

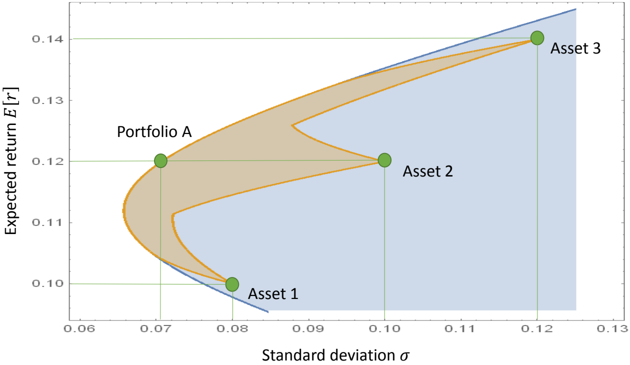
- Portfolio A with $w_1 = w_2 = w_3 = 1/3$ has
  - Expected return

$$\bar{r}_A = \sum_{i=1}^{3} \frac{1}{3} \, \mathbb{E}[r_i] = 0.12$$

  - Standard deviation

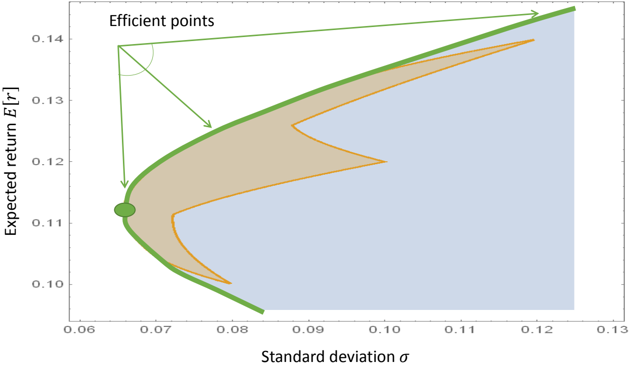
$$\sigma_A = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{1}{3^2} \sigma_{ij}} = 7.07\%$$

Mean-standard deviation diagram



- Yellow area = Set of all possible $(\sigma, \mathbb{E}[r])$ that can be obtained from portfolios such that $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$
- Blue area = As above but with shorting allowed (w_i's, $i = 1, 2, 3$, can be negative as well)

Efficient frontier



- Green curve = Minimum variance set (minimum variance attainable for a given return)
- Green point = Minimum variance point (minimum variance attainable using assets 1, 2 and 3)
- **Efficient frontier** = Curve above (and including) this point

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Markowitz model

- Portfolios of the efficient frontier $\bar{r}$ can be found by solving

$$\min_{\mathbf{w}} \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$
$$\text{s.t.} \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$$
$$\sum_{i=1}^{n} w_i = 1$$

Markowitz model

- Set up the Lagrangian

$$L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} - \lambda \left(\sum_{i=1}^{n} w_i \bar{r}_i - \bar{r} \right) - \mu \left(\sum_{i=1}^{n} w_i - 1 \right)$$

- Equations of the efficient set are solved by setting the partial derivatives of L to zero

$$\frac{\partial}{\partial w_i} L = \sum_{j=1}^{n} w_j \sigma_{ij} - \lambda \bar{r}_i - \mu = 0, \quad \forall i = 1, 2, \ldots, n$$

$$\frac{\partial}{\partial \lambda} L = \sum_{i=1}^{n} w_i \bar{r}_i - \bar{r} = 0$$

$$\frac{\partial}{\partial \mu} L = \sum_{i=1}^{n} w_i - 1 = 0$$

Example: Solving minimum variance portfolio

Asset i	1	2	3
$\mathbb{E}[r_i]$	1	2	3
σ_i	1	1	1

$$\rho = \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} - \lambda\left(\sum_{i=1}^{n} w_i \bar{r}_i - \bar{r}\right) - \mu\left(\sum_{i=1}^{n} w_i - 1\right)$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial w_1} L = w_1\sigma_1^2 - \lambda\bar{r}_1 - \mu = w_1 - \lambda - \mu = 0 & \text{(1a)} \\ \dfrac{\partial}{\partial w_2} L = w_2\sigma_2^2 - \lambda\bar{r}_2 - \mu = w_2 - 2\lambda - \mu = 0 & \text{(1b)} \\ \dfrac{\partial}{\partial w_3} L = w_3\sigma_3^2 - \lambda\bar{r}_3 - \mu = w_3 - 3\lambda - \mu = 0 & \text{(1c)} \\ \dfrac{\partial}{\partial \lambda} L = \displaystyle\sum_{i=1}^{3} w_i \bar{r}_i - \bar{r} = w_1 + 2w_2 + 3w_3 - \bar{r} = 0 & \text{(1d)} \\ \dfrac{\partial}{\partial \mu} L = \displaystyle\sum_{i=1}^{3} w_i - 1 = w_1 + w_2 + w_3 - 1 = 0 & \text{(1e)} \end{cases}$$

