ELEC-A7200

Signals and Systems

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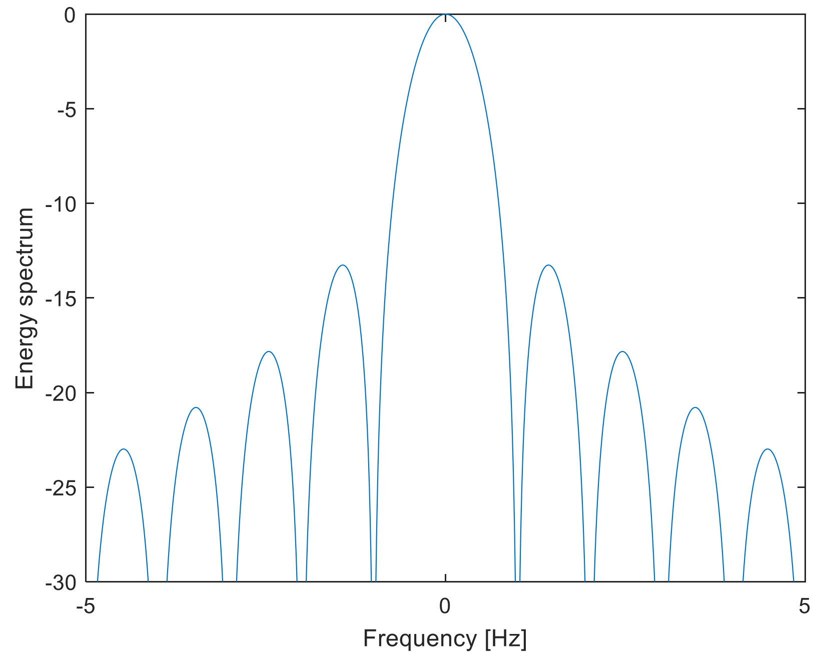
Fall 2022

Aalto University
School of Electrical
Engineering



Lecture 5 & 6
Fourier transform

Content

- Fourier transform
- Fourier transform vs Laplace transform
- Rayleigh’s energy theorem and energy spectrum
- Some properties of Fourier transform
- Bandwidth
- Fourier transform of special signals

From Fourier series to Fourier transform

- **Exponential Fourier series of a periodic signal** $x(t) = x(t + T_0)$

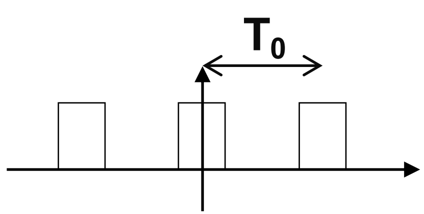
$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T_0}\int_{T_0} x(t)e^{-i\frac{2\pi k}{T_0}t}dt}_{X(k)}\, e^{i\frac{2\pi k}{T_0}t}$$

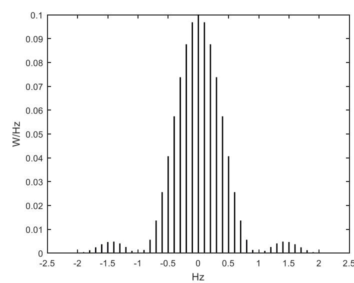
Coefficients of Exponential Fourier Series

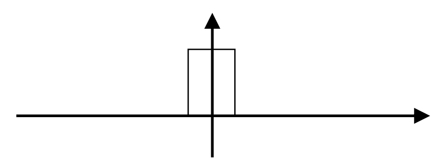
- **Aperiodic energy signal** $x(t)$

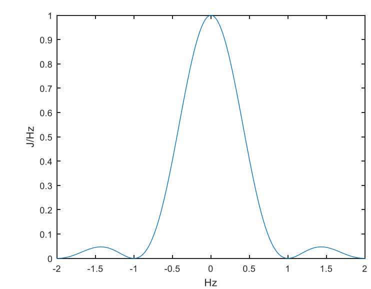
$$x(t) = \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} x(t)e^{-i2\pi f t}\,dt}_{X(f)}\; e^{i2\pi f t}df$$

Fourier transform









Fourier transform and inverse transform

- **Fourier transform**

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi f t} dt \overset{\text{def}}{=} \mathrm{F}[x(t)]$$

- **Inverse transform**

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi f t} df \overset{\text{def}}{=} \mathrm{F}^{-1}[x(t)]$$

The signal must satisfy $\int_{-\infty}^{\infty} |x(t)| dt < \infty$ in order the transform to exist

Alternative definitions used in the literature

$$X(\text{?}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-i\upsilon t} dt$$

$$\upsilon = \omega = 2\pi f$$

$$x(t) = \int_{-\infty}^{\infty} X(\text{?})e^{-i\upsilon t} dt$$

$$X(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)e^{-i\omega t} dt$$

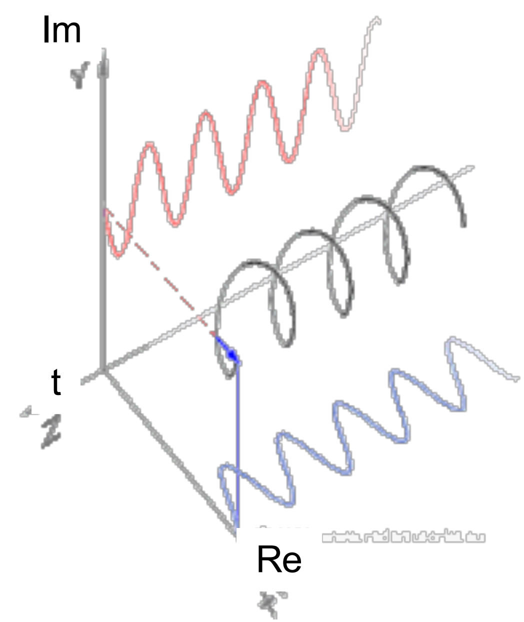
$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t} dt$$

Fourier transform

Inner product of the pulse with the phasor

$$X(f) = \langle x(t), e^{i2\pi ft} \rangle \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt$$

$e^{i2\pi ft}$



Laplace transform vs Fourier Transform

One-sided Laplace transform

$$\hat{X}(s) = \int_{0}^{\infty} x(t)e^{-st}dt \stackrel{\text{def}}{=} \mathrm{L}[x(t)]$$

$s = \gamma + i2\pi f$

Inverse transform

$$x(t) = \int_{\gamma+-i\infty}^{\gamma+i\infty} \hat{X}(s)e^{-st}ds \stackrel{\text{def}}{=} \mathrm{L}^{-1}[x(t)]$$

a.k.a. Fourier-Mellin integral

Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt \stackrel{\text{def}}{=} \mathrm{F}[x(t)]$$

$= \hat{X}(i2\pi f)$ If signal is causal $x(t)$=0 t<0 and $\int_{-\infty}^{\infty} |x(t)|dt < \infty$

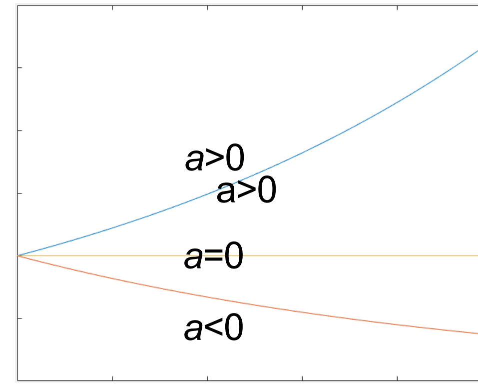
Inverse transform

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df \stackrel{\text{def}}{=} \mathrm{F}^{-1}[x(t)]$$

Set γ=0
and perform change of variables $s = i2\pi f$

Laplace transform vs Fourier Transform

Consider a signal $x(t) = e^{at}u(t)$



$$\int_{-\infty}^{\infty}|x(t)|dt = \int_{0}^{\infty} e^{at}dt = \lim_{t\to\infty}\frac{1}{a}e^{at} - \frac{1}{a}e^{a0} = \begin{cases} -\frac{1}{a} & a < 0 \\ \infty & a \geq 0 \end{cases}$$

$X(f) = \frac{1}{i2\pi f - a}$, $a<0$

$\hat{X}(s) = \frac{1}{s-a}$, $-\infty < a < \infty$

Fourier transform does not exist is $a \geq 0$

Laplace transform exists for all a

Laplace transform vs Fourier Transform

Consider a linear time invariant system $\frac{d}{dt}y(t) = ay(t) + x(t), \quad y(t) = 0, t \leq 0$

Impulse response $h(t) = e^{at}u(\mathrm{t})$

Response to complex phasor input $x(t) = e^{i2\pi ft}$ is of the form $y(t) = H(f)\, e^{i2\pi ft}$

$$\frac{d}{dt}y(t) = ay(t) + x(t)$$
$$i2\pi f\ H(f)e^{i2\pi ft} = aH(f)e^{i2\pi ft} + e^{i2\pi ft}$$

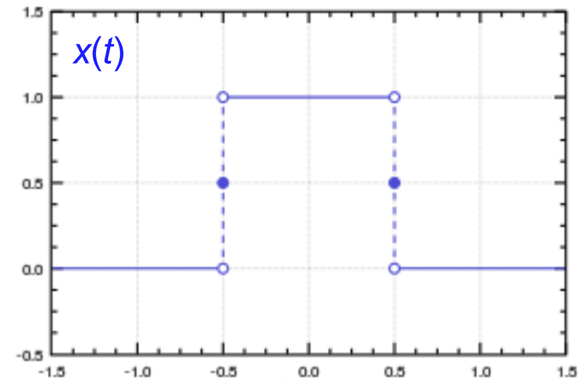
=> $H(f) = \frac{1}{i2\pi f - a}$

Frequency response of the system = Fourier transform *H*(*f*) *of h*(*t*) if *a*<0
=Laplace transform $\hat{H}\ (i2\pi f)$ for all *a*

In this course, we really do not need to care if Fourier transform exists or not!

