Aalto University
School of Science

MS-E2135
Decision Analysis
Lecture 6b

- *Ordinal weighting methods*
- *Incomplete preference statements*
- *Modeling incomplete information*
- *Dominance and non-dominated alternatives*
- *Computing dominance relations*
- *Decision rules*

Salo, Liesiö, Punkka, Vilkkumaa

A reminder

- The <u>only</u> meaningful interpretation for the attribute weight w_i:

The improvement in overall value when attribute a_i is changed from its worst level to its best **<u>relative to</u>** similar changes in other attributes

- Attribute weights cannot be interpreted without this interpretation
  - *Changing the measurement scale changes the weights*

- In trade-off weighting, specify equally preferred alternatives (or changes in alternatives) which differ with regard to two or more attributes
  - *Use trade-off weighting whenever possible*

$$x \sim y \Leftrightarrow \sum_{i=1}^{n} w_i v_i^N(x_i) = \sum_{i=1}^{n} w_i v_i^N(y_i)$$

Can we simplify weight elicitation?

- Specifying equally preferred alternatives requires quite an effort. *Do we need such an exhaustive representation of preferences to produce defensible decision recommendations*?
  - **Answer: Typically not**, we can for example derive decision recommendations based only on ordinal information – like SWING without giving the points to the attributes
    - *But… many such methods have **severe methodological problems***
  - **Answer2: Typically not**, we learn how to
    - Accommodate <u>incomplete preference statements</u> in the decision model
    - Generate <u>robust decision recommendations</u> that are compatible with such statements

Ordinal weighting methods

- The DM is only asked to rank the attributes in terms of their importance (i.e., preferences over changing the attributes from the worst to the best level, cf. SWING)
  - $R_j = 1$ for the most important attribute
  - $R_j = n$ for the least important attribute

- This ranking is then converted into numerical weights such that these weights are compatible with the ranking
  - $w_i > w_j \Leftrightarrow R_i < R_j$

Ordinal weighting methods

- **Rank sum** weights are proportional to the opposite number of the ranks

$$w_i \propto (n - R_i + 1)$$

- **Rank exponent** weights are relative to some power of $(n - R_i + 1)$

$$w_i \propto (n - R_i + 1)^z$$

  – If z > 1 (z < 1), the power increases (decreases) the weights of the most important attributes compared to rank sum weights.

e.g. attribute 1 more important

$$W_1 = 2 - 1 + 1 = 2$$
$$W_2 = 2 - 2 + 1 = 1$$

Normalize to get

$$w_1 = \frac{2}{3}, w_2 = \frac{1}{3}$$

Ordinal weighting methods

- **Rank reciprocal** weights are proportional to the inverse of the ranks

$$w_i \propto \frac{1}{R_i}$$

- **Centroid** weights are in the center of the set of weights that are compatible with the rank ordering
  - Order the attributes such that $w_1 \geq w_2 \geq \cdots \geq w_n$.
  - Then, the extreme points of the compatible weight set are (1,0,0,0…), (½, ½,0,0,…), (1/3, 1/3, 1/3,0,…),… ($1/n$,…,$1/n$).
  - The average of these extreme points is

$$w_i = \frac{1}{n}\sum_{j=i}^{n}\frac{1}{R_i}$$

Example: centroid weights

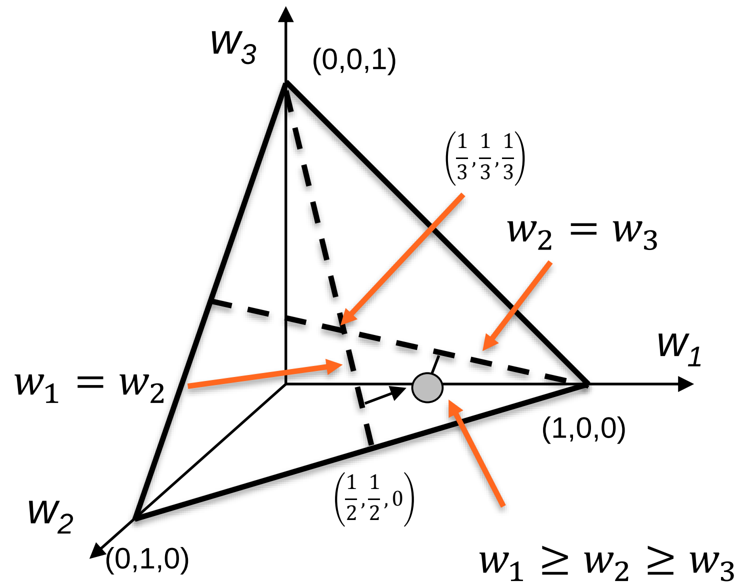
$$w_i = \frac{1}{n}\sum_{j=i}^{n}\frac{1}{R_i}$$