Example: Solving minimum variance portfolio

- Equations (1a)-(1c) yield

$$w_1 = \lambda + \mu, \quad w_2 = 2\lambda + \mu, \quad w_3 = 3\lambda + \mu$$

- Substituting these into (1d) and (1e) yields

$$\begin{cases} 14\lambda + 6\mu & = \bar{r} \\ 6\lambda + 3\mu & = 1 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{1}{2}\bar{r} - 1 \\ \mu = \frac{7}{3} - \bar{r} \end{cases},$$

and thus

$$w_1 = \frac{4}{3} - \frac{1}{2}\bar{r}, \quad w_2 = \frac{1}{3}, \quad w_3 = \frac{1}{2}\bar{r} - \frac{2}{3}$$

Example: Solving minimum variance portfolio

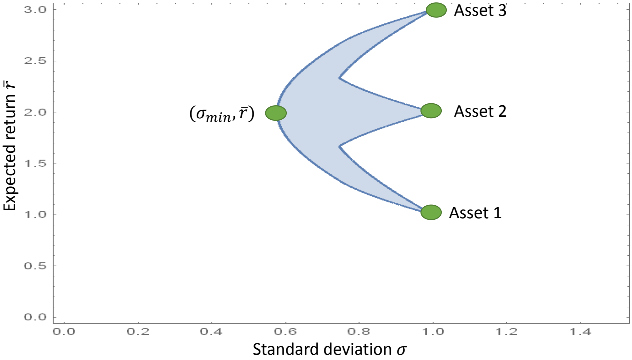
- Substituting the optimal weights w_1, w_2, w_3 into the objective of minimizing the portfolio variance yields

$$\min_{\mathbf{w}} \sigma^2 = \min_{\mathbf{w}} \sum_{i=1}^{3} w_i^2 = \frac{1}{2}\bar{r}^2 - 2\bar{r} + \frac{7}{3} \qquad (2)$$

- The expected return $\bar{r}$ for which variance is minimized can be found by differentiating (2) with respect to $\bar{r}$

$$\begin{aligned}
&\Rightarrow \bar{r} - 2 = 0 \Rightarrow \bar{r} = 2 \\
&\Rightarrow \sigma_{min}^2 = \frac{1}{3}, \quad \sigma_{min} = \frac{1}{\sqrt{3}} \approx 0.577
\end{aligned}$$

Example: Solving minimum variance portfolio



- Blue area = The set of all possible pairs $(\sigma, \mathbb{E}[r])$ that a portfolio can obtain for some $w_1, w_2, w_3 \geq 0, \sum_{i=1}^{n} w_i = 1$

Overview

Random returns

Portfolio mean and variance

Markowitz model

Two-fund theorem

One-fund theorem

Two-fund theorem

Theorem

*(**Two-fund theorem**) Given any two efficient funds (portfolios) with different expected returns, it is possible to duplicate any other efficient portfolio in terms of its mean and variance properties as a combination of these two.*

Proof: Let $\mathbf{w^1}$ and $\mathbf{w^2}$ be efficient portfolios with expected returns $\bar{r}^1$ and $\bar{r}^2$ and corresponding Lagrange multipliers λ^1, μ^1 and λ^2, μ^2. Construct the portfolio $\mathbf{w}^\alpha = \alpha\mathbf{w^1} + (1-\alpha)\mathbf{w^2}, \alpha \in \mathbb{R}$.

- Weights in $\mathbf{w}^\alpha$ sum to 1
- The expected return of $\mathbf{w}^\alpha$ is $\bar{r} = \alpha\bar{r}^1 + (1-\alpha)\bar{r}^2$
- If $\bar{r}^1 \neq \bar{r}^2$, then any $\bar{r}$ can be obtained by choosing a suitable α (this α may be negative)

