Aalto University
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# ELEC-E8125 Reinforcement Learning
# Policy gradient

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# Today

- Direct policy learning via policy gradient

# Learning goals

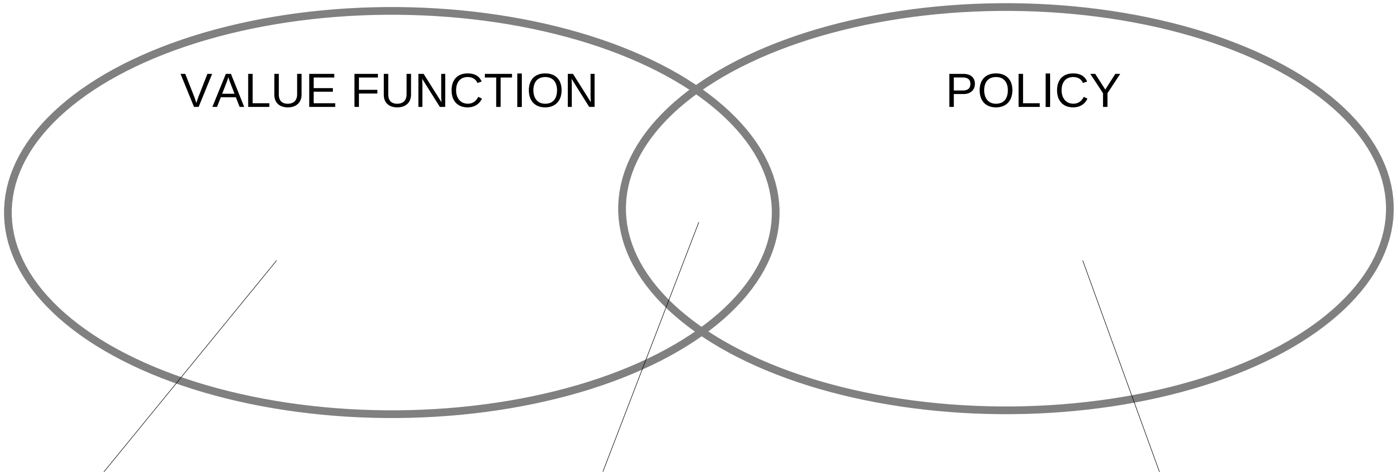
- Understand basis and limitations of policy gradient approaches

# Motivation

https://www.youtube.com/watch?v=xyJAvghtqIM

- Even with value function approximation, large state spaces can be problematic
- Learning parametric policies $\pi(a|s,\theta)$ directly without learning value functions sometimes easier
- Exploration or adversarial situations may benefit from stochastic policies

# Value-based vs policy-based RL




Value-based
- Learned value function
- Implicit policy

Actor-critic
- Learned value function
- Learned policy

Policy-based
- No value function
- Learned policy

- Can learn stochastic policies
- Usually locally optimal

# Stochastic policies

- Discrete actions: Soft-max policy

$$\pi_\theta(\boldsymbol{a}_t|\boldsymbol{s}_t) = 1/Z\, e^{\theta^T \varphi(\boldsymbol{s}_t, \boldsymbol{a}_t)}$$

Probability proportional to exponential linear combination of features.

Normalization constant

$$Z = \sum_a e^{\theta^T \varphi(\boldsymbol{s}_t, \boldsymbol{a}_t)}$$

- Continuous actions: Gaussian policy

$$\pi_\theta(a_t|\boldsymbol{s}_t) \sim N(\theta^T \varphi(\boldsymbol{s}_t), \sigma^2)$$

Mean is linear combination of features.

Can also be understood as linear policy plus exploration uncertainty

$$\pi_\theta(a_t|\boldsymbol{s}_t) = \theta^T \varphi(\boldsymbol{s}_t) + \epsilon \quad \epsilon \sim N(0, \sigma^2)$$



Note: Policies include exploration!

But how to fit these?

Note: This is not RL!

# Supervised policy learning – behavioral cloning

- Assume examples of policy are given in form of (s,a) pairs
- How to fit a stochastic policy to these?

$$\pi_{\boldsymbol{\theta}}(a_t|\boldsymbol{s}_t) \sim N(\boldsymbol{\theta}^T \boldsymbol{\varphi}(\boldsymbol{s}_t), \sigma^2)$$

Example

Note: This is not RL!

