MS-E2112 Multivariate Statistical Analysis (5cr)
Lecture 8: Canonical Correlation Analysis

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Canonical Correlation Analysis

Canonical Correlation Analysis

Canonical correlation analysis involves partition of variables into two vectors x and y. The aim is to find linear combinations $\alpha^T x$ and $\beta^T y$ that have the largest possible correlation.

Canonical Correlation Analysis

Let x be a p-variate random vector and let y be a q-variate random vector. The object in canonical correlation analysis is to find linear combinations

$$u_k = \alpha_k^T x$$

and

$$v_k = \beta_k^T y$$

that maximizes the correlation $|corr(u_k, v_k)|$ between u_k and v_k subject to

$$var(u_k) = var(v_k) = 1,$$

and

$$corr(u_k, u_t) = 0, corr(v_k, v_t) = 0, \ t < k.$$

Correlation

Quick reminder:

$$corr(w_1, w_2) = \frac{E[(w_1 - \mu_{w_1})(w_2 - \mu_{w_2})]}{\sigma_{w_1}\sigma_{w_2}}.$$

Canonical Correlation Analysis

The vectors α_k and β_k are called the kth canonical vectors and

$$\rho_k = |corr(u_k, v_k)|$$

are called canonical correlations.

Canonical Correlation Analysis

Whereas principal component analysis considers interrelationships within a set of variables, canonical correlation analysis considers relationships between two groups of variables.

Canonical Correlation Analysis, Examples

- Exercise — health.
- Open book exams — closed book exams.
- Job satisfaction — performance.

Canonical Correlation Analysis, Regression Analysis

Canonical correlation analysis can be seen as an extension of multivariate regression analysis. However, note that in canonical correlation analysis there is no assumption of causal asymmetry - x and y are treated symmetrically!

Canonical Correlation Analysis, Solution

Let $z = (x^T, y^T)^T$, and let

$$cov(z) = \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

Define

$$M_1 = \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21},$$

and

$$M_2 = \Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}.$$

Canonical Correlation Analysis, Solution

Now, the canonical vectors α_k are the eigenvectors of M_1 (α_k corresponds to the kth largest eigenvalue), the canonical vectors β_k are the eigenvectors of M_2, and ρ_k^2 are the eigenvalues of the matrix M_1 (and of M_2 as well). The proof of this solution can be found from pages 283-284 of [1].

Canonical Correlation Analysis, Solution

Note that the eigenvectors α_k and β_k do not have length$=1$!
Requirements

$$var(u_k) = var(\alpha_k^T x) = 1$$

and

$$var(v_k) = var(\beta_k^T y) = 1$$

define the lengths of the eigenvectors.

Canonical Correlation Analysis, Solution

If the covariance matrices Σ_{11} and Σ_{22} are not full rank, similar results may be obtained using generalized inverses. One may also consider dimension reduction as a first step.

Canonical Correlation Analysis, Sample Version

Sample estimates $\hat{\alpha}_k$, $\hat{\beta}_k$ and $\hat{\rho}_k$ of α_k, β_k and ρ_k, respectively, are obtained by using sample covariance matrices calculated from the samples $x_1, x_2, ..., x_n$, $y_1, y_2, ..., y_n$ and $z_1, z_2, ..., z_n$.

Testing Independence

Testing Independence

Assume that $z = (x^T, y^T)^T \sim N_{p+q}(\mu, \Sigma)$. Consider testing

H_0 : x and y are independent,

against

H_1 : x and y are not independent.

Testing Independence

Let $m = min\{p, q\}$, and let

$$T = -(n - \frac{1}{2}(p + q + 3)) \ln(\prod_{k=1}^{m}(1 - \hat{\rho}_k^2)).$$

Now, under H_0, and under the assumption of multivariate normality, the test statistic T is asymptotically distributed as $\chi^2(pq)$.

Testing Partial Independence

Assume that $z = (x^T, y^T)^T \sim N_{p+q}(\mu, \Sigma)$. Consider testing

H_0 : Only s of the canonical correlation coefficients are nonzero,

against

H_1 : The number of nonzero canonical correlation coefficients is larger than s.

Testing Partial Independence

Let $m = min\{p, q\}$, and let

$$T_s = -(n - \frac{1}{2}(p + q + 3)) \ln(\prod_{k=s+1}^{m} (1 - \hat{\rho}_k^2)).$$

Now, under H_0, and under the assumption of multivariate normality, the test statistic T is asymptotically distributed as $\chi^2((p - s)(q - s))$.

Independence Testing

If the normality assumption $z = (x^T, y^T)^T \sim N_{p+q}(\mu, \Sigma)$ does not hold, the p-values of the above mentioned test statistics can be approximated using permutations.