Fourier transform example 1: Rectangle pulse

Rectangle pulse $x(t)$=rect(t)

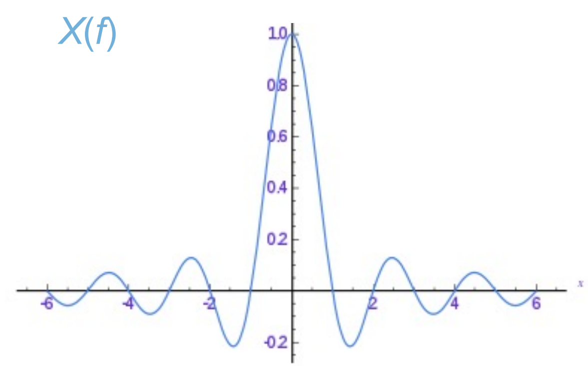


Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 e^{-i2\pi f t} dt$$

$$= -\frac{1}{i2\pi f}\left(e^{-i2\pi f \frac{1}{2}} - e^{-i2\pi f\left(-\frac{1}{2}\right)}\right)$$

$$= \frac{1}{i2\pi f}\left(e^{i\pi f} - e^{-i\pi f}\right) = \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f)$$



$$\sin x = \frac{1}{i2}\left(e^{ix} - e^{-x}\right)$$

$$\int_{x_0}^{x_1} e^{ax} dx = \frac{1}{a} e^{ax_1} - \frac{1}{a} e^{ax_0}$$

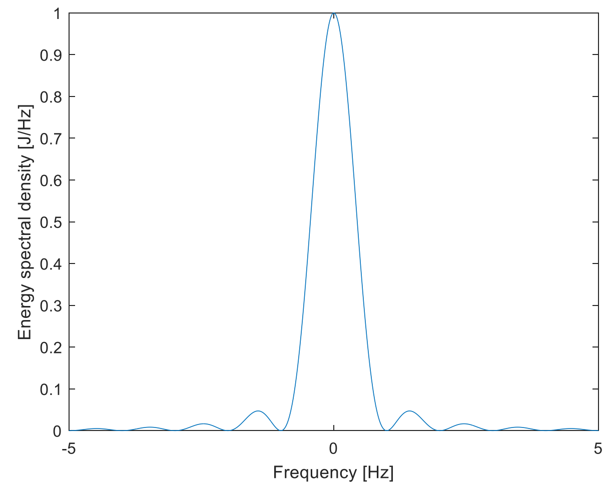
Rayleigh's energy theorem & energy spectrum (a.k.a spectral density)

Signal energy $E = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df$

$|X(f)|^2$ Energy density [J/Hz]

Energy spectrum describes

the division of the signal energy on different frequency components



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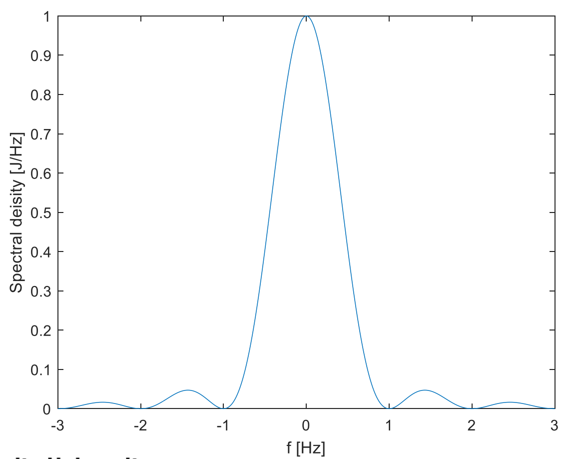
Single-sided and two-sided energy spectrum

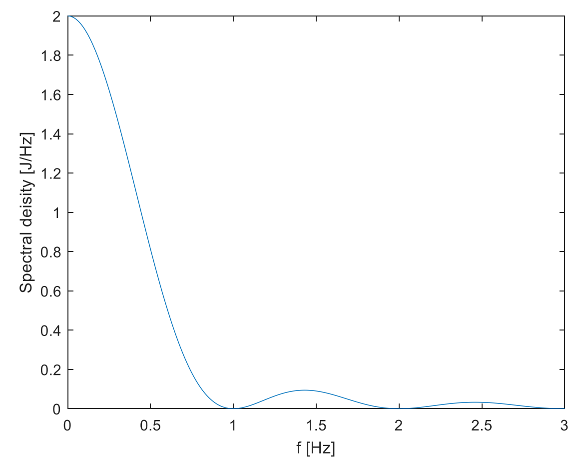
Rayleigh's energy theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X_{Two-sided}(f)|^2 \, df = 2\int_{0}^{\infty} |X_{Two-sided}(f)|^2 \, df = \int_{0}^{\infty} \left|X_{Single-sided}(f)\right|^2 \, df$$

Two-sided energy spectrum $|X_{Two-sided}(f)|^2$



Single-sided energy spectrum $\left|X_{Single-sided}(f)\right|^2$



Energy specrum

Properties of the Fourier-transform Part I

Superposition

$$\mathscr{F}\left[\sum_k a_k x_k(t)\right] = \sum_k a_k \mathscr{F}[x_k(t)]$$

Time and frequency scaling

$$\mathscr{F}[x(at)] = \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$$\mathscr{F}^{-1}[X(af)] = \frac{1}{|a|} x\left(\frac{t}{a}\right)$$

Duality

$$\mathscr{F}[X(t)] = x(-f)$$

$$\mathscr{F}^{-1}[x(f)] = X(-t)$$

Time and frequrncy shift

$$\mathscr{F}[x(t-\tau)] = e^{-j2\pi f\tau} X(f)$$

$$\mathscr{F}\left[x(t)e^{j2\pi f_0 t}\right] = X(f-f_0)$$

Linear modulation

$$\mathrm{F}[s(t)\cos(2\pi f_c t)] = \frac{1}{2}S(f f_c) + \frac{1}{2}S(f - f_c)$$

Convolution

$$\mathscr{F}[x(t) \otimes y(t)] = X(f)Y(f)$$

Derivative

$$\mathscr{F}\left[\frac{d}{dt}x(t)\right] = j2\pi f \cdot X(f)$$

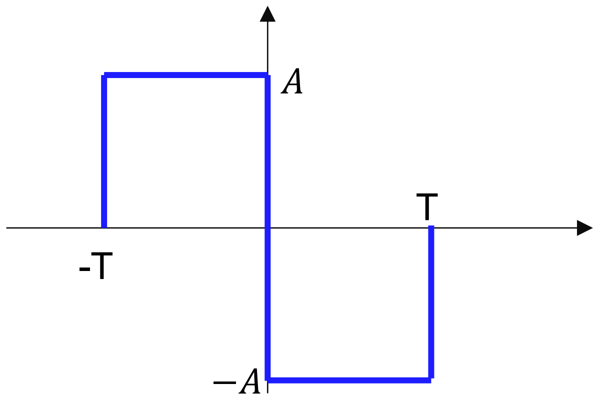
$$\mathscr{F}\left[\frac{d^n}{dt^n}x(t)\right] = (j2\pi f)^n \cdot X(f)$$

Anti-derivative (integral)

$$F\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{1}{j2\pi f}X(f) + \bar{x}\delta(f)$$

$$\bar{x} = \int_{-\infty}^{t} x(\tau)d\tau$$

Example 2: Pulse



Pulse

$x(t)$= $A\mathrm{rect}\left(\frac{t+\frac{1}{2}}{T}\right) - A\mathrm{rect}\left(\frac{t-1/2}{T}\right)$

Fourier transform

$$X(f) = AT\mathrm{sinc}(fT)e^{i2\pi f\frac{1}{2}T} - AT\mathrm{sinc}(fT)e^{-i2\pi f\frac{1}{2}T}$$

$F[\mathrm{rect}(t)] = \mathrm{sinc}(f)$

$\mathscr{F}[x(at)] = \frac{1}{|a|}X\left(\frac{f}{a}\right)$ $a=1/T$

$\mathscr{F}[x(t-\tau)] = e^{-j2\pi f\tau}X(f)$

$$X(f) = 2iAT\mathrm{sinc}(fT)\sin(\pi fT)$$

$\sin(x) = \frac{1}{i2}\left(e^{ix} - e^{-ix}\right)$

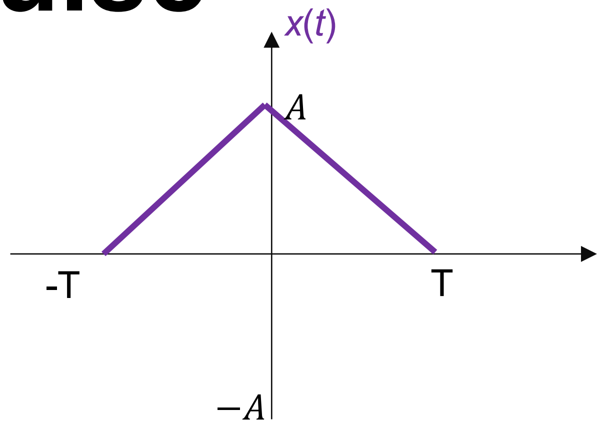
Example 3: Triangle pulse

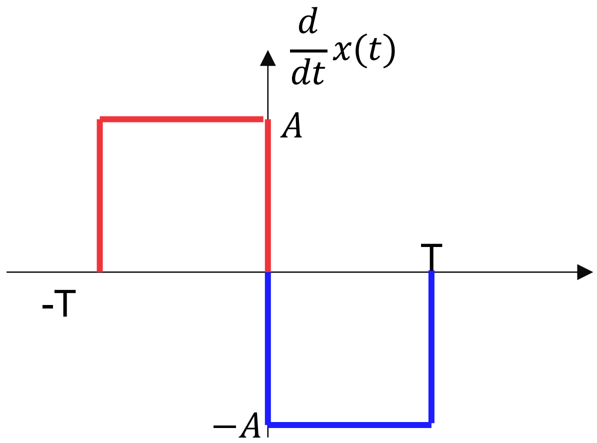
Triangle pulse

$$x(t) = A\mathrm{tria}\left(\frac{t}{T}\right)$$

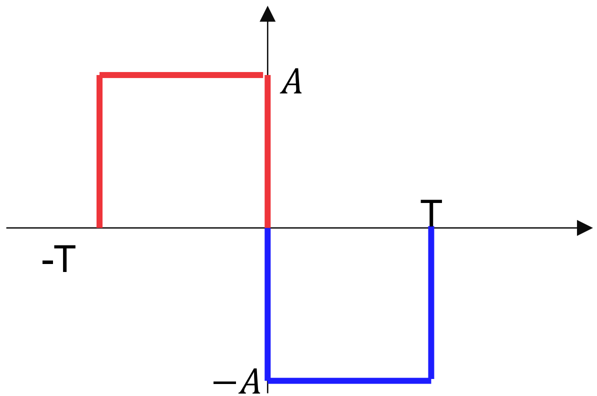
Time derivative

$$\frac{d}{dt}x(t) = A\mathrm{rect}\left(\frac{t+1/2}{T}\right) - A\mathrm{rect}\left(\frac{t-1/2}{T}\right)$$





Example 3: Triangle pulse



Pulse

$\frac{d}{dt}x(t) = A\mathrm{rect}\left(\frac{t+1/2}{T}\right) - A\mathrm{rect}\left(\frac{t-1/2}{T}\right)$