- Rank ordering $w_1 \geq w_2 \geq w_3$:

$$w_1 = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18} \approx 0.61$$

$$w_2 = \frac{1}{3}\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{5}{18} \approx 0.28$$

$$w_3 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \approx 0.11$$



Ordinal weighting methods: example

- Four attributes $\{a_1, a_2, a_3, a_4\}$ in descending order of importance ➔ $R_1 = 1, R_2 = 2, R_3 = 3, R_4 = 4$.

	a_1	a_2	a_3	a_4	$\sum$
Rank sum	4	3	2	1	10
weights	**0.4**	**0.3**	**0.2**	**0.1**	1
Rank exp(z=2)	16	9	4	1	30
weights	**0.53**	**0.30**	**0.13**	**0.03**	1
Rank reciprocal	1	1/2	1/3	1/4	25/12
weights	**0.48**	**0.24**	**0.16**	**0.12**	1
Centroid	25/48	13/48	7/48	3/48	1
weights	**0.52**	**0.27**	**0.15**	**0.06**	1

- Different methods produce different weights!

Ordinal weighting methods: example (cont'd)

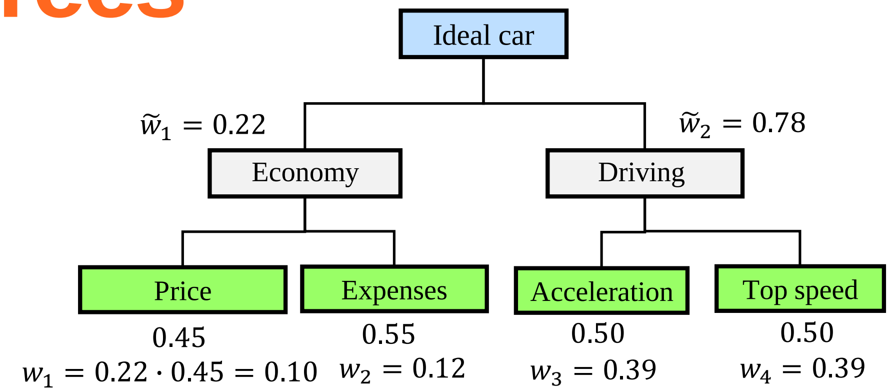
- Assume that the measurement scale of the most important attribute a_1 is changed from [0€,1000€] to [0€,2000€].
- Because $w_1 \propto v_1(x_1^*) - v_1(x_1^0)$, the weight of attribute a_1 should become much larger.
- Still,
  - Ranking among the attributes remains the same → rank-based weights remain the same
  - The alternatives' normalized scores on attribute a_1 become smaller → attribute a_1 has a smaller impact on the decision recommendation
- **Avoid using ordinal methods which produce a "point estimate" weight**

Weighting in value trees

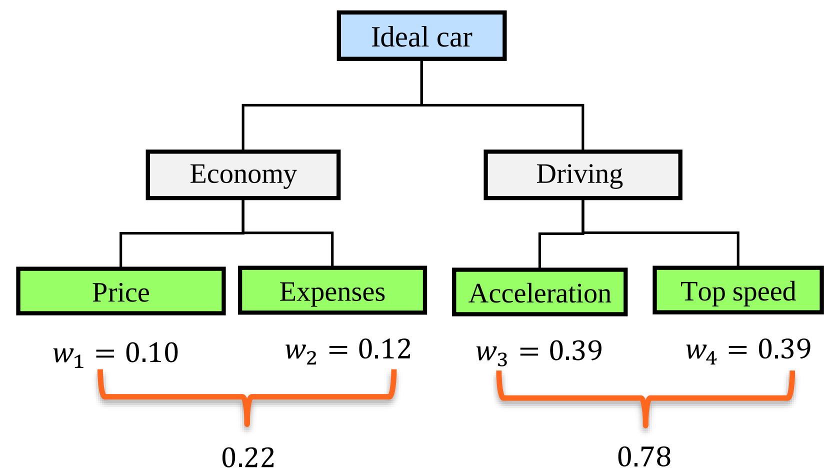
- Two modes of weighting
  - Hierarchical: all weights are elicited and then multiplied vertically
    - Problem: elicitation questions for the higher-level attributes are difficult to interpret:

