Aalto University
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MS-E135
Decision Analysis
Lecture 6a

- *Elicitation of attribute weights*
- *Trade-off methods*
- *SWING, SMART(S)*

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Elicitation of attribute weights

- Attribute weights are derived from the DM's preference statements

- Approaches to eliciting attribute weights:
  - Trade-off weighting
  - More straightforward techniques: SWING, SMART(S), and ordinal methods

Trade-off weighting

- The DM is asked to
  1. Set the performance levels of two imaginary alternatives x and y such that they are equally preferred ($x \sim y$):

$$w_1 v_1^N(x_1) + \cdots + w_n v_n^N(x_n) = w_1 v_1^N(y_1) + \cdots + w_n v_n^N(y_n),\ \textbf{or}$$

  2. Set the performance levels of four imaginary alternatives x, x', y, and y' such that changes $x \leftarrow x'$ and $y \leftarrow y'$ are equally preferred ($x \leftarrow x' \sim_d y \leftarrow y'$):

$$w_1(v_1^N(x_1) - v_1^N(x_1')) + \cdots + w_n(v_n^N(x_n) - v_n^N(x_n')) = w_1(v_1^N(y_1) - v_1^N(y_1')) + \cdots + w_n(v_n^N(y_n) - v_n^N(y_n'))$$

Trade-off weighting

- n-1 pairs of equally preferred alternatives/changes → n-1 linear constraints + 1 normalization constraint (0weights add up to one)

- If the pairs are suitably selected (no linear dependencies), the system of n linear constraints has a unique solution
  - E.g., select a reference attribute and compare the other attributes against it
  - E.g., compare the "most important" attribute to the second most important, the second most important to the third most important etc

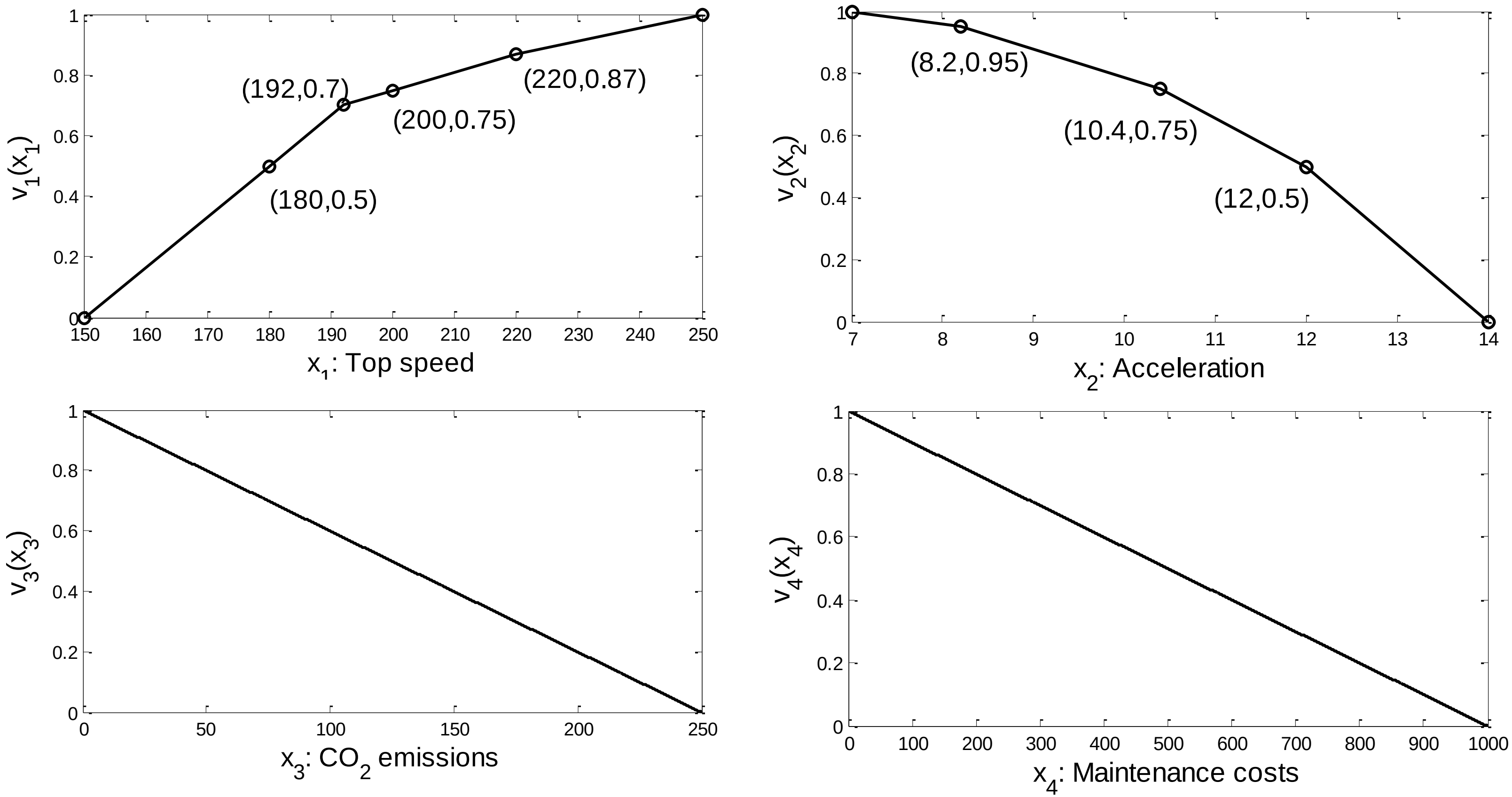
Trade-off weighting: example (1/7)