Fourier transform

$$j2\pi f X(f) = AT\mathrm{sinc}(fT)e^{i2\pi f\frac{1}{2}T} - AT\mathrm{sinc}(fT)e^{-i2\pi f\frac{1}{2}T}$$

$$\mathscr{F}\left[\frac{d}{dt}x(t)\right] = j2\pi f \cdot X(f)$$

$$F[\mathrm{rect}(t)] = \mathrm{sinc}(f)$$

$$\mathscr{F}[x(at)] = \frac{1}{|a|}X\left(\frac{f}{a}\right)$$

a=1/T

$$\mathscr{F}[x(t-\tau)] = e^{-j2\pi f\tau}X(f)$$

$$\Leftrightarrow\ j2\pi f X(f) = 2iAT\mathrm{sinc}(fT)\sin(\pi f T)$$

$$\Leftrightarrow X(f) = AT\mathrm{sinc}(fT)\frac{\sin(\pi f T)}{\pi f}$$

$$\Leftrightarrow X(f) = AT\mathrm{sinc}^2(fT)$$

$$\sin(x) = \frac{1}{i2}\left(e^{ix} - e^{-ix}\right)$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Example: Triangle pulse vs rect pulse

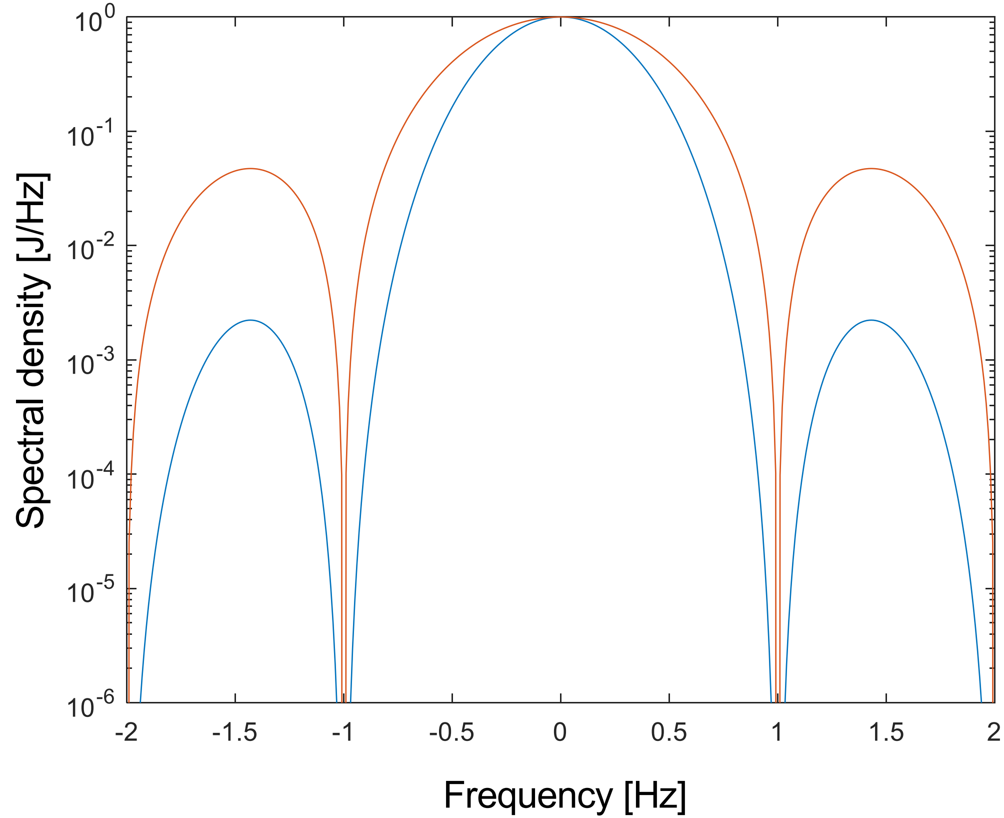
Spectral density of tria pulse

$$|Xtri_a(f)|^2 = A^2T^2\text{sinc}^4(fT)$$

Spectral density of rect pulse

$$|Xrec_t(f)|^2 = A^2T^2\text{sinc}^2(fT)$$

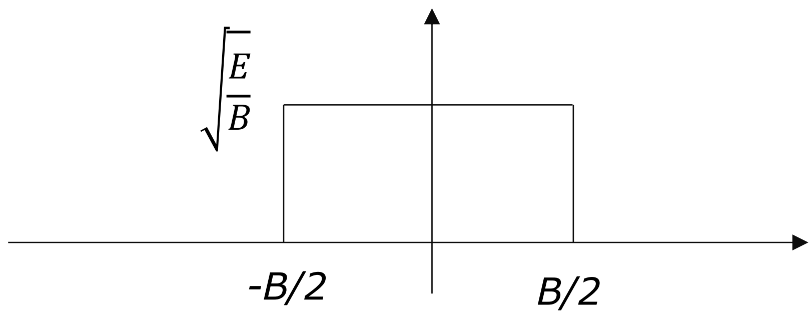
Rect changes faster than tria.
Hence, it has wider spectrum.



Inverse Fourier transform example

Bandwith limted signal

$X(f) = \sqrt{\frac{E}{B}}\,\mathrm{rect}\left(\frac{f}{B}\right)$



Inverse Fourier transform

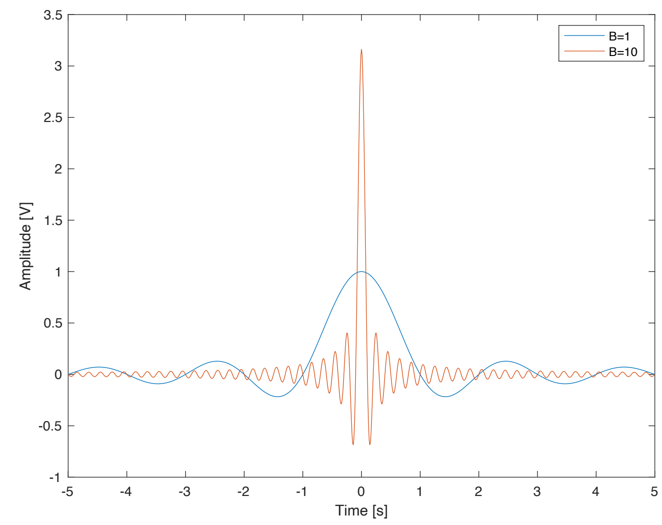
$$x(t) = \int_{-\infty}^{\infty} x(t) e^{i2\pi f t} df = \int_{-\frac{1}{2}B}^{\frac{1}{2}B} \sqrt{\frac{E}{B}} e^{i2\pi f t} df$$

$$= \sqrt{\frac{E}{B}} \frac{1}{i2\pi f} \left(e^{i2\pi t \frac{1}{2}B} - e^{i2\pi t \left(-\frac{1}{2}B\right)} \right)$$

$$= \sqrt{\frac{E}{B}} \frac{B}{i2\pi f B} \left(e^{i\pi t B} - e^{-i\pi t B} \right) = \sqrt{EB} \frac{\sin(\pi B t)}{\pi f B} = \sqrt{EB}\,\mathrm{sinc}(Bt)$$

$$\sin x = \frac{1}{i2}\left(e^{ix} - e^{-x}\right)$$

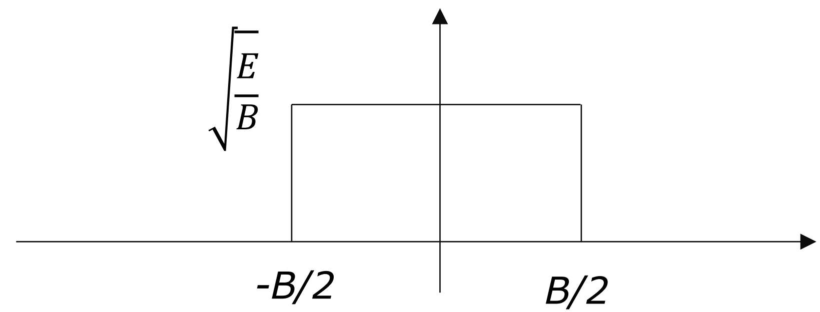
$$\int_{x_0}^{x_1} e^{ax} dx = \frac{1}{a} e^{a x_1} - \frac{1}{a} e^{a x_0}$$



Inverse Fourier transform example (again)

Bandwith limted signal

$X(f) = \sqrt{\frac{E}{B}}\,\mathrm{rect}(\frac{f}{B})$



Inverse Fourier transform

$$F\big(\mathrm{rect}(t)\big) = \mathrm{sinc}(f)$$

$$F^{-1}\big(\mathrm{rect}(f)\big) = \mathrm{sinc}(-t)$$

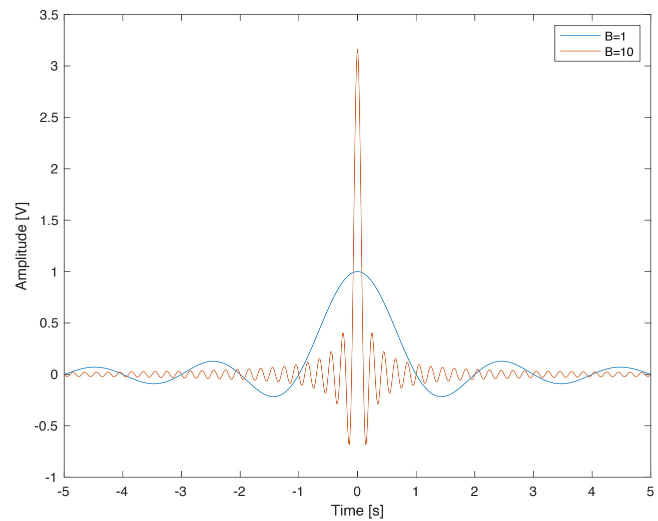
$$\sqrt{\frac{E}{B}}$$

$$F^{-1}\left(\sqrt{\frac{E}{B}}\,\mathrm{rect}\left(\frac{f}{B}\right)\right) =$$

