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Differential and Integral Calculus 1

Exercises, Week 5

Work on Warm-up 1–4 during the exercise sessions of Week 5.
Submit on MyCourses your solutions for Homework 1 and 2 by Sunday, October 9th.

Warm-up 1: Consider the function $f(x) = \frac{1}{1-x}$.

1. Show that, for $x \neq 1$,
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1-x}. \qquad (*)$$
2. Show that $\frac{x^{n+1}}{1-x} = O(x^{n+1})$ as $x \to 0$.
3. Use this to find the n-th Maclaurin polynomial for f by using the alternative definition of Taylor polynomial (with the big O).

Warm-up 2: Compute the following limits by using suitable Taylor polynomials:

$$(a) \quad \lim_{x\to 0} \frac{1-\cos(x^2)}{(1-\cos x)^2} \qquad (b) \quad \lim_{x\to 0} \frac{\sin(\sin x) - x}{x(\cos(\sin x) - 1)} \qquad (c) \quad \lim_{x\to 0} \frac{(e^x - 1 - x)^2}{x^2 - \ln(1+x^2)}.$$

Warm-up 3: Solve the following differential equations and initial-value problems:

$$(a) \quad y' = (2y-5)\cos(x) \qquad (c) \quad y(x) = 2 + \int_0^x \frac{t}{y(t)}\,dt$$

$$(b) \quad y' = 2\left(1 - \frac{y}{3}\right) \qquad (d) \quad \begin{cases} x^2y' + y &= x^2e^{1/x} \\ y(1) &= 3e. \end{cases}$$

Warm-up 4: Consider the following differential equation:
$$y' = 2y\left(1 - \frac{y}{3}\right).$$

1. Find the general solution.
2. Find a particular solution y that minimizes the definite integral $\int_0^1 y(x)\,dx$.

Submit on MyCourses your solutions **only** for the following two problems. Explain the reasoning behind your solutions, do not just return the final result. If you make use of important results from the lectures, state what they are.

Homework 1: Consider the function $g(x) = \frac{1}{(1-x)^2}$.

1. Show that
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \frac{n+2-(n+1)x}{(1-x)^2}x^{n+1}.$$

2. Find the n-th Maclaurin polynomial for g by using the alternative definition of Taylor polynomial (with the big O).

Hints: Differentiate both sides of $(*)$ in Warm-up 1, with $n+1$ instead of n. If $h(x) = O(u(x))$ as $x \to a$ and $k(x) = O(u(x))$ as $x \to a$, then $h(x) + k(x) = O(u(x))$ as $x \to a$. [2 points]

Homework 2: Consider the following differential equation:
$$\frac{dy}{dx} + y = x.$$

1. Find the general solution.

2. The general solution will involve some parameter $C \in \mathbb{R}$. That is, for each $C \in \mathbb{R}$, you have a particular solution $y_C(x)$. Define the function
$$F(C) = \int_0^1 y_C(x)\,dx.$$
Sketch by hand the graph of F.

3. There is some freedom in the way you write the general solution. For instance instead of giving the general solution in the form
$$y(x) = \sin(x) + C\cos(x), \text{ for } C \in \mathbb{R}$$
(but the general solution in this problem looks different!), you could write
$$y(x) = \sin(x) - 2C\cos(x), \text{ for } C \in \mathbb{R},$$
because any real number C can be expressed as twice the opposite of some real number, or
$$y(x) = \sin(x) + (C - 27)\cos(x), \text{ for } C \in \mathbb{R},$$
because any real number is equal to some real number minus 27. Depending on the way you write the general solution, the function F in the part 2 of this problem might look different. Swap C with $-C$ in the general solution you gave in part 1, and sketch the graph of the new function $F(C) = \int_0^1 y_C(x)\,dx$ that you get this way.

4. Find a way to write the general solution so that the origin of the plane lies on the graph of the corresponding function F. [2 points]