- Consider two situation where there are two magazines A and B reporting a comparison of cars x^1 , x^2 , and x^3 , **based on the same expert appraisal**, using the **same attributes**:

	a_1: Top speed km/h	a_2: Acceleration 0-100 km/h	a_3: CO_2 emissions g/km	a_4: Maintenance costs €/year
x^1	192 km/h	12.0 s	120 g/km	400 €/year
x^2	200 km/h	10.4 s	140 g/km	500 €/year
x^3	220 km/h	8.2 s	150 g/km	600 €/year

Trade-off weighting: example (2/7)

- Attribute-specific value functions elicited from the expert:



Trade-off weighting: example (3/7)

- Consider changing top speed (reference attribute) **from 150 to 250 km/h**. All other things being equal, what would be an equally preferred change in:
  - Acceleration time? Expert's answer: **from 14 to 7 s** ⇒

$$w_1\left(v_1^N(250)-v_1^N(150)\right)=w_2\left(v_2^N(7)-v_2^N(14)\right)\Rightarrow w_1(1-0)=w_2(1-0)\Leftrightarrow w_2=w_1$$

  - CO_2 emissions? Expert's answer: **from 100 to 0 g/km** ⇒

$$w_1\left(v_1^N(250)-v_1^N(150)\right)=w_3\left(v_3^N(0)-v_3^N(100)\right)\Rightarrow w_3=\frac{1}{v_3^N(0)-v_3^N(100)}w_1=\frac{1}{1-0.6}w_1=2.5\,w_1$$

  - Maintenance costs? Expert's answer: **from 800 to 0 €/year** ⇒

$$w_1\left(v_1^N(250)-v_1^N(150)\right)=w_4\left(v_4^N(0)-v_4^N(800)\right)\Rightarrow w_4=\frac{1}{v_4^N(0)-v_4^N(800)}w_1=\frac{1}{1-0.2}w_1=1.25\,w_1$$

After normalization, we get $w_1=w_2=0.17, w_3=0.44, w_4=0.22$

Trade-off weighting: example (4/7)

- **Magazine A** uses the following measurement scales:

Attribute	Measurement scale	v_i^N
a_1: Top speed (km/h)	[150, 250]	$v_1^N(180) = 0.5,\ v_1^N(192) = 0.7,\ v_1^N(200) = 0.75,\ v_1^N(220) = 0.87$
a_2: Acceleration time (s)	[7, 14]	$v_2^N(12) = 0.5,\ v_2^N(10.4) = 0.75,\ v_2^N(8.2) = 0.95$
a_3: CO_2 emissions (g/km)	[120, 150]	$5 - x_3/30$
a_4: Maintenance costs (€/year)	[400,600]	$3 - x_4/200$

- $\frac{w_1}{w_2} = \frac{v_2^N(7) - v_2^N(14)}{v_1^N(250) - v_1^N(150)} = \quad 1$
- $\frac{w_1}{w_3} = \frac{v_3^N(0) - v_3^N(100)}{v_1^N(250) - v_1^N(150)} = \frac{\frac{100}{30}(v_3^N(120) - v_3^N(150))}{1} = \frac{10}{3}$
- $\frac{w_1}{w_4} = \frac{v_4^N(0) - v_4^N(800)}{v_1^N(250) - v_1^N(150)} = \frac{\frac{800}{200}(v_3^N(400) - v_3^N(600))}{1} = 4$

- These equalities and $\sum_{i=1}^{4} w_i = 1$
  give $w_1 = w_2 = 0.39,\ w_3 = 0.12,\ w_4 = 0.10.$

Trade-off weighting: example (5/7)