$$\mathscr{F}^{-1}[X(af)] = \frac{1}{|a|}x\left(\frac{t}{a}\right)$$

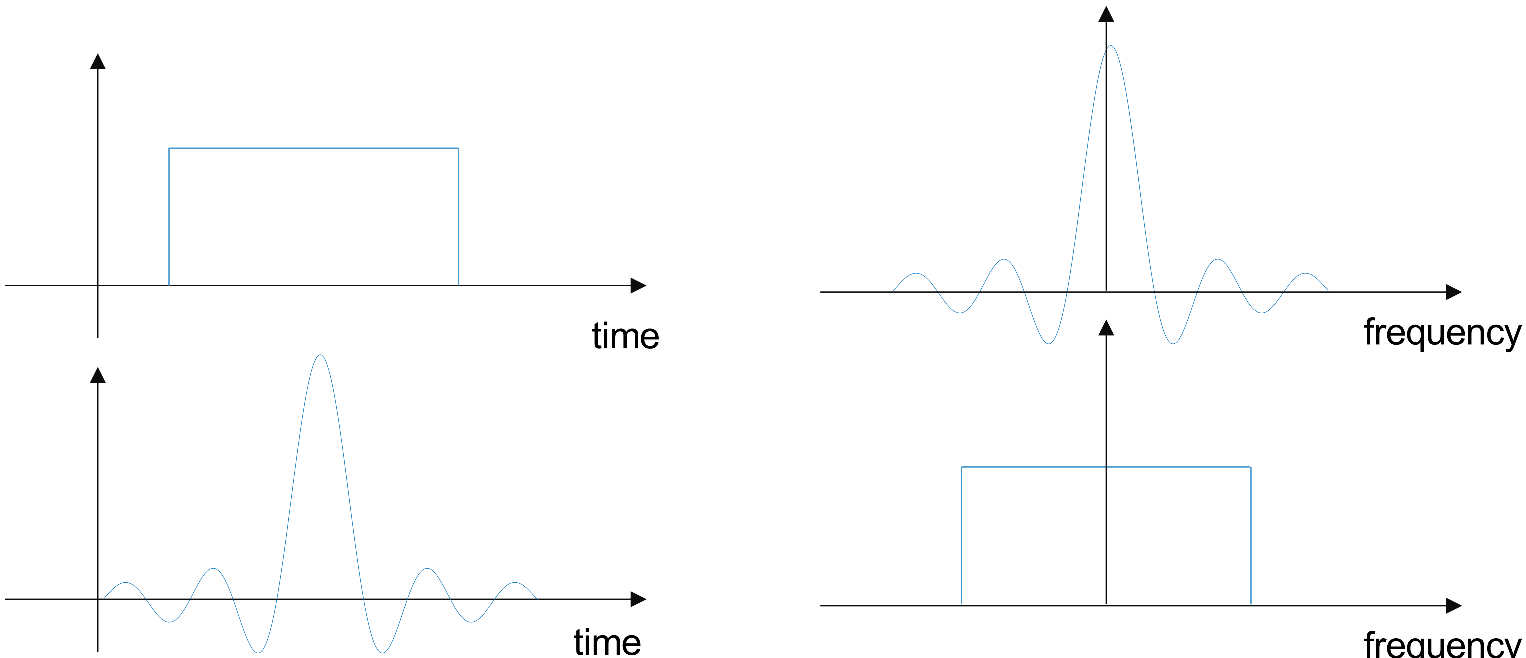
$$\sqrt{\frac{E}{B}}\,B\mathrm{sinc}(-tB) = \sqrt{EB}\,\mathrm{sinc}(Bt)$$

$$\mathrm{sinc}(t) = \mathrm{sinc}(-t)$$



Time-frequency localization

Signal cannot be localized both in time and frequency domain

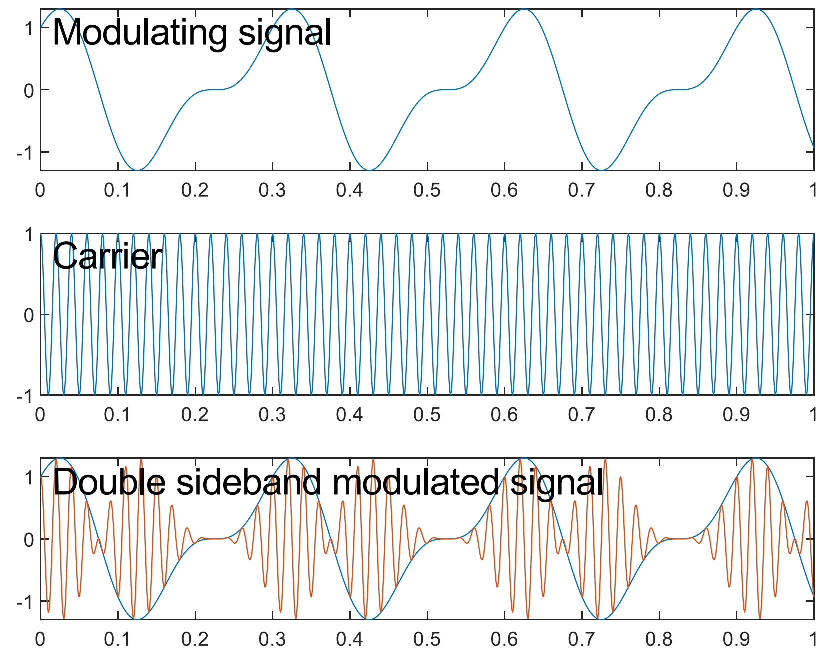


Linear modulation

- In electronics and telecommunications, modulation is the process of varying one or more properties of a periodic waveform, called the carrier signal ($\cos(2\pi f_c t)$), with a separate signal $s(t)$ called the modulation signal that typically contains information to be transmitted.
- In linear modulation, the modulating signal controls the amplitude of the carrier

$$x(t) = s(t)\cos(2\pi f_c t)$$

This modulation method is also known as ***Double Sideband (DSB) modulation*** in communications engineering literature.



Example: Double sideband modulation of a rect pulse

Pulse

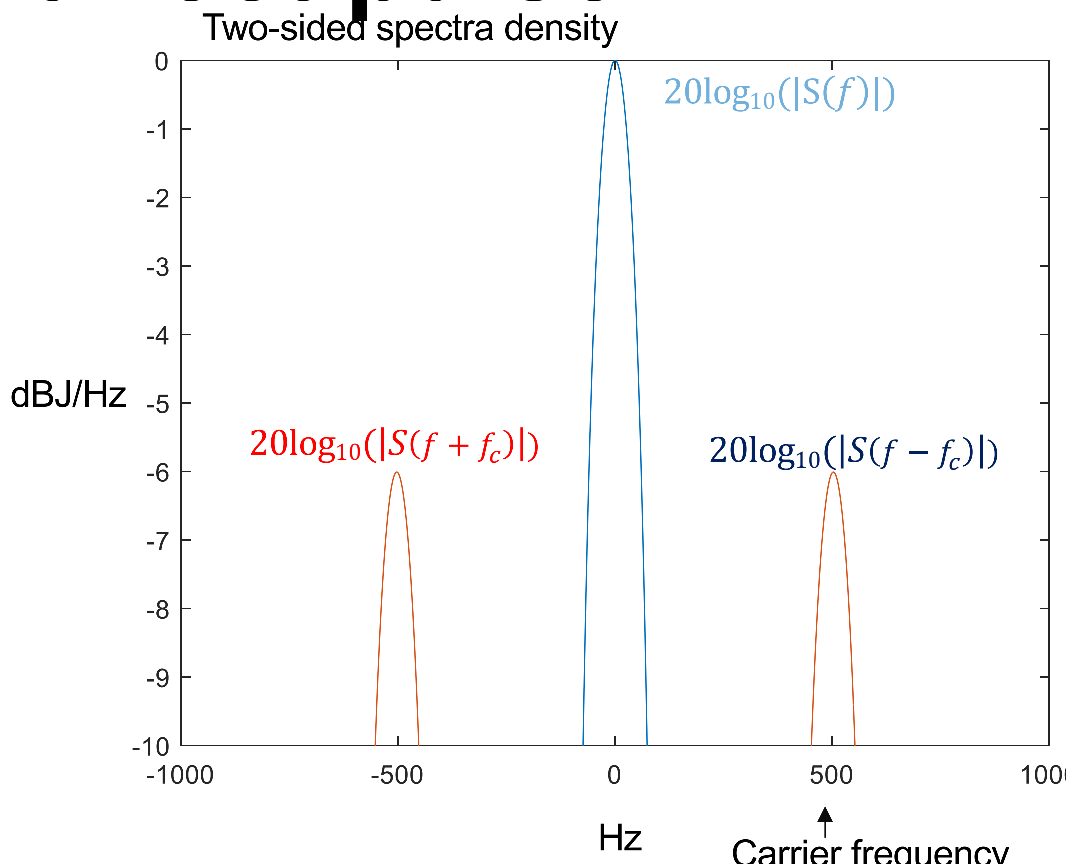
$$s(t) = A\mathrm{rect}\left(\frac{t}{T}\right) \Leftrightarrow S(f) = AT\mathrm{sinc}(fT)$$

$$\mathscr{F}[x(at)] = \frac{1}{|a|} X\left(\frac{f}{a}\right)$$ $a=1/T$

Linear modulation

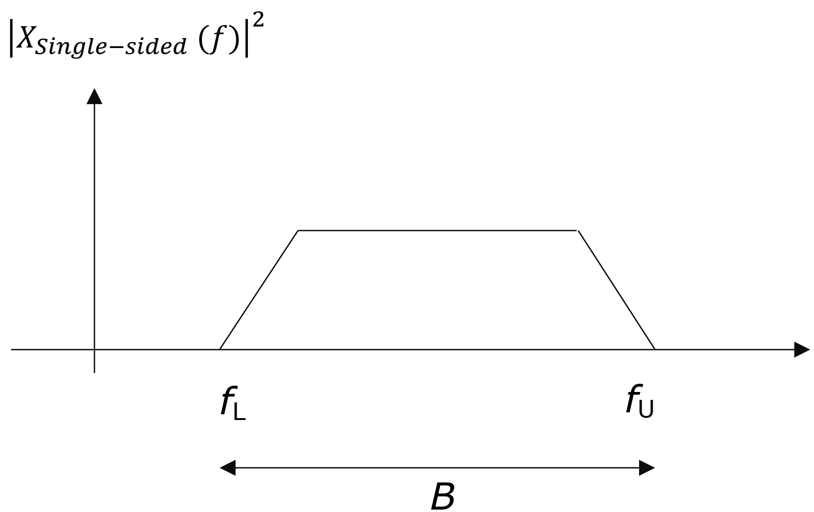
$$x(t) = s(t)\cos(2\pi f_c t)$$
$$= s(t)\left(\frac{1}{2}e^{i2\pi f_c t} + \frac{1}{2}e^{-i2\pi f_c t}\right)$$
$$\Leftrightarrow X(f) = \frac{1}{2}S(f + f_c) + \frac{1}{2}S(f - f_c)$$

