Calculus 1
Why, where, who, when, how. . . ?

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Key point

If you want to remember just one thing from this course:

The derivative is the slope of the function.

Theorem

- $f'(x_0) > 0 \quad \Rightarrow \quad$ *f is increasing at* x_0
- $f'(x_0) < 0 \quad \Rightarrow \quad$ *f is decreasing at* x_0
- *the local maxima and minima of f are found by seeing where* $f(x_0)' = 0$ *(and inspecting case by case).*

If you need to maximize/minimize something, you can do it with derivatives.

Maximizing/minimizing stuff with derivatives

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(In the book: Section 4.8)

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(And check that it really is a minimum!)

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Calculus in science and real life

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Famous limit

$$\lim_{n \to +\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad \Rightarrow \qquad \lim_{n \to +\infty} \left(1 + \frac{8/100}{n}\right)^n = e^{8/100}.$$

Derivatives in physics

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Derivatives in physics (Sect. 2.11)

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How does your speed change?

$$\text{acceleration at time } t_0 = \lim_{t \to t_0} \frac{v(t) - v(t_0)}{t - t_0} = v'(t_0) = f''(t_0).$$

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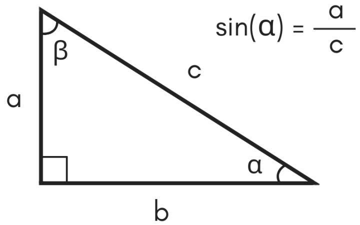
$$\text{total work from } a \text{ to } b = \int_a^b F(x)\,dx.$$

Taylor polynomials and calculators

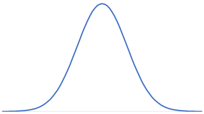


A computer doesn't really know what angles are...

Use Taylor approximations!

Probability – normal distribution

Special function that we saw a lot in the course: $f(x) = e^{-x^2}$.



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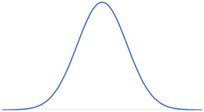

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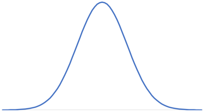

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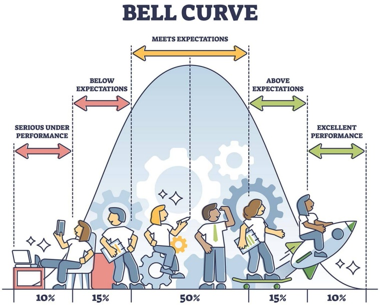

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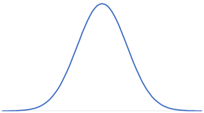

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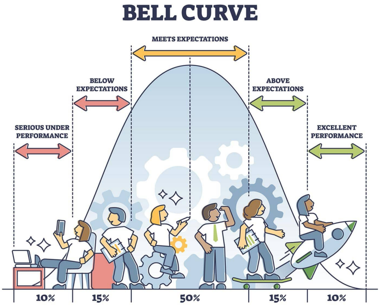

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx = 1.$$

Taylor series to the rescue

$$\int_0^x e^{-t^2}\,dt = \int_0^2 \left(1 - t^2 + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \dots\right) dt$$

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$$= \left(t - \frac{t^3}{3} + \frac{t^5}{5 \times 2!} - \frac{t^7}{7 \times 3!} + \frac{t^9}{9 \times 4!} - \dots\right)\Big|_0^x$$

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$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}$$

Differential equations in science

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The island can grow enoufh food to supply a population of L rabbits indefinitely.

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This is modeled by

$$\frac{dy}{dt} = ky\Big(1 - \frac{y}{L}\Big).$$

Newton's second law of motion:

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Simple harmonic motion (mass suspended by a spring):

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

where k is the spring constant.

History

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When you study calculus from a more conceptual point of view, it’s called “(mathematical) analysis”.

History of calculus

- Ancient Greeks
- ...
- 17th century (Newton, Leibniz,...)

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- Who won? Newton.
- Who was big friends with the Royal Society? Also Newton.

Galileo Galilei

(Before Newton and Leibniz)

General stuff about math

Two kinds of math

- math that is developed because it is needed.
- math that is developed because it is fun.

What is truth?

- (In science) 99/100 experiments work $\rightarrow$ True.
- (In math) It has to be always true. And you need to prove it.

If you just *believe* that something is correct and it's not, you end up making mistakes.

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2. Any line segment may be extended indefinitely.
3. A circle may be drawn with any given point as center and any given radius.
4. All right angles are equal.
5. If two straight lines in a plane are met by another line, and if the sum of the internal angles on one side is less than two right angles, then the straight lines will meet if extended sufficiently on the side on which the sum of the angles is less than two right angles.
   (Translated: Two non-parallel lines will meet at some point.)

   Bla bla bla