Two-fund theorem

- Is $\mathbf{w}^{\alpha}$ efficient at its return level $\bar{r} = \alpha\bar{r}^1 + (1-\alpha)\bar{r}^2$?
- Optimality conditions are:

$$\frac{\partial}{\partial w_i} L = \sum_{j=1}^{n} w_j \sigma_{ij} - \lambda \bar{r}_i - \mu = 0, \quad \forall i = 1, 2, \ldots, n$$

$$\frac{\partial}{\partial \lambda} L = \sum_{i=1}^{n} w_i \bar{r}_i - \bar{r} = 0$$

$$\frac{\partial}{\partial \mu} L = \sum_{i=1}^{n} w_i - 1 = 0$$

- Since $\mathbf{w^i}$ are efficient, $(\mathbf{w^i}, \lambda^i, \mu^i), i = 1, 2$ satisfy these equations when $\bar{r} = \bar{r}^i$
- We want to now show that the variables $(\mathbf{w}^{\alpha}, \lambda^{\alpha}, \mu^{\alpha}) = \alpha(\mathbf{w^1}, \lambda^1, \mu^1) + (1-\alpha)(\mathbf{w^2}, \lambda^2, \mu^2)$ satisfy these optimality conditions for $\bar{r} = \alpha\bar{r}^1 + (1-\alpha)\bar{r}^2$.

Two-fund theorem

- Two last equations are clearly satisfied:
  - Sum of weights is 1 by construction
  - The return constraint holds for $\bar{r} = \bar{r}^\alpha = \alpha\bar{r}^1 + (1-\alpha)\bar{r}^2$
- For $\mathbf{w}^\alpha$, the first set of equations is:

$$\frac{\partial}{\partial w_i^\alpha} L^\alpha = \sum_{j=1}^{n} w_j^\alpha \sigma_{ij} - \lambda^\alpha \bar{r}_i - \mu^\alpha = 0, \quad \forall i = 1, 2, \ldots, n$$

- If we substitute the variables $(\mathbf{w}^\alpha, \lambda^\alpha, \mu^\alpha)$ into this, we get

$$\sum_{j=1}^{n} \left(\alpha w_j^1 + (1-\alpha) w_j^2\right) \sigma_{ij} - \left(\alpha\lambda^1 + (1-\alpha)\lambda^2\right) \bar{r}_i - \left(\alpha\mu^1 + (1-\alpha)\mu^2\right) = 0$$

Two-fund theorem

- Rearranging the terms with α and $(1-\alpha)$ together, the left-hand side of the equation can be expressed as

$$\frac{\partial}{\partial w_i^\alpha} L^\alpha = \alpha \frac{\partial}{\partial w_i^1} L^1 + (1-\alpha) \frac{\partial}{\partial w_i^2} L^2$$

- $\frac{\partial}{\partial w_i^\alpha} L^\alpha$ is equal to zero, because we know that $\frac{\partial}{\partial w_i^1} L^1$ and $\frac{\partial}{\partial w_i^2} L^2$ are zero
- Thus, all of the optimality conditions are satisfied with the variables $(\mathbf{w}^\alpha, \lambda^\alpha, \mu^\alpha)$
- Hence $\mathbf{w}^\alpha$ is the optimal portfolio at return level $\bar{r} = \alpha \bar{r}^1 + (1-\alpha)\bar{r}^2$.
- Since we can obtain any return level by changing α, any optimal portfolio (corresponding to any return level) can be constructed using the two funds $\mathbf{w^1}$ and $\mathbf{w^2}$, which completes the proof of the two fund theorem. $\square$

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One-fund theorem

Risk-free asset

- What if there is a risk-free asset?
  - Return r_f and variance $\sigma_f^2 = 0$
  - Unlimited lending and borrowing are possible at the risk-free rate r_f
- Let us invest the share $1 - \alpha$ in a portfolio of risky assets A with expected return $\bar{r}_A$ and variance σ_A^2, and the share α in the risk-free asset
- The expected return is

$$\bar{r}_\alpha = \alpha r_f + (1 - \alpha)\bar{r}_A$$

- Standard deviation is

$$\sigma_\alpha = \sqrt{(1-\alpha)^2 \sigma_A^2} = (1 - \alpha)\sigma_A$$

One-fund theorem (with a risk-free asset)

Theorem

*(**One-fund theorem**) When there is a risk-free asset, there is a single fund F of risky assets such that any efficient portfolio can be constructed as a combination of the fund F and the risk-free asset.*

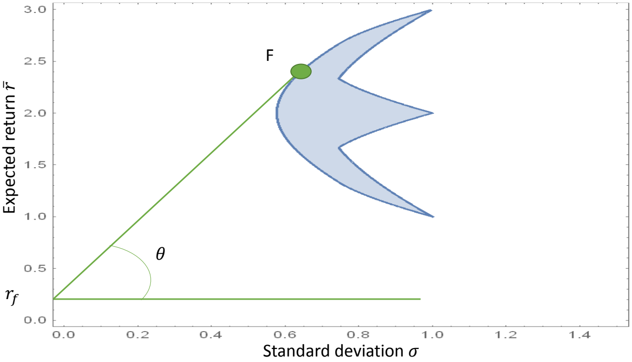
Proof:

- $(\sigma_\alpha, \bar{r}_\alpha)$ forms a line in σ-$\bar{r}$-space as a function of α
- $(\sigma_\alpha, \bar{r}_\alpha)$ should be selected so that the line is as steep as possible, i.e., its slope $k = (\bar{r}_\alpha - r_f)/\sigma_\alpha$ is at maximum

$$\Rightarrow \max_{\mathbf{w}} \frac{\sum_{i=1}^{n} w_i(\bar{r}_i - r_f)}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}}} \quad \square$$

Let $S = \sum_{i=1}^{n} w_i$. We need not constrain S to 1, since S will cancel out from the above expression, which makes solving the problem easier

One-fund theorem (with a risk-free asset)



- Green point = The portfolio that maximizes the slope k of line $\bar{r} = r_f + k\sigma$ from $(\sigma_\alpha, \bar{r}_\alpha)$ through the feasible set

Solving for the one fund

- How to determine this portfolio F?
- At optimum, the partial derivative of the slope with respect to each weight w_k is zero:

$$0 = \frac{\partial}{\partial w_k} \frac{\sum_{i=1}^n w_i(\bar{r}_i - r_f)}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}}}, \quad k = 1, 2, \dots, n$$

$$0 = \frac{\bar{r}_k - r_f}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}}} - \frac{1}{2} \frac{\sum_{i=1}^n w_i(\bar{r}_i - r_f)}{\left(\sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}}\right)^3} 2 \sum_{i=1}^n w_i \sigma_{ik}$$

$$\Rightarrow \bar{r}_k - r_f = \frac{\sum_{i=1}^n w_i(\bar{r}_i - r_f)}{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}} \sum_{i=1}^n w_i \sigma_{ik}, \quad k = 1, 2, \dots, n$$

Solving for the one fund

- Note that each partial derivative equation has the same term at the beginning of the right-hand side (independent of the w_k used to take derivative). Let us denote this term by $\lambda(\mathbf{w})$

$$\lambda(\mathbf{w}) = \frac{\sum_{i=1}^{n} w_i(\bar{r}_i - r_f)}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}}$$

- The partial derivative equation for each w_k can now be written as

$$\bar{r}_k - r_f = \lambda(\mathbf{w}) \sum_{i=1}^{n} w_i \sigma_{ik}, \quad k = 1, 2, \ldots, n$$

Solving for the one fund

- While this system of equations may look challenging, it can easily be solved by a change of variables
- Define a new variable v_k as a function of $w_i, i = 1, ..., n$ as

$$v_k = \lambda(\mathbf{w}) w_k = w_k \frac{\sum_{i=1}^{n} w_i(\bar{r}_i - r_f)}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}}$$

- With the new variables v_k, the partial differential equations become

$$\bar{r}_k - r_f = \sum_{i=1}^{n} v_i \sigma_{ik}, \quad k = 1, 2, \ldots, n$$

- This system of equations can easily be solved for v_k
- This solution approach works here well due to the specific nature of the equations

Solving for the one fund

- From the solution v_k, we can compute the optimal w_k using the definition of v_k
$$v_k = \lambda(\mathbf{w})w_k$$
- Note that v_k satisfy
$$\sum_{i=1}^{n} v_i = \lambda(\mathbf{w}) \sum_{i=1}^{n} w_i = \lambda(\mathbf{w})$$
- Thus, we can solve w_k from known v_k by normalization
$$\frac{v_k}{\sum_{i=1}^{n} v_i} = \frac{\lambda(\mathbf{w})w_k}{\lambda(\mathbf{w})} = w_k, \quad k = 1, 2, \ldots, n$$

Example: One-fund theorem

Asset i	1	2	3	Risk-free
$\mathbb{E}[r_i]$	1	2	3	1/2
σ_i	1	1	1	0

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Optimality conditions

$$\bar{r}_k - r_f = \sum_{i=1}^{3} v_i \sigma_{ik} \Rightarrow \begin{cases} v_1 = 1 - 1/2 = 1/2 \\ v_2 = 2 - 1/2 = 3/2 \\ v_3 = 3 - 1/2 = 5/2 \end{cases}$$

- Normalization of weights

$$w_k = \frac{v_k}{\sum_{i=1}^{3} v_i}, \quad \sum_{i=1}^{3} v_i = 9/2 \Rightarrow \begin{cases} w_1 = (1/2)/(9/2) = 1/9 \\ w_2 = (3/2)/(9/2) = 3/9 \\ w_3 = (5/2)/(9/2) = 5/9 \end{cases}$$

Overview

Random returns

Portfolio mean and variance

Markowitz model

Two-fund theorem

One-fund theorem