$$\mathscr{F}\left[x(t)e^{j2\pi f_0 t}\right] = X(f - f_0)$$



Bandwidth

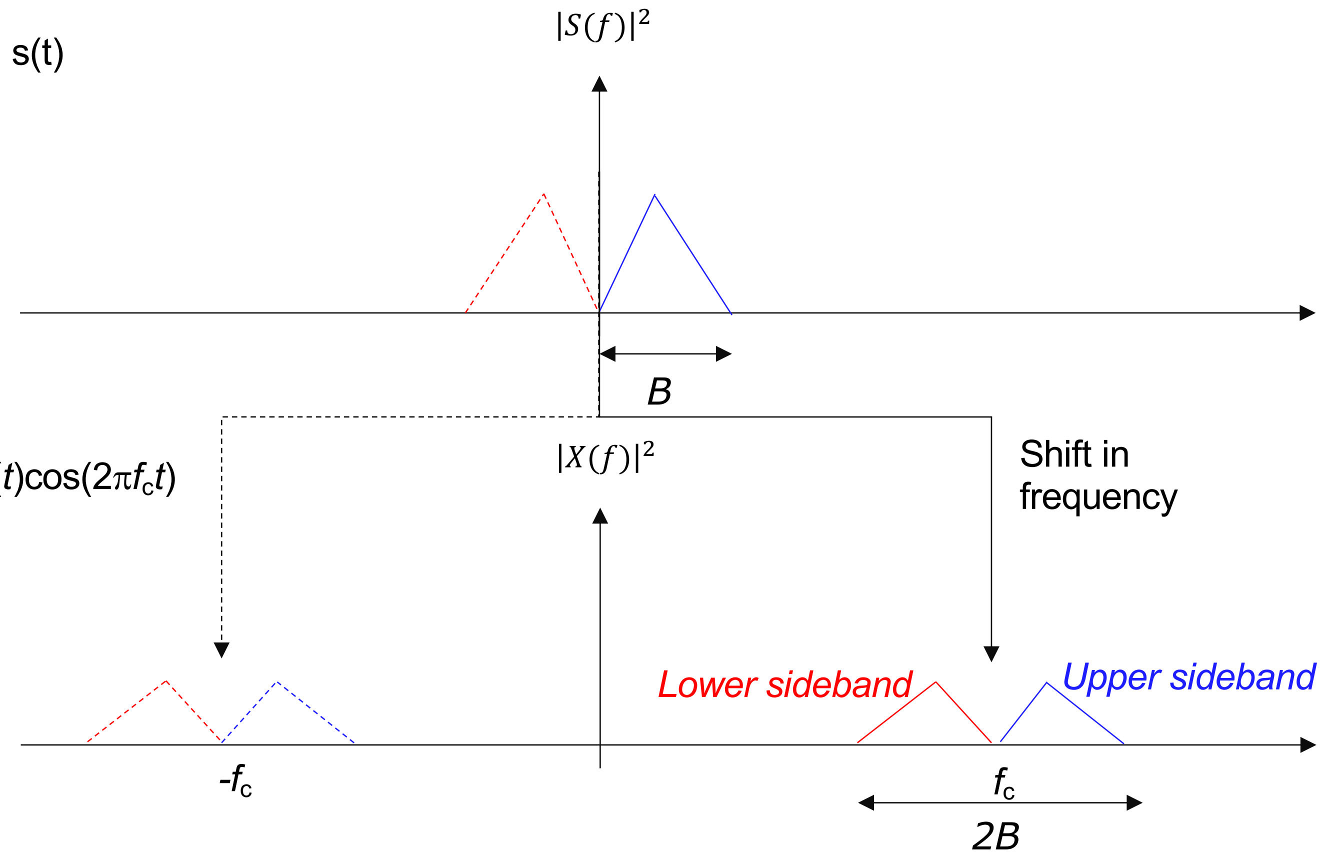
A frequency domain signal X(f) is said to be *bandlimited* if there is two frequencies f_L and f_U such that $|X(f)|>0$ for $f_L \leq f \leq f_U$. The *bandwith* of the signal is then $B=f_U-f_L$



Bandwidht: Double sideband suppressed carrier modulation

Bandwidth limited signal s(t)

Modulated signal $x(t)=s(t)\cos(2\pi f_c t)$



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Bandwidth

Most practical signals are not bandlimited, hence a more general definition is needed. Even though signals analyzed in the real world are not exactly bandlimited, they are often “essentially bandlimited” in a way that the energy spectrum of them is mostly concentrated on a finite frequency interval. Common definitions for bandwidth

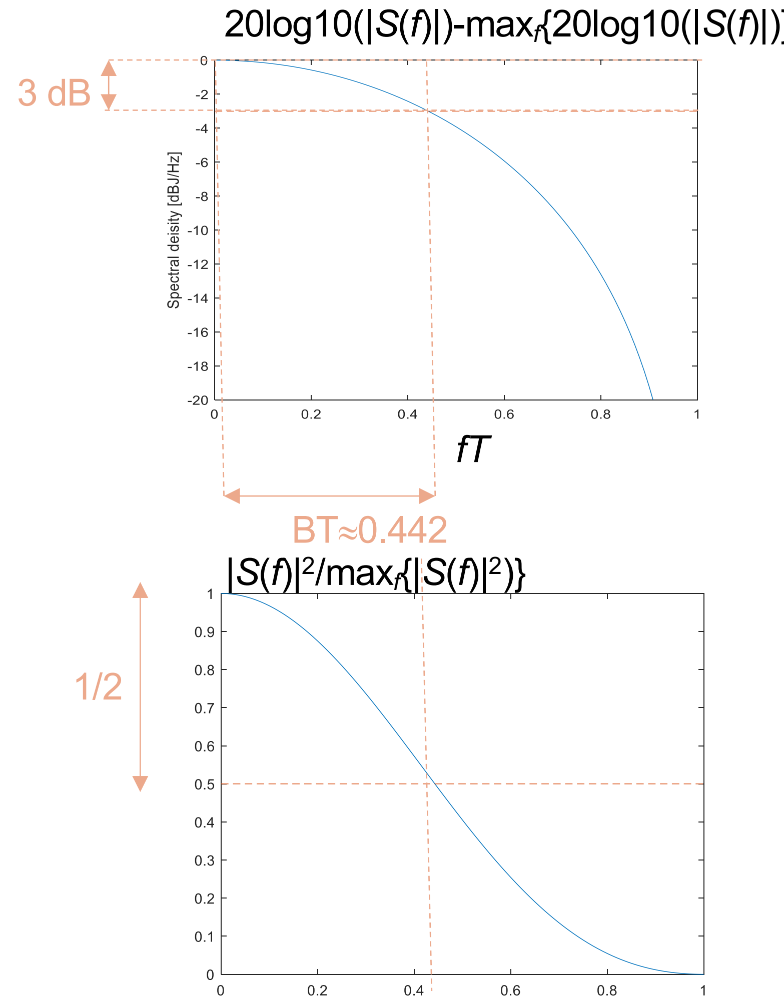
- **95%-Bandwidth**: Frequency range containing 95% of the signal energy
- **3dB-Bandwidth**: Frequency range, where the spectral density is at most 3dB less than its peak value

Example 3dB-Bandwidth of rectangle pulse

Single-sided spectrum

$$s(t) = A\text{rect}\left(\frac{t}{T}\right) \Leftrightarrow S(f) = AT\text{sinc}(fT)$$

Bandwidth $B \approx \frac{1}{2T}$



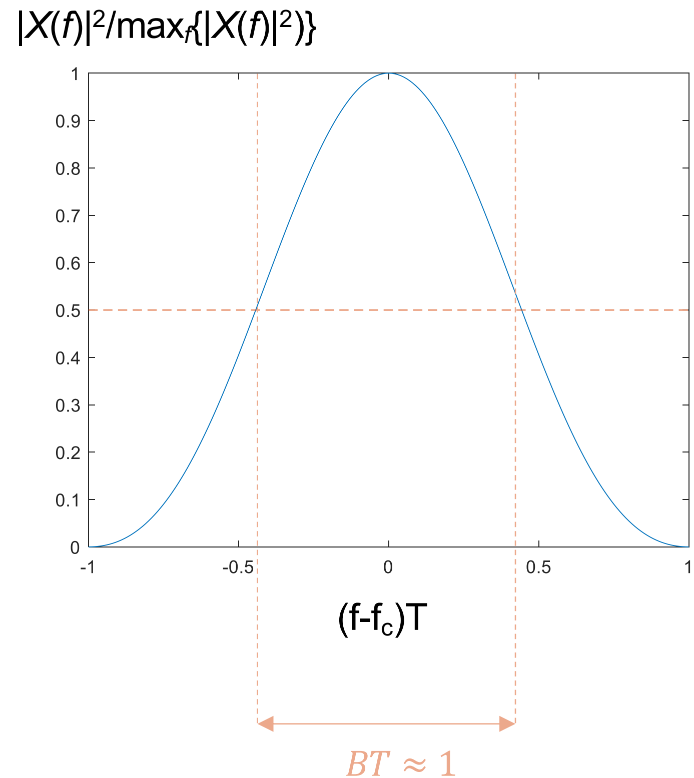
Example 3dB-Bandwidth of modulated rectangle pulse

Linearly modulated rectangle pulse

$$x(t) = s(t)\cos(2\pi f_c t)$$
$$\Leftrightarrow X(f) = \frac{1}{2}S(f + f_c) + \frac{1}{2}S(f - f_c)$$

Bandwidth $B \approx \frac{1}{T}$

Modulated signal has twice the bandwidth of the base band signal



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Example: Bluetooth signal 99% Bandwidth