      $\tilde{w}_1 = w_1 + w_2 \propto (v_1(x_1^*) - v_1(x_1^0)) + (v_2(x_2^*) - v_2(x_2^0))$

    $\rightarrow$ Avoid!



  - Non-hierarchical: weights are only elicited for the twig-level attributes at the lowest level of the hierarchy



Recap: elements of MAVT

- Elements of MAVT:
  - Alternatives $X = \{x^1, \dots, x^m\}$
  - Attributes $A = \{a_1, \dots, a_n\}$
  - Attribute weights $w = [w_1, \dots, w_n] \in \mathbb{R}^n$
  - Attribute-specific (normalized) values $v \in \mathbb{R}^{m \times n}$, $v_{ji} = v_i^N\left(x_i^j\right) \in [0,1]$
  - Overall values of alternatives $V\left(x^j, w, v\right) = \sum_{i=1}^n w_i v_{ji}$, $j = 1, \dots, m$

Recap: Elicitation of attribute weights

- Defining **equally preferred** alternatives / changes between alternatives leads on a **linear equation** on the weights
  - E.g., "All else being equal, a change 150 → 250 km/h in top speed is *equally preferred* to a change 14 → 7 s in acceleration time" ⇒

$$w_1 v_1^N(250) + w_2 v_2^N(14) + w_3 v_3^N(x_3) + w_4 v_4^N(x_4) - V(150{,}14, x_{3,} x_4) =$$
$$w_1 v_1^N(150) + w_2 v_2^N(7) + w_3 v_3^N(x_3) + w_4 v_4^N(x_4) - V(150{,}14, x_{3,} x_4)$$
$$\Leftrightarrow w_1 v_1^N(250) - w_1 v_1^N(150) = \boldsymbol{w_2 v_2^N}(7) - \boldsymbol{w_2 v_2^N}(14)$$

- **Question:** What if the DM finds it difficult or is unable to define such alternatives / changes?
  - E.g., she can only state that a change 150 → 250 km/h in top speed is *preferred* to a change 14 → 7 s in acceleration time?

Incomplete preference statements

- Set the performance levels of two imaginary alternatives x and y such that $x \succcurlyeq y \Rightarrow$

$$w_1 v_1^N(x_1) + \cdots + w_n v_n^N(x_n) \geq w_1 v_1^N(y_1) + \cdots + w_n v_n^N(y_n).$$

Attribute	Measurement scale
a_1: Top speed (km/h)	[150, 250]
a_2: Acceleration time (s)	[7, 14]
a_3: CO_2 emissions (g/km)	[120, 150]
a_4: Maintenance costs (€/year)	[400,600]

- For instance, a change 150 → 250 km/h in top speed is preferred to a change 14 → 7 s in acceleration time:

$$w_1 v_1^N(250) + w_2 v_2^N(14) + w_3 v_3^N(x_3) + w_4 v_4^N(x_4) - V(150{,}14, x_3, x_4) \geq$$
$$w_1 v_1^N(150) + w_2 v_2^N(7) + w_3 v_3^N(x_3) + w_4 v_4^N(x_4) - V(150{,}14, x_3, x_4)$$
$$\Leftrightarrow w_1 \geq w_2$$

- Incomplete preference statements result in **linear inequalities** between the weights

Incomplete preference statements: example

- Consider attributes
  - CO_2 emissions $a_3 \in [120g, 150g]$
  - Maintenance costs $a_4 \in [400€, 600€]$

- Preferences are elicited with SMARTS:
  - Q: "If the reduction 600€ → 400€ in maintenance costs is worth 10 points, how valuable is the lowering of 150g → 120g in CO_2 emissions?"
  - A: "Between 15 and 20 points"

$$1.5w_4[v_4^N(400) - v_4^N(600)] \leq w_3[v_3^N(120) - v_3^N(150)] \leq 2w_4[v_4^N(400) - v_4^N(600)]$$

$$\Rightarrow 1.5w_4 \leq w_3 \leq 2w_4$$

Incomplete preference statements: example

- Preferences are elicited with trade-off methods:
  - Q: "Define an interval for x such that the reduction 600€ → 400€ in maintenance costs is as valuable as 150 g → x g in CO_2 emissions."
  - A: "x is between 130 and 140 g"