# Supervised policy learning – behavioral cloning

- Assume examples of policy are given in form of (s,a) pairs. Assume independent examples
- How to fit a stochastic policy to these?

$$\pi_{\boldsymbol{\theta}}(a_t|\boldsymbol{s}_t) \sim N(\boldsymbol{\theta}^T \boldsymbol{\varphi}(\boldsymbol{s}_t), \sigma^2)$$

Example

- Maximum likelihood parameter estimation
  - Here: maximize probability of actions given states and parameters

$$P(A|S;\theta) = \prod_t \pi_{\boldsymbol{\theta}}(a_t|\boldsymbol{s}_t)$$



How to proceed?

# Example: Maximum likelihood estimation

- Maximize log-likelihood

$$P(A|S;\theta)=\prod_t \pi_{\boldsymbol{\theta}}(a_t|\boldsymbol{s}_t)$$

$$N(\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(a-\mu)^2}{\sqrt{2}\sigma}}$$

# Example: Maximum likelihood estimation

- Maximize log-likelihood

$$P(A|S;\theta)=\prod_t \pi_{\boldsymbol{\theta}}(a_t|\boldsymbol{s}_t)$$

$$N(\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(a-\mu)^2}{2\sigma}}$$

$$\log P(A|S;\theta)=\sum_t \log \pi_{\boldsymbol{\theta}}(a_t|\boldsymbol{s}_t)$$
$$\nabla \log P(A|S;\theta)=\sum_t \nabla \log \pi_{\boldsymbol{\theta}}(a_t|\boldsymbol{s}_t)$$



But we don’t have examples!

# What is a good policy?

- How to measure policy quality?

$$R(\boldsymbol{\theta}) = E\left[\sum_{t=0}^{T} \gamma^t r_t\right]$$

- More generally,

$$R(\boldsymbol{\theta}) = E\left[\sum_{t=0}^{T} c_t r_t\right]$$

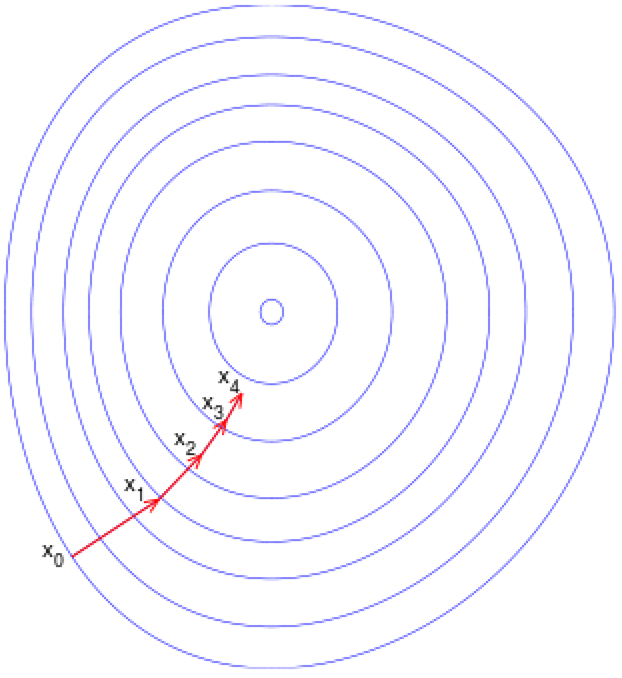
Can also represent average reward per time step.

General time scaling factor



How to optimize parameters?

# Policy gradient




- Use gradient ascent on *R(θ)*

- Update policy parameters by

$$\theta_{m+1} = \theta_m + \alpha_m \nabla_\theta R|_{\theta=\theta_m}$$

$$\sum_{m=0}^{\infty} \alpha_m > 0 \qquad \sum_{m=0}^{\infty} \alpha_m^2 < \infty$$

Guarantees convergence to local minimum.

- How to calculate gradient?

$$R(\theta) = E\left[\sum_{t=0}^{T} c_t r_t\right]$$

Depends on θ.

How to estimate gradient from data (if we have a chance to try different policies)?

# Finite difference gradient estimation

- What is gradient?
  - Vector of partial derivatives
- How to estimate derivative?
  - Finite difference: $f'(x) \approx \frac{f(x+dx)-f(x)}{dx}$
- For policy gradient:
  - Generate variation $\Delta\boldsymbol{\theta}_i$
  - Estimate experimentally $R(\boldsymbol{\theta}+\Delta\boldsymbol{\theta}_i) \approx \hat{R}_i = \sum_{t=0}^{H} c_t r_t$
  - Compute gradient $[\boldsymbol{g}_{FD}^T, R_{ref}]^T = (\Delta\boldsymbol{\Theta}^T \Delta\boldsymbol{\Theta})^{-1} \Delta\boldsymbol{\Theta}^T \hat{\boldsymbol{R}}$
  - Repeat until estimate converged