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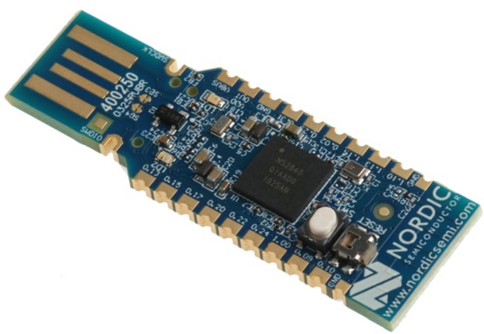
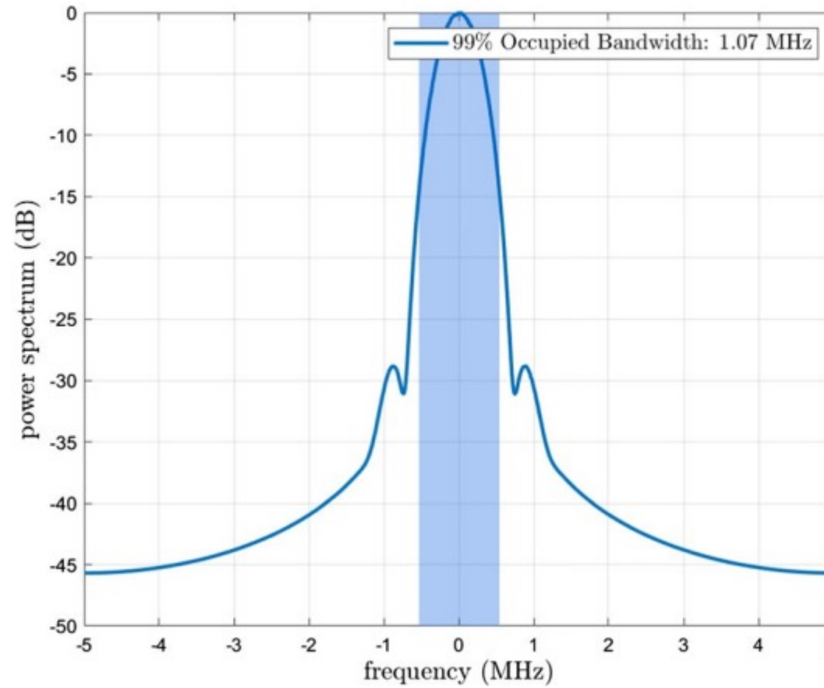
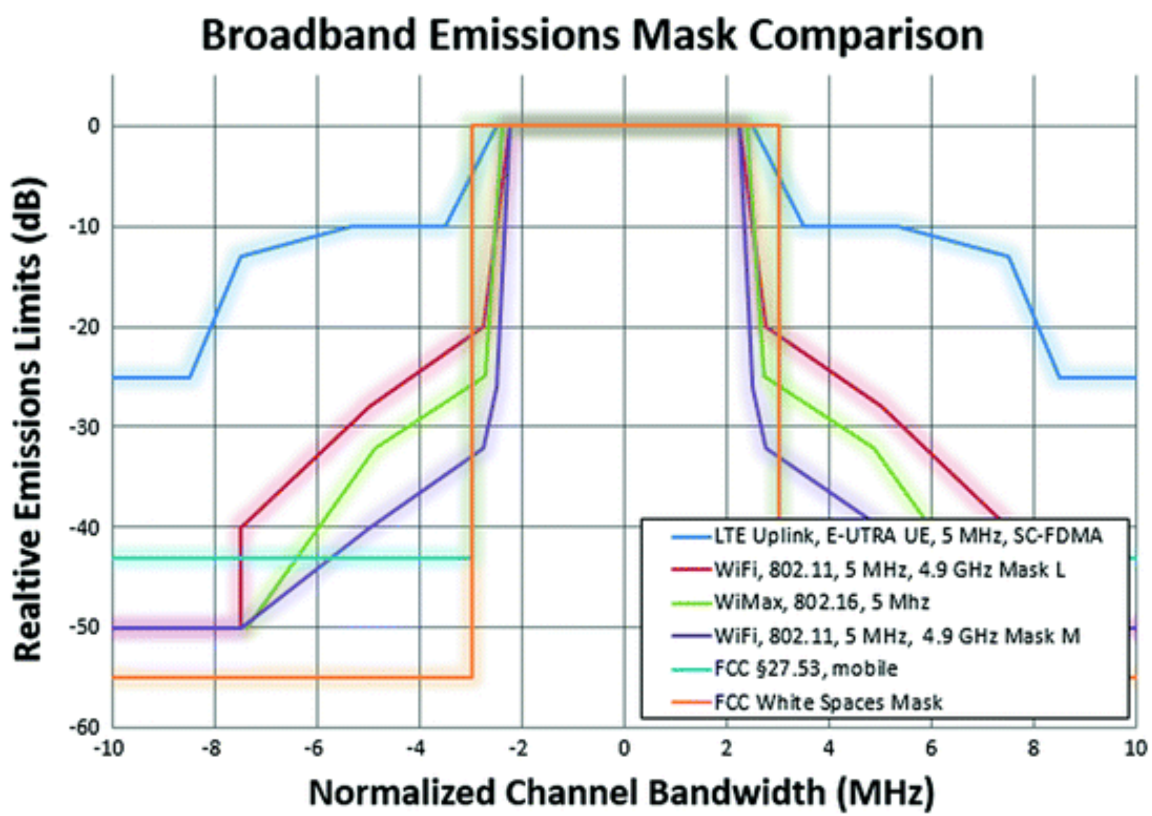




Kavousi Ghafi, H., Spindelberger, C. and Arthaber, H., 2021. Modeling of co-channel interference in bluetooth low energy based on measurement data. *EURASIP Journal on Wireless Communications and Networking*, *2021*(1), pp.1-17.

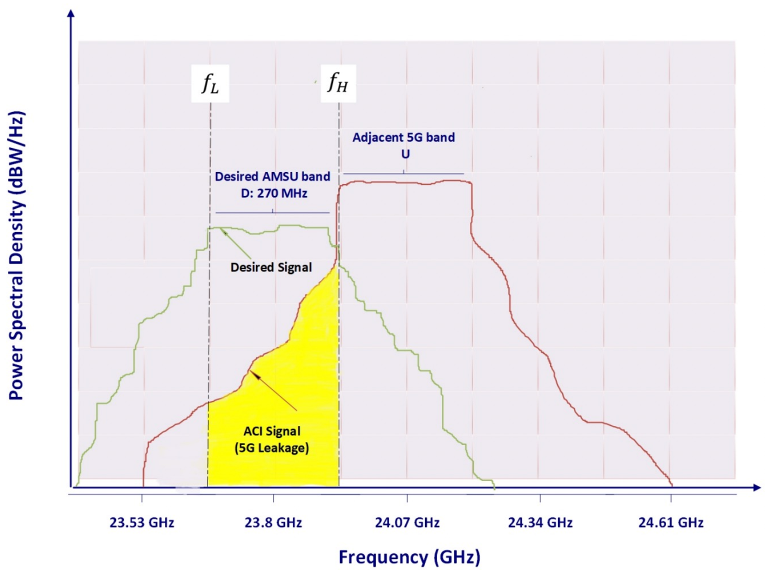
Spectral masks

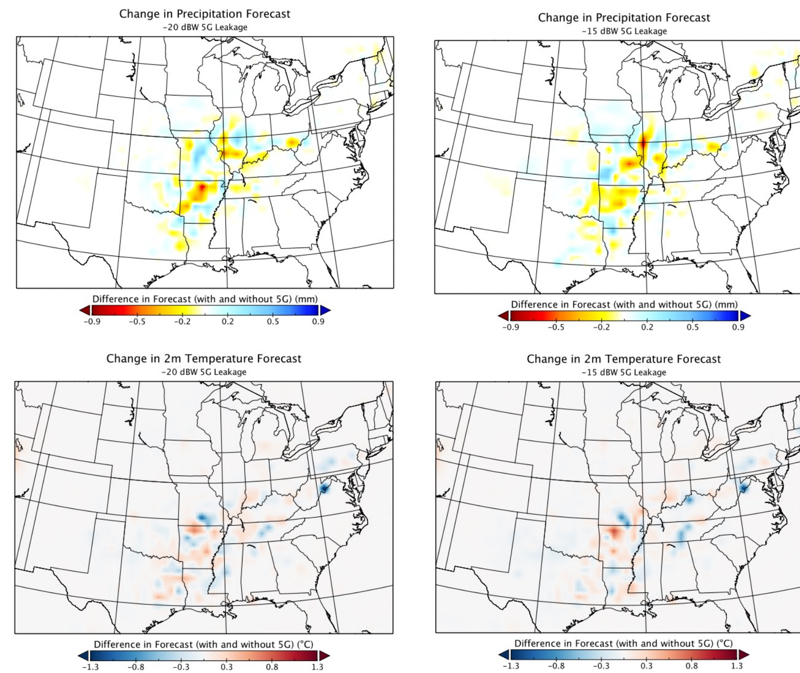
- Frequency regulator sets limits on how much enery can be leaked to adjacent frequency bands.
- This limits the pulse waveforms that can be used.



Spectral emissions can matter…

Impact of 5G mmWave on weather radars





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Yousefvand, M., Wu, C.T.M., Wang, R.Q., Brodie, J. and Mandayam, N., 2020, September. Modeling the Impact of 5G Leakage on Weather Prediction. In *2020 IEEE 3rd 5G World Forum (5GWF)* (pp. 291-296). IEEE.

Convolution example: Two RC filters in series

Impulse response of two RC filters in series

- Impulse response of the first stage

$$h_1(t) = e^{-\frac{1}{T_1}t}u(t) \Leftrightarrow H_1(f) = \frac{T_1}{T_1 + j2\pi f}$$

- Impulse response of the second stage

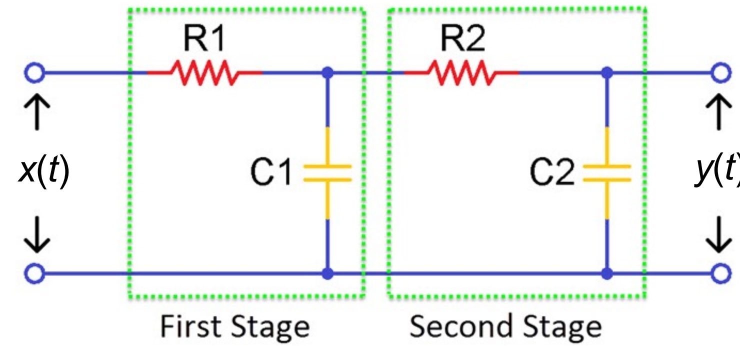
$$h_2(t) = e^{-\frac{1}{T_2}t}u(t) \Leftrightarrow H_2(f) = \frac{T_2}{T_2 + j2\pi f}$$

- Impulse response of the filter

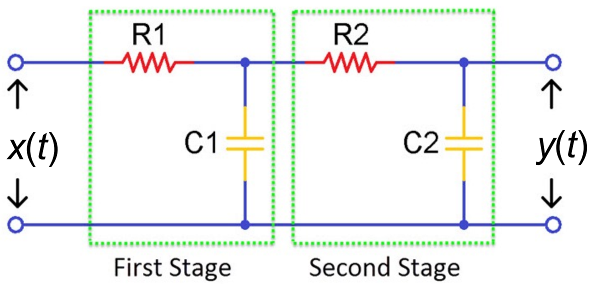
$$h(t) = \int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau)d\tau$$