Attribute	Measurement scale
a_1: Top speed (km/h)	[150, 250]
a_2: Acceleration time (s)	[7, 14]
a_3: CO_2 emissions (g/km)	[120, 150]
a_4: Maintenance costs (€/year)	[400,600]

For x>140, the reduction in maintenance costs is more valuable

For x<130, the lowering of CO2 emissions is more valuable

$$w_3[v_3^N(\boxed{140}) - v_3^N(150)] \leq w_4[v_4^N(400) - v_4^N(600)] \leq w_3[v_3^N(\boxed{130}) - v_3^N(150)]$$

$$\Rightarrow v_3^N(140)w_3 \leq w_4 \leq v_3^N(130)w_3$$

$$\Rightarrow \frac{1}{3}w_3 \leq w_4 \leq \frac{2}{3}w_3, \text{ if } v_3^N \text{ is linear and decreasing.}$$

Highest and lowest x for which equality possible

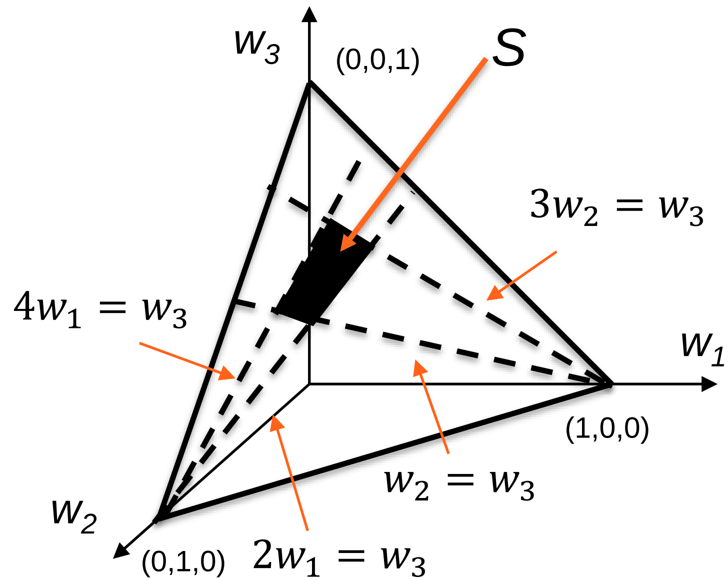
Modeling incomplete information

- Incomplete information about attribute weights can be modeled as a set S of feasible weights that are consistent with the DM's preference statements:

$$S \subseteq S^0 = \left\{ w \in \mathbb{R}^n \middle| \sum_{i=1}^{n} w_i = 1, w_i \geq 0 \;\; \forall i \right\}$$

Modeling incomplete information

- Linear inequalities on weights can correspond to
  1. Weak ranking $w_i \geq w_j$
  2. Strict ranking $w_i - w_j \geq \alpha$ where $\alpha > 0$
  3. Ranking with multiples $w_i \geq \alpha w_j$ (equivalent to incompletely defined weight ratios $w_i / w_j \geq \alpha$)
  4. Interval form $\alpha \leq w_i \leq \alpha + \varepsilon$
  5. Ranking of differences $w_i - w_j \geq w_k - w_l$



