ELEC-E7130
Example on parameter estimation: One unknown parameter

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Discrete uniform distribution $U(1, N)$

Let $U(1, N)$ denote the discrete uniform distribution, where $N \in \{1, 2, \ldots\}$. The point probabilities are given by

$$p(i) = \mathrm{P}\{X = i\} = \frac{1}{N}, \quad i \in \{1, \ldots, N\},$$

and the (theoretical) mean value by

$$\mathrm{E}(X) = \sum_{i=1}^{N} \mathrm{P}\{X = i\} i = \frac{1}{N} \sum_{i=1}^{N} i = \frac{1}{N} \frac{N(N+1)}{2} = \frac{N+1}{2}.$$

Estimation of N: Method of Moments (MoM)

Consider a sample without replacement of size n, $(x_1, \ldots, x_n)$, from set $\{1, 2, \ldots, N\}$. The first (theoretical) moment μ_1 equals

$$\mu_1 = \mathrm{E}(X) = \frac{N+1}{2},$$

and the corresponding sample moment m_1 is the sample mean $\bar{x}_n$:

$$m_1 = \bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Let $\hat{N}$ denote the estimator of the unknown parameter N. By solving the requirement (for the first moment) that

$$m_1 = \mu_1 = \frac{\hat{N}+1}{2},$$

we get the *MoM estimator*:

$$\hat{N}^{\mathrm{MoM}} = 2m_1 - 1 = 2\bar{x}_n - 1. \tag{1}$$

Estimation of N: Maximum Likelihood (ML)

Consider again a sample without replacement of size n, $(x_1, \ldots, x_n)$, from set $\{1, 2, \ldots, N\}$. Let M_n denote the maximum value of this sample,

$$M_n = \max\{x_1, \ldots, x_n\} \geq n.$$

The likelihood function for this discrete distribution with unknown parameter N equals

$$\begin{aligned} L(x_1, \ldots, x_n; N) &= P\{X_1 = x_1, \ldots, X_n = x_n; N\} \\ &= \begin{cases} \prod_{i=0}^{n-1} \left(\frac{1}{N-i}\right) = \frac{(N-n)!}{N!}, & \text{if } N \geq M_n; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\hat{N}$ denote the estimator of the unknown parameter N. Since

$$\begin{aligned} &\max_{N \in \{1,2,\ldots\}} L(x_1, \ldots, x_n; N) \\ &= \max\left\{0, \frac{(M_n - n)!}{M_n!}, \frac{(M_n + 1 - n)!}{(M_n + 1)!}, \ldots\right\} \\ &= \frac{(M_n - n)!}{M_n!}, \end{aligned}$$

we get the following *ML estimator*:

$$\hat{N}^{\mathrm{ML}} = \arg\max_{N \in \{1,2,\ldots\}} L(x_1, \ldots, x_n; N) = M_n. \tag{2}$$

Numerical example

Assume that your sample without replacement of size $n = 5$ from set $\{1, 2, \ldots, N\}$ (with unknown N) is as follows:

$$(x_1, \ldots, x_n) = (5, 2, 7, 12, 3).$$

Now, the sample mean is $\bar{x}_n = \frac{29}{5} = 5.8$, and the sample maximum equals $M_n = 12$.

MoM estimate: By (1), we get

$$\hat{N}^{\text{MoM}} = 2\bar{x}_n - 1 = 10.6 \ (\approx 11)$$

ML estimate: By (2), we get

$$\hat{N}^{\text{ML}} = M_n = 12$$

The following two estimates are taken from the *German tank problem* discussed in lecture slides 41-42.

Bayesian estimate:

$$\hat{N}^{\text{Bayes}} = \frac{(M_n - 1)(n - 1)}{n - 2} = \frac{44}{3} = 14.7 \ (\approx 15)$$

Minimum-variance unbiased estimate:

$$\hat{N}^{\text{MinV}} = M_n + \frac{M_n}{n} - 1 = \frac{67}{5} = 13.4 \ (\approx 13)$$