- Frequency response of the filter

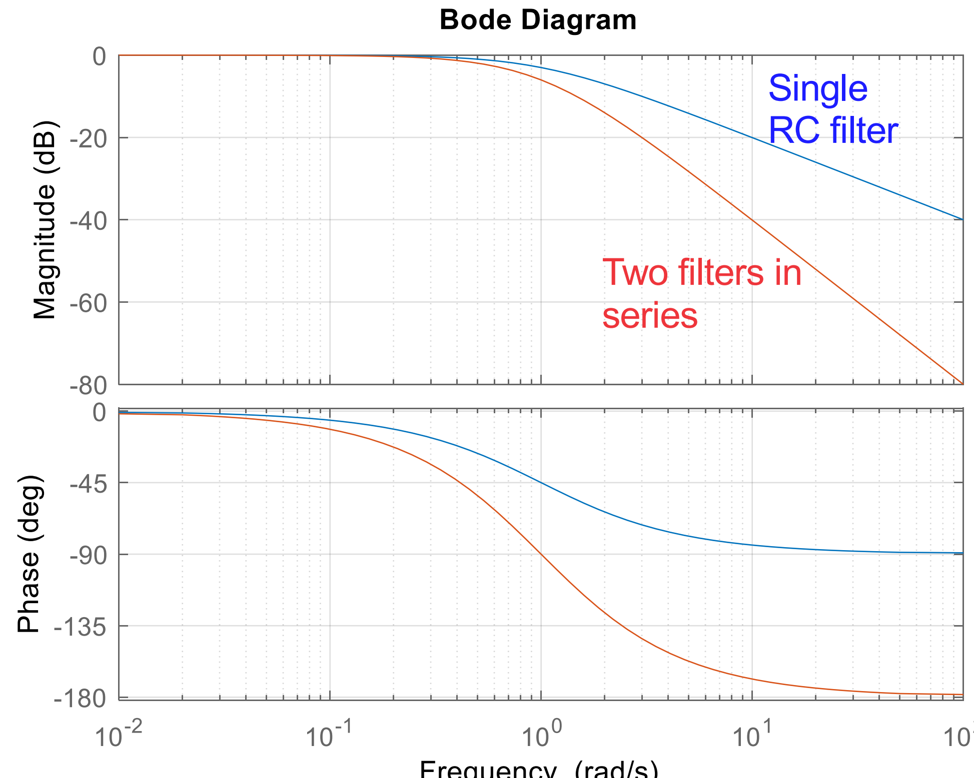
$$H(f) = H_1(f)H_2(f) = \frac{T_1T_2}{(T_1 + j2\pi f)(T_2 + j2\pi f)}$$



Convolution example: Two RC filters in series



- First stage $H_1(f) = \frac{1}{1+j2\pi f}$
- 2nd stage $H_2(f) = \frac{1}{1+j2\pi f}$
- Overall 2nd order filer
  $H(f) = H_1(f)H_2(f) = \frac{1}{(1+j2\pi f)^2}$



Fourier transform of special functions

Fourier transform of Dirac's delta function

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt = e^{-j2\pi f 0} = 1$$

Fourier transform of a constant (by duality from the above)

$$F[1] = \delta(-f) = \delta(f)$$

$$\mathscr{F}^{-1}[x(f)] = X(-t)$$

Fourier transform of a phasor = Frequency shift

(by duality from the time shift)

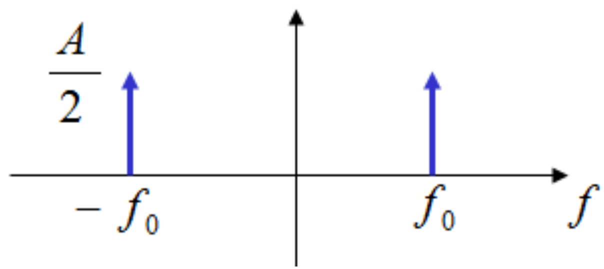
$$F\left[e^{j2\pi f_0 t}\right] = \delta(f - f_0)$$

$$F[\delta(t - t_0)] = e^{-j2\pi f t_0}$$

Fourier transform of sinusoidal

Fourier transform of a cosine

$$\mathrm{F}[A\cos(2\pi f_0 t)] = F\left[A\frac{1}{2}\left(e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}\right)\right] = \frac{A}{2}F[e^{i2\pi f_0 t}] + \frac{A}{2}F\left[e^{-i2\pi f_0 t}\right] = \frac{A}{2}\delta(f - f_0) + \frac{A}{2}\delta(f + f_0)$$

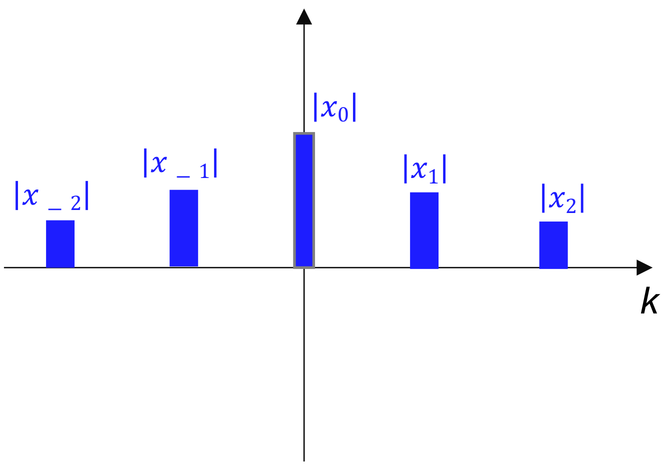


Fourier transform of an exponential Fourier series

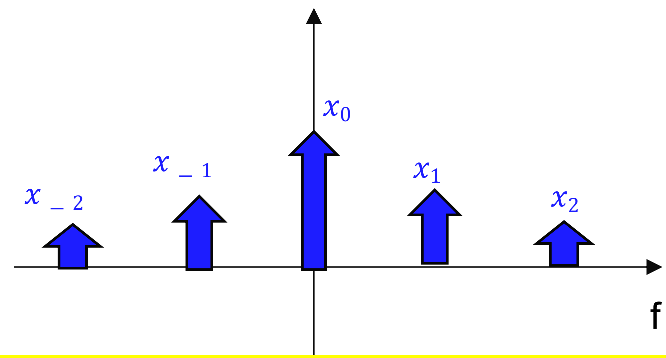
Exponential Fourier series of a periodic signal x(t) = x(t+T_0)

$$x(t) = \sum_{k=-\infty}^{\infty} x_k \, e^{i\frac{2\pi k}{T_0}t} \Leftrightarrow X(f) = \sum_{k=-\infty}^{\infty} x_k \, \delta\left(f - \frac{k}{T_0}\right)$$

Amplitude spectrum



Visualization of Fourier-transform



Amplitude and power spectra are not well-defined for Dirac’s delta function

Truncating a continuous time signal

Let us take a time interval –T/2≤ t≤ T/2 of a periodic signal

$$x(t) = A\cos(2\pi f_0 t)$$

This can be written as

$$\tilde{x}(t) = A\cos(2\pi f_0 t)\,\mathrm{rect}\left(\frac{t}{T}\right)$$

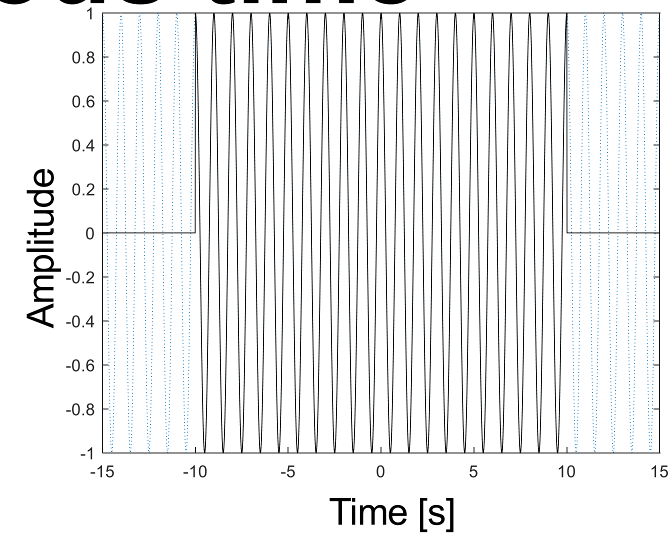
Multiplication in time domain => Convolution in frequency domain

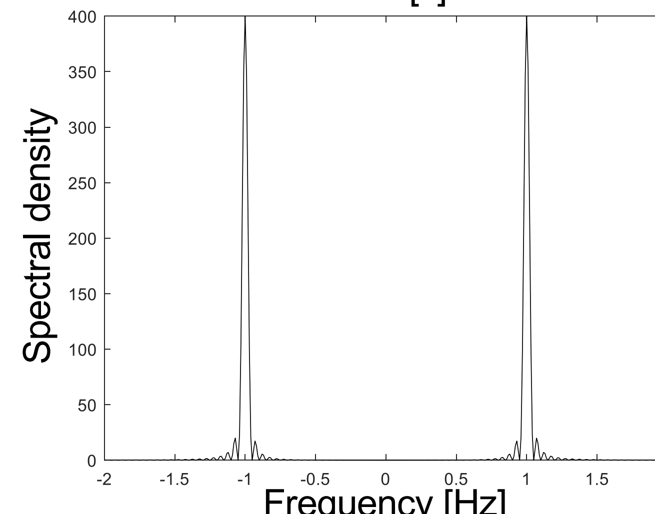
$$X(f) = \left\{\frac{A}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right)\right\} \otimes \{T\mathrm{sinc}(fT)\}$$

$$= \frac{AT}{2}\left(\mathrm{sinc}\left((f - f_0)T\right) + \mathrm{sinc}\left((f + f_0)T\right)\right)$$

$$\mathscr{F}[x(t) \otimes y(t)] = X(f)Y(f)$$

$$\mathscr{F}[X(t)] = x(-f)$$





Truncating a continuous time signal

Let us take a time interval –T/2≤ t≤ T/2 of a periodic signal

$$x(t) = A\cos(2\pi f_0 t)$$

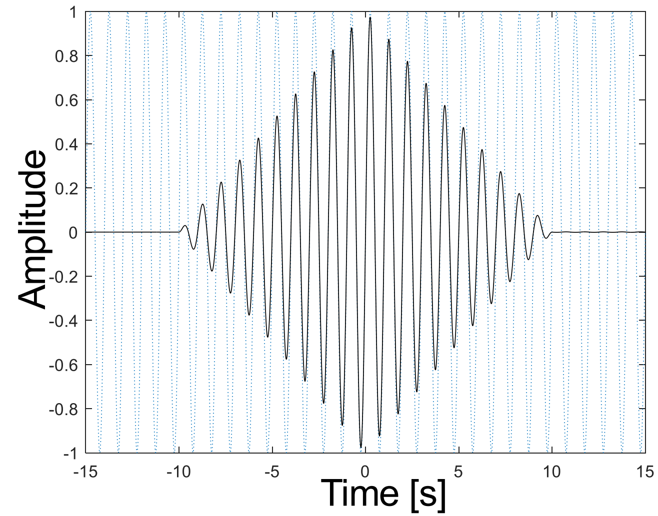
But this time weight the signal while cutting

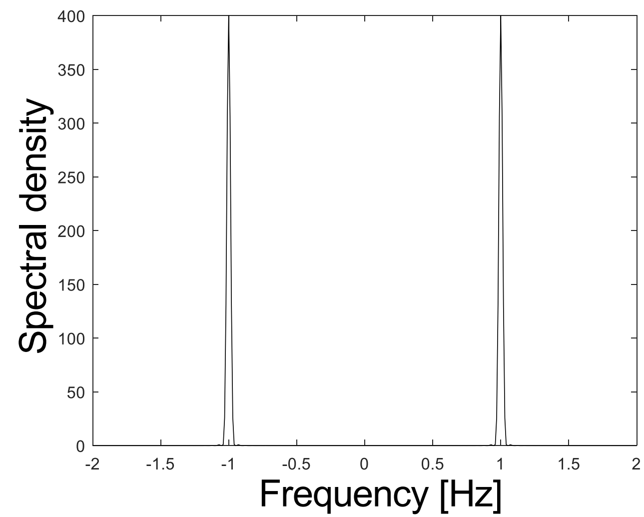
$$\tilde{x}(t) = A\cos(2\pi f_0 t)\,\mathrm{tria}\left(\frac{t}{T}\right)$$