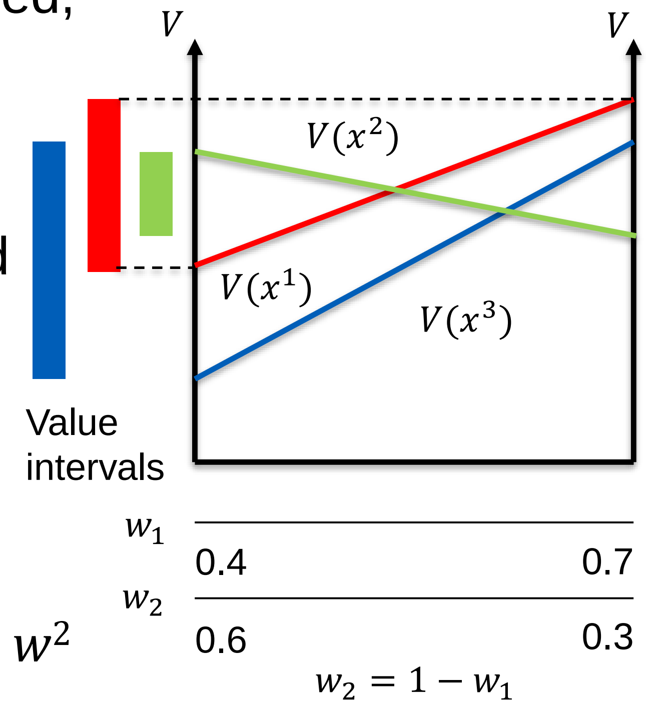
$w_2 \leq w_3 \leq 3w_2$,
$2w_1 \leq w_3 \leq 4w_1$

Overall value intervals

- Because the weights are incompletely specified, the alternatives' overall values are <u>intervals:</u>

$$V(x,w,v) \in \left[\min_{w\in S} V(x,w,v), \max_{w\in S} V(x,w,v)\right]$$

- Note: linear functions obtain their minima and maxima at an extreme point of S
  – E.g., $S = \{w \in S^0 \subseteq \mathbb{R}^2 | 0.4 \leq w_1 \leq 0.7\} \Rightarrow ext(S) = \{(0.4, 0.6), (0.7, 0.3)\}$



- Note: $w \in ext(S)$ is an extreme point of $S \Leftrightarrow$ $\nexists\, w^1, w^2 \in S, w^1 \neq w^2$ such that $w = t\, w^1 + (1\text{-}t)\, w^2$ for some $t \in (0,1)$

Dominance

- Preference over interval-valued alternatives can be established through a *dominance relation*

- **Definition:** x^k dominates x^j in S, denoted $x^k \succ_S x^j$, if and only if (=iff)

$$\begin{cases} V(x^k, w, v) \geq V(x^j, w, v) & \text{for all } w \in S \\ V(x^k, w, v) > V(x^j, w, v) & \text{for some } w \in S \end{cases}$$

  i.e., iff the overall value of x^k is greater than or equal to that of x^j for all feasible weights and strictly greater for some.

Non-dominated alternatives

- An alternative is *non-dominated* if no other alternative dominates it

- The set of non-dominated alternatives is

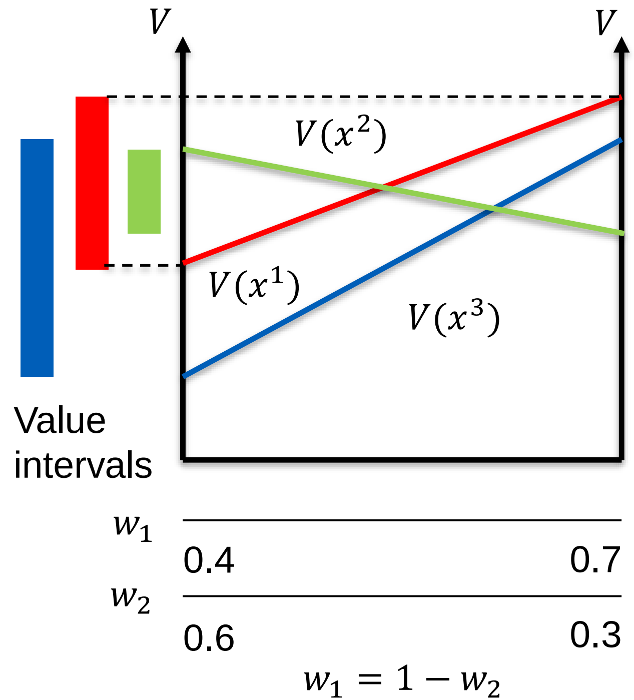
$$X_{ND} = \left\{x^k \in X \mid \nexists j \ \text{ such that } x^j \succ_S x^k\right\}$$

- X_{ND} contains all alternatives that would be meaningful recommendations
  - These are the alternatives for which there is no other alternative that has at least as high value for all feasible weights and strictly higher for some feasible weights

Non-dominated alternatives

x^k is non-dominated if no other alternative has higher value than x^k for all feasible weights