- **Magazine A** reports the alternatives' attribute-specific values multiplied by 10 (i.e., scaled to interval [0,10]) and the attribute

	v_1 : Top speed	v_2 : Acceleration	v_3 : CO_2	v_4 : Maintenance	Overall value
x^1	7	5	10	10	6.86
x^2	7.5	7.5	3.3	5	6.76
$\boldsymbol{x^3}$	**8.7**	**9.5**	**0**	**0**	**7.14**
Weights w_i	39%	39%	12%	10%	

- Possible (mis)interpretations / "headlines":
  - "Only power matters – minor emphasis on costs and environment"
  - "Car x^3 terrible w.r.t. CO_2 emissions and maintenance costs – yet, it's the expert's choice!"
  - "No significant differences in top speed – differences are in CO_2 emissions and maintenance costs"

Trade-off weighting: example (6/7)

- **Magazine B** uses the following measurement scales:

Attribute	M. scale	v_i^N
a_1: Top speed	[192, 220]	$v_1^N(150) = -4.12,\ v_1^N(180) = -1.18,\ v_1^N(192) = 0,\ v_1^N(200) = 0.29,\ v_1^N(220) = 1,\ v_1^N(250) = 1.76$
a_2: Acceleration	[8.2, 12]	$v_2^N(14) = -1.11, v_2^N(12) = 0,\ v_2^N(10.4) = 0.56,\ v_2^N(8.2) = 1,\ v_2^N(7) = 1.11$
a_3: CO_2 emissions	[0, 250]	$1 - x_3/250$
a_4: Maintenance	[0,1000]	$1 - x_4/1000$

- $w_1\left(v_1^N(250) - v_1^N(150)\right) = w_2\left(v_2^N(7) - v_2^N(14)\right) \Rightarrow \frac{w_1}{w_2} = \frac{v_2^N(7)-v_2^N(14)}{v_1^N(250)-v_1^N(150)} = \frac{1.11+1.11}{1.76+4.12} = 0.378$
- $\frac{w_1}{w_3} = \frac{v_3^N(0)-v_3^N(100)}{v_1^N(250)-v_1^N(150)} = \frac{1-\frac{150}{250}}{1.76+4.12} = 0.068$
- $\frac{w_1}{w_4} = \frac{v_4^N(0)-v_4^N(800)}{v_1^N(250)-v_1^N(150)} = \frac{1-\frac{200}{1000}}{1.76+4.12} = 0.136$

- The three equalities and $\sum_{i=1}^4 w_i = 1$ give $w_1 = 0.039, w_2 = 0.103,\ w_3 = 0.572,\ w_4 = 0.286.$

Trade-off weighting: example (7/7)

- **Magazine B** reports the alternatives' attribute-specific values multiplied by 10 (i.e., scaled to interval [0,10]) and the attribute

	v_1 : Top speed	v_2 : Acceleration	v_3 : CO_2	v_4 : Maintenance	Overall value
x^1	0	0	5.2	6	4.7
x^2	2.9	5.6	4.4	5	4.6
$\boldsymbol{x^3}$	**10**	**10**	**4**	**4**	**4.9**
Weights w_i	3.9%	10.3%	57.2%	28.6%	

- Possible (mis)interpretations:
  - "Emphasis on costs and environmental issues"
  - "x^3 wins only on the least important attributes – yet, it's the expert's choice!"
  - "Car x^1 terrible w.r.t. top speed and acceleration time"

Trade-off weighting

- ❑ Weights reflect value differences over the measurement scales → changing the measurement scales changes the weights
- ❑ The attribute-specific values used in trade-off weighting account for the measurement scales explicitly → weights represent the DM’s preferences regardless of the measurement scales
- ❑ Trade-off weighting has a solid theoretical foundation and requires thinking; use whenever possible

SWING

- **Swing-weighting process**
  1. Consider alternative $x^0 = (x_1^0, \ldots, x_n^0)$ (each attribute on the worst level).
  2. Choose the attribute a_j that you would first like to change to its most preferred level x_j^* (i.e., the attribute for which such a change is the most valuable). Give that attribute a (non-normalized) weight $W_j = 100$.
  3. Consider x^0 again. Choose the next attribute $a_k$that you would like to change to its most preferred level. Give it weight $W_j \in (0,100]$ that reflects this improvement relative to the first one.
  4. Repeat step 3 until all attributes have been weighted.
  5. Obtain weights w_j by normalizing W_j.