Not easy to choose.

$$\Delta\boldsymbol{\Theta}^T = \begin{bmatrix} \Delta\boldsymbol{\theta}_1, \ldots, \Delta\boldsymbol{\theta}_I \\ 1, \ldots, 1 \end{bmatrix}$$

$$\hat{\boldsymbol{R}}^T = [\hat{R}_1, \ldots, \hat{R}_I]$$

Where does this come from?

$$\hat{R}_i \approx R_{ref} + \boldsymbol{g}^T \Delta\boldsymbol{\theta}_i$$

# Likelihood-ratio approach

- Assume trajectories tau are generated by roll-outs, thus

$$\boldsymbol{\tau} \sim p_\theta(\boldsymbol{\tau}) = p(\boldsymbol{\tau}|\boldsymbol{\theta}) \quad R(\boldsymbol{\tau}) = \sum_{t=0}^{H} c_t r_t$$

- Expected return can then be written

$$R(\boldsymbol{\theta}) = E_{\boldsymbol{\tau}}[R(\boldsymbol{\tau})] = \int p_\theta(\boldsymbol{\tau}) R(\boldsymbol{\tau}) d\boldsymbol{\tau}$$

- Gradient is thus

$$\nabla_\theta R(\boldsymbol{\theta}) = \int \nabla_\theta p_\theta(\boldsymbol{\tau}) R(\boldsymbol{\tau}) d\boldsymbol{\tau}$$

$$= \int p_\theta(\boldsymbol{\tau}) \nabla_\theta \log p_\theta(\boldsymbol{\tau}) R(\boldsymbol{\tau}) d\boldsymbol{\tau}$$

Likelihood ratio “trick”: Substitute

- Why do that?

$$= E_{\boldsymbol{\tau}}[\nabla_\theta \log p_\theta(\boldsymbol{\tau}) R(\boldsymbol{\tau})]$$

$$\nabla_\theta p_\theta(\boldsymbol{\tau}) = p_\theta(\boldsymbol{\tau}) \nabla_\theta \log p_\theta(\tau)$$

$$p_\theta(\boldsymbol{\tau}) = p(\boldsymbol{s}_0) \prod_{t=0}^{H} p(\boldsymbol{s}_{t+1}|\boldsymbol{s}_t, \boldsymbol{a}_t) \pi_\theta(\boldsymbol{a}_t|\boldsymbol{s}_t)$$



Try substitution for log-gradient! $\nabla_\theta \log p_\theta(\boldsymbol{\tau}) = \sum_{t=0}^{H} \nabla_\theta \log \pi_\theta(\boldsymbol{a}_t|\boldsymbol{s}_t)$

We know this!

# Example differentiable policies

Normalization constant missing.

- Soft-max policy

$$\pi_{\theta}(\boldsymbol{a}_t|\boldsymbol{s}_t) \propto e^{\theta^T \varphi(\boldsymbol{s}_t, \boldsymbol{a}_t)}$$

Probability proportional to exponential linear combination of features.

  - Log-policy (*score function*)

$$\nabla_{\theta} \log \pi_{\theta}(\boldsymbol{a}_t|\boldsymbol{s}_t) = \varphi(\boldsymbol{s}_t, \boldsymbol{a}_t) - E_{\pi_{\theta}}[\varphi(\boldsymbol{s}_t, \cdot)]$$

- Gaussian policy

$$\pi_{\theta}(a_t|\boldsymbol{s}_t) \sim N(\theta^T \varphi(\boldsymbol{s}_t), \sigma^2)$$

Mean is linear combination of features.

  - Log-policy

$$\nabla_{\theta} \log \pi_{\theta}(a_t|\boldsymbol{s}_t) = \frac{(a_t - \theta^T \varphi(\boldsymbol{s}_t))\varphi(\boldsymbol{s}_t)}{\sigma^2}$$



Can also be understood as linear policy plus exploration uncertainty

$$\pi_{\theta}(a_t|\boldsymbol{s}_t) = \theta^T \varphi(\boldsymbol{s}_t) + \epsilon \quad \epsilon \sim N(0, \sigma^2)$$

# Example differentiable policies

Normalization constant missing.

- Discrete neural net policy

$$\pi_{\boldsymbol{\theta}}(\boldsymbol{a}_t|\boldsymbol{s}_t) \propto e^{f_\theta(\boldsymbol{s}_t, \boldsymbol{a}_t)}$$

Probability proportional to exponential neural network output.