- Alternative x^1 dominates x^3
- Alternatives x^1 and x^2 are non-dominated



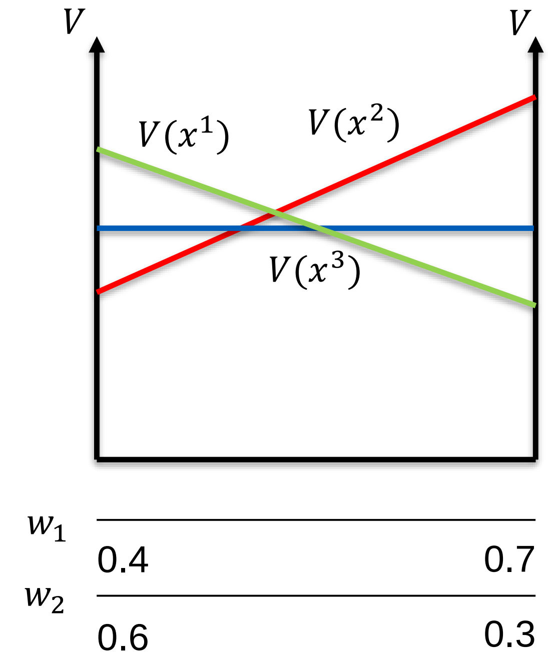
Non-dominated vs. potentially optimal alternatives

- A non-dominated alternative is not necessarily optimal for any $w \in S$

– x^1, x^2 and x^3 are all non-dominated

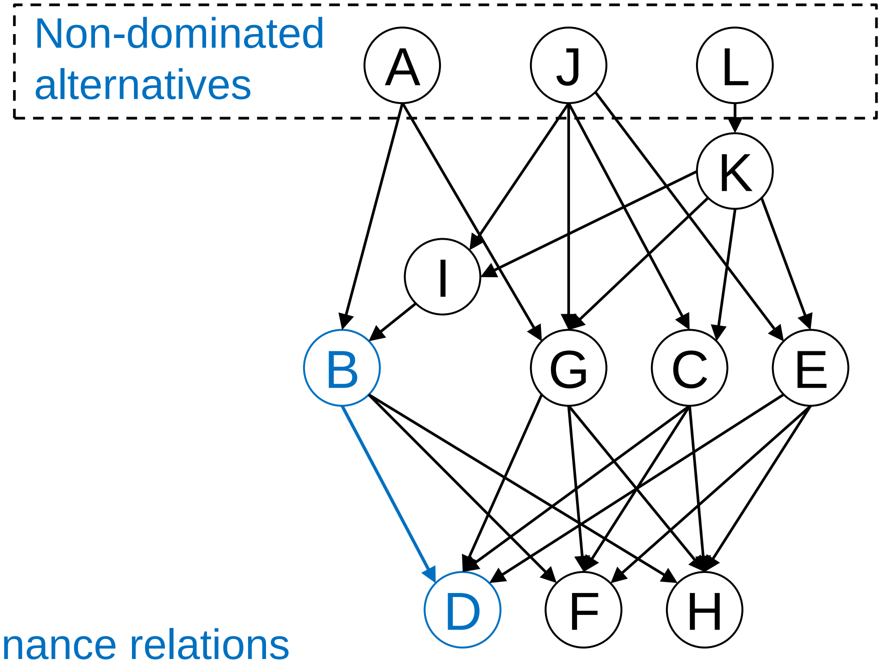
– **Only** x^1 and x^2 are *potentially optimal*: they maximize *V* for some $w \in S$

– Still, neither of them is guaranteed to be better than x^3



Properties of dominance relation

- Transitive
  - If A dominates B and B dominates C, then A dominates C
- Asymmetric
  - If A dominates B, then B does not dominate A
- Irreflexive
  - A does not dominate itself



Dominance relations expressed with a directed arc: B dominates D

Computing dominance relations

- If x^k dominates x^j:
  1. $V(x^k, w, v) \geq V(x^j, w, v)$ for all $w \in S$

     $\Leftrightarrow \min_{w \in S}\left[V(x^k, w, v) - V(x^j, w, v)\right] \geq 0 \Leftrightarrow \min_{w \in S}\left[\sum_{i=1}^{n} w_i (v_{ki} - v_{ji})\right] \geq 0$