Scoring and Predicting

Scoring and Predicting

Let X and Y denote the $n \times p$ and $n \times q$ data matrices for n individuals, and let $\hat{\alpha}_k$ and $\hat{\beta}_k$ denote the kth (sample) canonical vectors. Then the $n \times 1$ vectors

$$\eta_k = X\hat{\alpha}_k$$

and

$$\phi_k = Y\hat{\beta}_k$$

denote the scores of the n individuals on the kth canonical correlation variables.

Scoring and Predicting

If the x and y variables are interpreted as the "predictor" and "predicted" variables, respectively, then the η_k score vector can be used to predict the ϕ_k score vector by using least square regression:

$$(\tilde{\phi_k})_i = \hat{\rho}_k((\eta_k)_i - \hat{\alpha}_k^T \bar{x}) + \hat{\beta}_k^T \bar{y}.$$

The canonical correlation $\hat{\rho}_k$ estimates the proportion of the variance of ϕ_k that is explained by the regression on x.

Example

Example: closed book exams — open book exams.

Example

Marks in open-book (O) and closed-book (C) exams:

i	Mechanics (C)	Vectors (C)	Algebra (O)	Analysis (O)	Statistics (O)
1	77	82	67	67	81
2	63	78	80	70	81
3	75	73	71	66	81
⋮	⋮	⋮	⋮	⋮	⋮
100	46	52	53	41	40

Source: K. V. Mardia, J. T. Tent, J. M. Bibby, Multivariate analysis, Academic Press, London, 2003 (reprint of 1979).

Example

Means:

Variable	Mean
x_1	38.9545
x_2	50.5909
y_1	50.6023
y_2	46.6818
y_3	42.3068

Example

Covariance matrix

$$
\begin{array}{c}
\begin{array}{cc} \qquad\quad \Sigma_{11} & \qquad\qquad\qquad \Sigma_{12} \end{array} \\
\Sigma = \left[\begin{array}{cc|ccc}
302.3 & 125.8 & 100.4 & 105.1 & 116.1 \\
 & 170.9 & 84.2 & 93.6 & 97.9 \\
\hline
 & & 111.6 & 110.8 & 120.5 \\
 & & & 217.9 & 153.8 \\
 & & & & 294.4
\end{array}\right] \\
\begin{array}{cc} \qquad\quad \Sigma_{21} & \qquad\qquad\qquad \Sigma_{22} \end{array}
\end{array}
$$

Example

Calculate the eigenvectors

$$M_1 = \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \Rightarrow \hat{\alpha}_k$$

and

$$M_2 = \Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \Rightarrow \hat{\beta}_k.$$

Here

$$\hat{\alpha}_1 = \begin{bmatrix} 0.0260 \\ 0.0518 \end{bmatrix}$$

and

$$\hat{\beta}_1 = \begin{bmatrix} 0.0824 \\ 0.0081 \\ 0.0035 \end{bmatrix}.$$

Example

$$u_1 = 0.0260x_1 + 0.0518x_2$$

and

$$v_1 = 0.0824y_1 + 0.0081y_2 + 0.0035y_3.$$

The highest correlation occurs between an average of x_1 and x_2 weighted on x_2 and an average of $y1$, $y2$ and y_3, heavily weighted on y_1
The canonical correlations

$$\rho_1 = 0.6630$$

and

$$\rho_2 = 0.0412.$$

Example

Predicting

$$\left(\tilde{\phi}_k\right)_i = \hat{\rho}_k\left((\eta_k)_i - \hat{\alpha}_k^T \bar{x}\right) + \hat{\beta}_k^T \bar{y}.$$

Here

$$\begin{aligned}\left(\tilde{\phi}_1\right)_i &= 0.6630\left((\eta_1)_i - (0.0260 * 38.9545 + 0.0518 * 50.5909)\right) \\ &+(0.0824 \cdot 50.6023 + 0.0081 \cdot 46.6818 + 0.0035 \cdot 42.3068)\end{aligned}$$

$$\approx 0.6630(\eta_1)_i + 2.2905$$

$$\approx 0.6630(0.0260(x_1)_i + 0.0518(x_2)_i) + 2.2905$$

$$\approx 0.0172(x_1)_i + 0.0343(x_2)_i + 2.2905.$$

Note that this almost predicts y_1.

Words of Warning

Some Words of Warning

- The procedure maximizes the correlation between the linear combination of variables — it can be more than difficult to interpret the results.
- Correlation does not automatically imply causality.

Next Week

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Next week we will talk about discriminant analysis and classification.

References

References I

K. V. Mardia, J. T. Kent, J. M. Bibby, Multivariate Analysis, Academic Press, London, 2003 (reprint of 1979).

References II

- R. V. Hogg, J. W. McKean, A. T. Craig, Introduction to Mathematical Statistics, Pearson Education, Upper Sadle River, 2005.
- R. A. Horn, C. R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
- R. A. Horn, C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.