**Mathematics for Economists**

**Problem Set 4**
**Due date: Wednesday 12.10 at 12.15**

### Exercise 1

Find all the *local* and *global* maximizers of $f(x_1, x_2) = 3x_1x_2 - x_1^3 - x_2^3$.

### Exercise 2

The following two functions are defined on $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Determine whether each of them is a concave or a convex function.

1. $f(x, y) = -2x + y + x^2 - 2xy + y^2$
2. $f(x, y, z) = 100 - 2x^2 - y^2 - 3z - xy - e^{x+y+z}$

### Exercise 3

Assume that a monopoly sells a good that has the demand function $D(P, A) = (\alpha - P)A^\beta$ (quantity demanded), where $\alpha > 0$ and $\beta \in (0, 1)$ are exogenous parameters, $P$ is the price of the good, and $A$ is the money invested in marketing. Assume that $c > 0$ is the unit cost of producing the good.

1. Write down the firms profit function.
2. Find the price $P$ and the money spent on advertising $A$ that satisfy the first order optimality conditions.
3. Verify that the second order sufficient optimality conditions are satisfied.

### Exercise 4

A firm produces two commodities whose quantities are denoted as $q_1$ and $q_2$. The corresponding commodity prices are $p_1$ and $p_2$, respectively. The cost of producing quantities $(q_1, q_2)$ is given by the cost function $C(q_1, q_2) = 2q_1^2 + q_1q_2 + 2q_2^2$. The firm's profit maximization problem is

$$\max_{q_1, q_2} \pi = p_1q_1 + p_2q_2 - \left(2q_1^2 + q_1q_2 + 2q_2^2\right) = \mathbf{p}^T\mathbf{q} - \mathbf{q}^T A\mathbf{q},$$

where $\mathbf{p} = (p_1, p_2)^T$, $\mathbf{q} = (q_1, q_2)^T$ and $A = \begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{bmatrix}$.

Find the profit-maximizing quantities $q_1^*$ and $q_2^*$.

**Exercise 5**

Consider a problem of fitting a linear model $y_i = \alpha + \beta x_i$ into a data with observations $(x_i, y_i)$, $i = 1, 2, \ldots, n$, where $x_i, y_i \in \mathbb{R}$:

$$\min_{\alpha,\beta} \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2.$$

a) Use the first order optimality conditions to find the optimal $\hat{\alpha}$ and $\hat{\beta}$:

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x},$$
$$\hat{\beta} = \frac{(\sum_{i=1}^{n} x_i y_i) - n\bar{x}\bar{y}}{(\sum_{i=1}^{n} x_i^2) - n\bar{x}^2}.$$

Here $\bar{x}$ and $\bar{y}$ are the arithmetic mean of $x_i$ and $y_i$, respectively.
b) Find the Hessian matrix of the objective function and use it to show that the objective function is strictly convex.

**Exercise 6**

a) Consider the following budget set $B = \{(x, y) \in \mathbb{R}^2 : p_x x + p_y y \leq w, \quad x, y \geq 0\}$ where $p_x, p_y, w > 0$. Show that this set is convex and compact. Tip: you may use the result that $\{x \in \mathbb{R}^n : f(x) \leq 0\}$ is closed for any continuous function $f$ and that any intersection of closed sets is closed.
b) Assume that $U(x, y) = x^a y^b$, $a, b > 0$. Argue why the maximizers of $U$ over $B$ are the same as maximizers of $\ln(U(x, y))$ over $\hat{B} = \{(x, y) \in \mathbb{R}^2 : p_x x + p_y y \leq w, \quad x, y > 0\}$.
c) What can you say about the existence and uniqueness of the solutions of

$$\max_{(x,y)\in B} U(x, y)$$