- Gaussian neural network policy

$$\pi_{\boldsymbol{\theta}}(a_t|\boldsymbol{s}_t) \sim N(f_\theta(\boldsymbol{s}_t), \sigma^2)$$

$$\nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(a_t|\boldsymbol{s}_t) = \frac{(a_t - f_\theta(\boldsymbol{s}_t)) \nabla_\theta f_\theta(\boldsymbol{s}_t)}{\sigma^2}$$



OK, now to applying the policy gradient:

$$\nabla_{\boldsymbol{\theta}} R(\boldsymbol{\theta}) = E_{\boldsymbol{\tau}}[\nabla_{\boldsymbol{\theta}} \log p_{\boldsymbol{\theta}}(\boldsymbol{\tau}) R(\boldsymbol{\tau})]$$

# MC policy gradient – REINFORCE

- Episodic version shown here

Reward for trial *i.*

- Approach:
  - Perform episode *J (=1,2,3,...)*
  - Estimate gradient $\boldsymbol{g}_{RE} = E_{\boldsymbol{\tau}}\left[\left(\sum_{t=0}^{H} \nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}_t | \boldsymbol{s}_t)\right) R(i)\right]$ Use empirical mean.

$$\approx \frac{1}{J} \sum_{i=1}^{J} \left[\left(\sum_{t=0}^{H} \nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}_t^{[i]} | \boldsymbol{s}_t^{[i]})\right)\left(\sum_t r_{t,i}\right)\right]$$

  - Update policy and repeat with new trial(s) until convergence
- No need to generate policy variations because of stochastic policy

# Limitations so far

- High variance (uncertainty) in gradient estimate due to stochastic policy
- Slow convergence, hard to choose learning rate
  - Parametrization dependent gradient estimate
- On-policy method

# Decreasing variance by adding baseline

- Constant baseline can be added to reduce *variance* of the gradient estimate

$$\nabla_{\theta} R(\boldsymbol{\theta}) = E_{\boldsymbol{\tau}}\left[\nabla_{\theta} \log p_{\theta}(\boldsymbol{\tau})(R(\boldsymbol{\tau}) - b)\right]$$
$$= E_{\boldsymbol{\tau}}\left[\nabla_{\theta} \log p_{\theta}(\boldsymbol{\tau}) R(\boldsymbol{\tau})\right]$$

- Does not cause bias because

$$E_{\boldsymbol{\tau}}\left[\nabla_{\theta} \log p_{\theta}(\boldsymbol{\tau}) b\right] = \int p_{\theta}(\boldsymbol{\tau}) \nabla_{\theta} \log p_{\theta}(\boldsymbol{\tau}) b \, d\boldsymbol{\tau} =$$

$$\int \nabla_{\theta} p_{\theta}(\boldsymbol{\tau}) b \, d\boldsymbol{\tau} = b \nabla_{\theta} \int p_{\theta}(\boldsymbol{\tau}) \, d\boldsymbol{\tau} = b \nabla_{\theta} 1 = 0$$



Intuition:
Modifying rewards by a constant
does not change optimal policy.

# Episodic REINFORCE with optimal baseline

- Optimal baseline for episodic REINFORCE (minimize variance of estimator):

$$b_h = \frac{E_{\boldsymbol{\tau}}\left[\left(\sum_{t=0}^{H} \nabla_{\theta_h} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}_t | \boldsymbol{s}_t)\right)^2 R_{\boldsymbol{\tau}}\right]}{E_{\boldsymbol{\tau}}\left[\left(\sum_{t=0}^{H} \nabla_{\theta_h} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}_t | \boldsymbol{s}_t)\right)^2\right]}$$

In practice, approximate by empirical mean (average over trials).

- Approach:
  - Perform trial *J (=1,2,3,...)*
  - For each gradient element *h*
    - Estimate optimal baseline $b_h$
    - Estimate gradient
  - Repeat until convergence

Component-wise!

$$g_h = \frac{1}{J} \sum_{i=1}^{J} \left[\left(\sum_{t=0}^{H} \nabla_{\theta_h} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}_t^{[i]} | \boldsymbol{s}_t^{[i]})\right)(R(i) - b_h^{[i]})\right]$$

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Even with optimal baseline, variance can be an issue.