SWING: example

- **Magazine A**'s measurement scales
  - Alternative $x^0 = \left(150\frac{km}{h}, 14s, 150\frac{g}{km}, 600\frac{€}{year}\right)$
  - The first attribute to be changed from the worst to the best level: $a_1 \rightarrow W_1 = 100$
  - The second attribute: $a_2 \rightarrow W_2 = 100$
  - The third attribute: $a_3 \rightarrow W_3 = 30$
  - The fourth attribute: $a_4 \rightarrow W_4 = 20$
  - Normalized weights: $w_1 = w_2 = 40\%$
    $w_3 = 12\%, w_4 = 8\%$

Attribute	Measurement scale
a_1: Top speed	[150, 250]
a_2: Acceleration	[7, 14]
a_3: CO_2 emissions	[120, 150]
a_4: Maintenance	[400,600]

About SWING weighting

- **The mode of questioning explicitly (but only) considers the least and most preferred levels of the attributes**
  - Assumes that the DM can directly numerically assess the strength of preference of changes between these levels

- **NOTE that we only have two preference relations: $\succcurlyeq$ and $\succcurlyeq_d$**
  - For example preference statement $W_1 = 100, W_4 = 20$ is equal to $v_1(x_1^*) - v_1(x_1^0) = 5[v_4(x_4^*) - v_4(x_4^0)]$, which assumes that there exist levels $x_1^{0.2}$, $x_1^{0.4}$, $x_1^{0.6}$, $x_1^{0.8}$ so that $(x_1^{0.2} \leftarrow x_1^0) \sim_d (x_1^{0.4} \leftarrow x_1^{0.2}) \sim_d \ldots \sim_d (x_1^* \leftarrow x_1^{0.8})$
    - Then $v_1(x_1^*) - v_1(x_1^0) = 5[v_1(x_1^{0.2}) - v_1(x_1^0)] = 5[v_4(x_4^*) - v_4(x_4^0)]$ if $(x_1^{0.2}, x_2, x_3, x_4\ ) \leftarrow (x_1^0, x_2, x_3, x_4\ ) \sim_d (x_1\ , x_2, x_3, x_4^*) \leftarrow (x_1\ , x_2, x_3, x_4^0)$

SMART

- Simple Multi-Attribute Rating Technique process:
  1. Select the least important attribute and give it a weight of 10 points.
  2. Select the second least important attribute and give it a weight (≥10 points) that reflects its importance compared to the least important attribute.
  3. Go through the remaining attributes in ascending order of importance and give them weights that reflect their importance compared to the less important attributes.
  4. Normalize the weights.

- This process does not consider the measurement scales at all
  → interpretation of weights is questionable

SMARTS

SMARTS = SMART using <u>S</u>wings

1. Select the attribute corresponding to the least preferred change from worst to best level and give it a weight of 10 points.
2. Go through the remaining attributes in ascending order of preference over changing the attribute from the worst to the best level, and give them weights that reflect their importance compared to the less preferred changes.
3. Normalize the weights.

SMARTS: example

- **Magazine A**'s measurement scales
  - Alternative $x^0 = \left(150\,\frac{km}{h}, 14s, 150\,\frac{g}{km}, 600\,\frac{€}{year}\right)$
  - Least preferred change from the worst to the best level: $a_4 \rightarrow W_4 = 10$
  - The second least preferred change: $a_3 \rightarrow W_3 = 20$
  - The third least preferred change : $a_2 \rightarrow W_2 = 40$
  - The fourth least preferred change: $a_1 \rightarrow W_1 = 40$
  - Normalized weights: $w_1 = w_2 = 36\%,\ w_3 = 18\%,\ w_4 = 9\%$.

Attribute	Measurement scale
a_1: Top speed	[150, 250]
a_2: Acceleration	[7, 14]
a_3: CO_2 emissions	[120, 150]
a_4: Maintenance	[400,600]

Empirical problems related to SWING & SMARTS

- People tend to use only multiples of 10 when assessing the weights, e.g.,
  - SWING: $W_1 = W_2 = 100, W_3 = 30, W_4 = 20 \rightarrow w_1 = w_2 = 0.40, w_3 = 0.12, w_4 = 0.08$
  - SMARTS: $W_1 = W_2 = 40, W_3 = 20, W_4 = 10 \rightarrow w_1 = w_2 = 0.36, w_3 = 0.18, w_4 = 0.09$
  - SWING and SMARTS typically produce different weights

- Assessments may reflect only ordinal, not cardinal information about the weights
  - E.g., SMARTS weights $W_4 = 10$ and $W_3 = 20$ only imply that $W_4 < W_3$, not that $W_3/W_4 = 2$

Summary

- Elicitation of the attribute-specific value functions
  - *Use indifference methods if possible*
- The <u>only</u> meaningful interpretation for attribute weight w_i:

The improvement in overall value when attribute a_i is changed from its worst level to its best **relative to** similar changes in other attributes

- Additive value function describes the DM's preferences if and only if the attributes are mutually preferentially independent and each attribute is difference independent of the others
- Prefer weighting methods that apply indifference relations between alternatives / changes in alternatives
  - *Trade-off methods*
  - *SWING and SMARTS – but be aware of associated problems*