Multiplication in time domain => Convolution in frequency domain

$$X(f) = \left\{\frac{A}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right)\right\} \otimes \{T\,\mathrm{sinc}^2(fT)\}$$

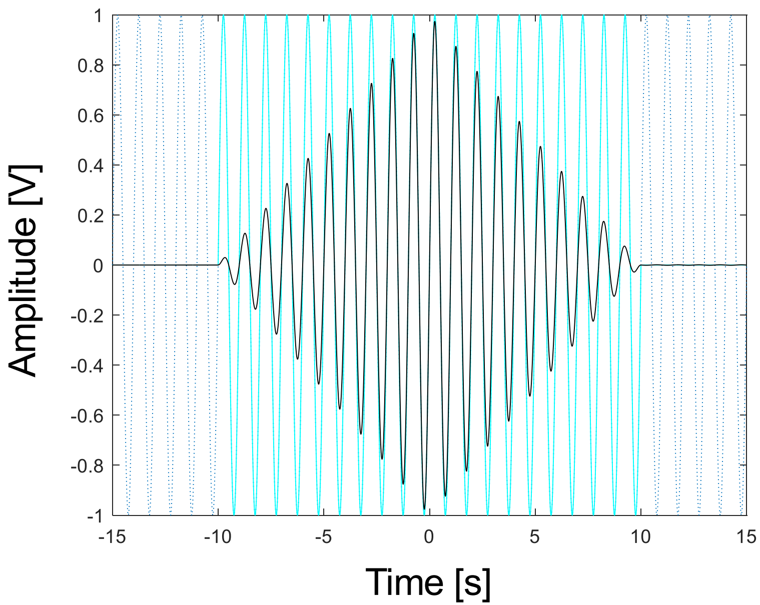
$$= \frac{AT}{2}\left(\mathrm{sinc}^2\left((f - f_0)T\right) + \mathrm{sinc}^2\left((f + f_0)T\right)\right)$$

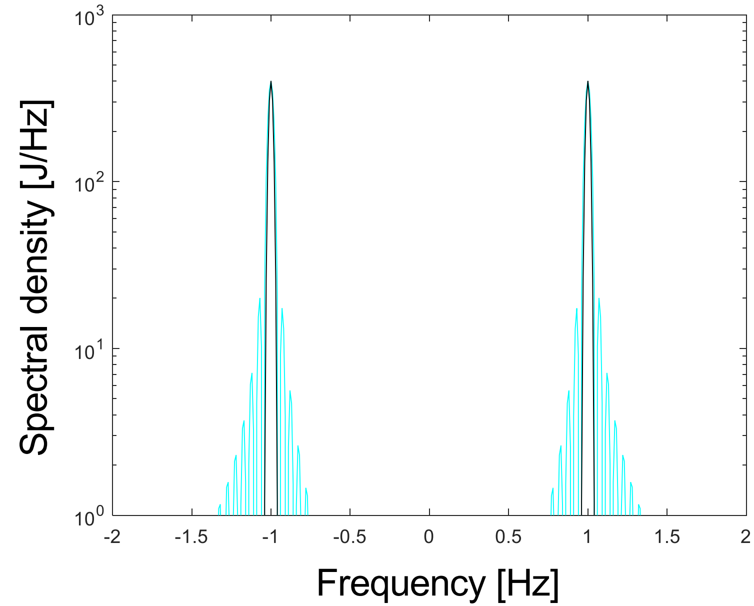




Truncating a continuous time signal

Rectangular window and triangular window





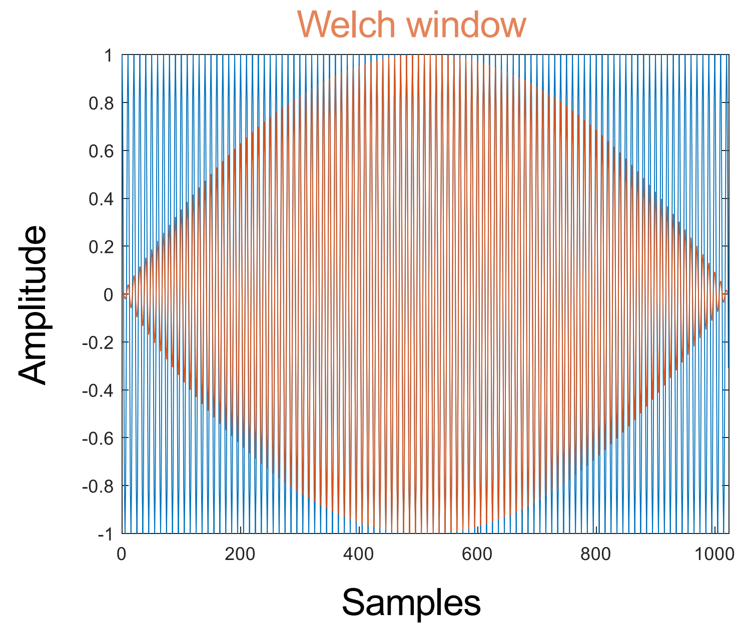
Windowing

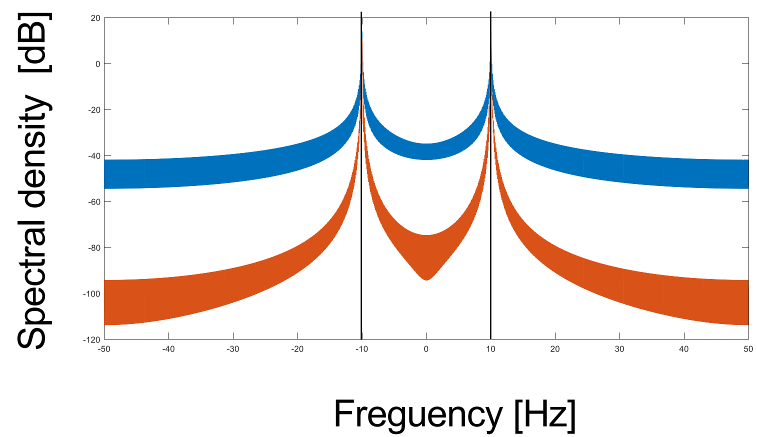
When truncating the signal, we can select the window function w(t) truncating the signal

$$\tilde{x}(t) = x(t)w(t)$$

This is typically done after sampling the signal

$$\tilde{x}_s(nT_s) = x(nT_s)w(n)$$





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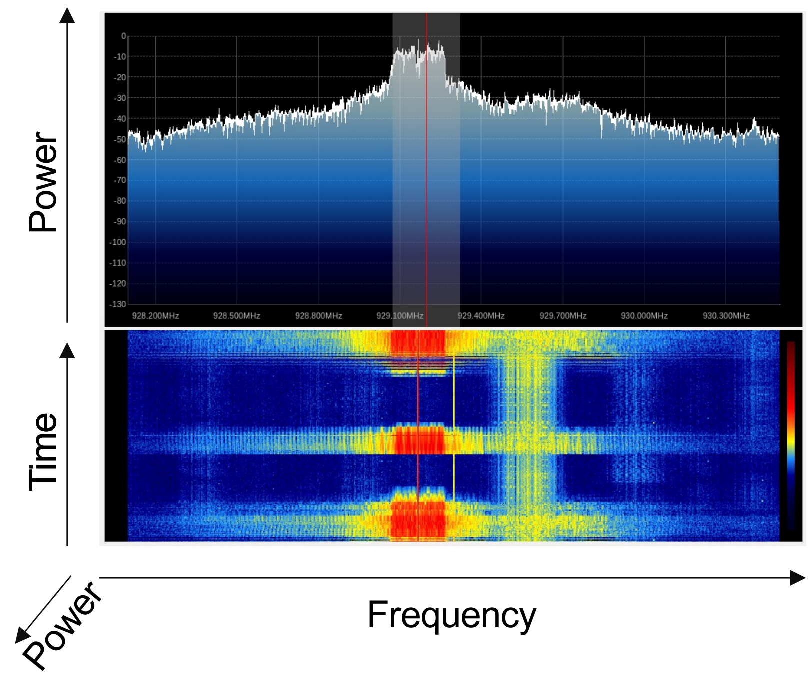
Time-frequency analysis & Spectrogram

In signal processing, time–frequency analysis comprises those techniques that study a signal in both the time and frequency domains simultaneously.

A spectrogram is a visual representation of the spectrum of frequencies of a signal as it varies with time.

Example: NB-IoT base station signal



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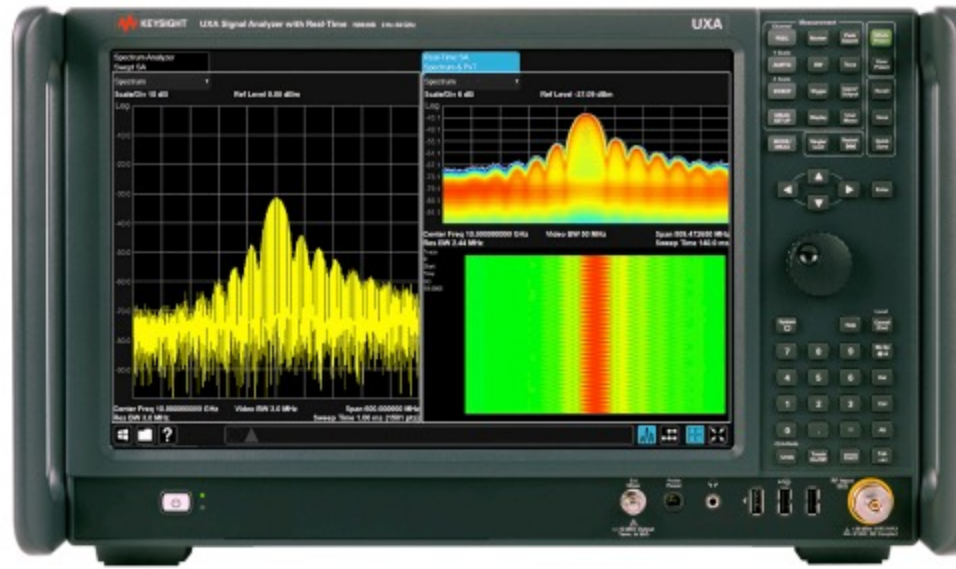
Real-time spectrum analyzer

- **Real-time spectrum analyzers are utilized to do the spectrum and time-frequency analysis of signals.**



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