# Policy gradient theorem

- Observation: Future actions do not depend on past rewards.

$$E\left[\nabla_{\boldsymbol{\theta}} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}_t|\boldsymbol{s}_t) r_k\right] = 0 \quad \forall t > k$$

"don't take into account past rewards when evaluating the effect of an action" (causality, taking an action can only affect future rewards)

- PGT:
  - Reduces variance of estimate → Fewer samples needed on average

$$\boldsymbol{g}_{PGT} = E_{\boldsymbol{\tau}}\left[\sum_{k=0}^{H}\left(\sum_{t=0}^{k} \nabla_{\boldsymbol{\theta}_h} \log \pi_{\boldsymbol{\theta}}(\boldsymbol{a}_t|\boldsymbol{s}_t)\right)(c_k r_k - b_k^h)\right]$$



Note: If rewards only at the final time step, this is equivalent to REINFORCE.

# Off-policy policy gradient

- What if we have samples from another policy (e.g. earlier time steps?

  Optimize $E_{\tau\sim\pi_\theta(\tau)}[R(\tau)]$

  using samples from $\pi'(\tau)$

- Use importance sampling!

Where does this come from?

$$E_{s\sim p(s)}[f(s)] = \int p(s) f(s)\, ds$$
$$= E_{s\sim q(s)}\left[\frac{p(s)}{q(s)} f(s)\right]$$

# Off-policy policy gradient

- What if we have samples from another policy (e.g. earlier timesteps?

  Optimize $E_{\tau \sim \pi_\theta(\tau)}[R(\tau)]$

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$$E_{s \sim p(s)}[f(s)] = \int p(s) f(s) ds$$
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Thus, optimize $E_{\tau \sim \pi'(\tau)}\left[\frac{\pi_\theta(\tau)}{\pi'(\tau)} R(\tau)\right]$

# Off-policy policy gradient

$$E_{\tau\sim\pi'(\tau)}\left[\frac{\pi_\theta(\tau)}{\pi'(\tau)}R(\tau)\right]$$

- We had earlier

$$p_\theta(\boldsymbol{\tau})=p(\boldsymbol{s}_0)\prod_{t=0}^{H}p(\boldsymbol{s}_{t+1}|\boldsymbol{s}_t,\boldsymbol{a}_t)\pi_\theta(\boldsymbol{a}_t|\boldsymbol{s}_t)$$

- Thus

$$\frac{\pi_\theta(\tau)}{\pi'(\tau)}=\frac{p(\boldsymbol{s}_0)\prod_{t=0}^{H}p(\boldsymbol{s}_{t+1}|\boldsymbol{s}_t,\boldsymbol{a}_t)\pi_\theta(\boldsymbol{a}_t|\boldsymbol{s}_t)}{p(\boldsymbol{s}_0)\prod_{t=0}^{H}p(\boldsymbol{s}_{t+1}|\boldsymbol{s}_t,\boldsymbol{a}_t)\pi'(\boldsymbol{a}_t|\boldsymbol{s}_t)}=\frac{\prod_{t=0}^{H}\pi_\theta(\boldsymbol{a}_t|\boldsymbol{s}_t)}{\prod_{t=0}^{H}\pi'(\boldsymbol{a}_t|\boldsymbol{s}_t)}$$

# Off-policy policy gradient

- Now the gradient

$$\nabla_\theta E_{\tau\sim\pi'(\tau)}\left[\frac{\pi_\theta(\tau)}{\pi'(\tau)}R(\tau)\right]=E_{\tau\sim\pi'(\tau)}\left[\frac{\nabla_\theta\pi_\theta(\tau)}{\pi'(\tau)}R(\tau)\right]$$

$$=E_{\tau\sim\pi'(\tau)}\left[\frac{\pi_\theta(\tau)}{\pi'(\tau)}\nabla_\theta\log\pi_\theta(\tau)R(\tau)\right]$$

$$=\boxed{E_{\tau\sim\pi'(\tau)}\left[\left(\prod_t\frac{\pi_\theta(a_t|s_t)}{\pi'(a_t|s_t)}\right)\left(\sum_t\nabla_\theta\log\pi_\theta(a_t|s_t)\right)\left(\sum_t r_t\right)\right]}$$

Compare to on-policy (REINFORCE)

$$\nabla_\theta E_{\tau\sim\pi_\theta(\tau)}[R(\tau)]=E_{\tau\sim\pi_\theta(\tau)}\left[\left(\sum_t\nabla_\theta\log\pi_\theta(a_t|s_t)\right)\left(\sum_t r_t\right)\right]$$

# Summary

- Policy gradient methods can be used for stochastic policies and continuous action spaces
- Finite-difference approaches approximate gradient by policy adjustments
- Likelihood ratio-approaches calculate gradient through known policy
- Policy gradient often requires very many updates because of noisy gradient and small update steps resulting in slow convergence

# Next: Actor-critic approaches

- Can we combine policy learning with value-based methods?

- Readings
  - Sutton&Barto Ch 13.5