  2. $V(x^k, w, v) > V(x^j, w, v)$ for some $w \in S$

     $\Leftrightarrow \max_{w \in S}\left[V(x^k, w, v) - V(x^j, w, v)\right] > 0 \Leftrightarrow \max_{w \in S}\left[\sum_{i=1}^{n} w_i (v_{ki} - v_{ji})\right] > 0$

- Dominance relations between two alternatives can be established by comparing their minimum and maximum value differences

Computing dominance relations: example

- Consider three cars with normalized attribute-specific values:

Car	v_1^N: Top speed	v_2^N: Acceleration	v_3^N: CO_2 emissions	v_4^N: Maintenance
x^1	0.7	0.5	1	1
x^2	0.75	0.75	0.33	0.5
x^3	0.87	0.95	0	0

- Assume that the feasible weights S are characterized by the following inequalities

$$S = \{w \in S^0 \subseteq \mathbb{R}^4 | w_1 = w_2 \geq 3w_3,\ w_3 \geq w_4 \geq 0.1\}$$

Computing dominance relations: example

```
Values=[0.7 0.5 1 1; 0.75 0.75 0.33 0.5; 0.87 0.95 0 0];
A=[0 -1 3 0;0 0 -1 1;0 0 0 -1];
b=[0;0;-0.1];
Aeq=[1 -1 0 0;1 1 1 1];
beq=[0;1];
MinValueDiff=zeros(3,3);
MaxValueDiff=zeros(3,3);

for i=1:3
    for j=i+1:3
        [w,fval]=linprog((Values(i,:)-Values(j,:))',A,b,Aeq,beq);
        MinValueDiff(i,j)=fval;
        [w,fval]=linprog((Values(j,:)-Values(i,:))',A,b,Aeq,beq);
        MaxValueDiff(i,j)=-fval;
        MinValueDiff(j,i)=-MaxValueDiff(i,j);
        MaxValueDiff(j,i)=-MinValueDiff(i,j);
        if MinValueDiff(i,j)>=0 && MaxValueDiff(i,j)>0
           disp(['Alternative ' num2str(i) ' dominates ' num2str(j) '.'])
        elseif MinValueDiff(j,i)>=0 && MaxValueDiff(j,i)>0
           disp(['Alternative ' num2str(j) ' dominates ' num2str(i) '.'])
        end

    end
end
```

Matlab function linprog(f,A,b,Aeq,beq) solves the optimization problem:

$$\min_x f^T x \text{ such that } \begin{cases} A \cdot x \leq b \\ Aeq \cdot x = beq \end{cases}$$

Computing dominance relations: example

- Minimum and maximum value differences

$$\min_{w\in S}[V(x^1,w,v)-V(x^2,w,v)] = -0.003 < 0$$
$$\max_{w\in S}[V(x^1,w,v)-V(x^2,w,v)] = 0.0338 > 0$$

$\rightarrow$ Neither x^1 nor x^2 dominate the other

$$\min_{w\in S}[V(x^2,w,v)-V(x^3,w,v)] = -0.045 < 0$$
$$\max_{w\in S}[V(x^2,w,v)-V(x^3,w,v)] = -0.0163 < 0$$

$\rightarrow$ x^3 dominates x^2

$$\min_{w\in S}[V(x^1,w,v)-V(x^3,w,v)] = -0.048 < 0$$
$$\max_{w\in S}[V(x^1,w,v)-V(x^3,w,v)] = 0.0175 > 0$$

$\rightarrow$ Neither x^1 nor x^3 dominate the other

- $X_{ND} = \{x^1, x^3\}$

Computing dominance relations: example

- Note: because value differences are linear in *w*, minimum and maximum value differences are obtained at the extreme points of the set of feasible weights S

	w^1	w^2	w^3
$V(x^1)$-$V(x^2)$	-0.003	0.0204	0.0338
$V(x^2)-V(x^3)$	-0.045	-0.031	-0.0163
$V(x^1)-V(x^3)$	-0.048	-0.0106	0.0175

$$w^1 = (0.4\ 0.4\ 0.1\ 0.1)$$

$$w^2 = \left(\frac{27}{70}, \frac{27}{70}, \frac{9}{70}, \frac{1}{10}\right) \approx (0.386, 0.386, 0.129, 0.10)$$

$$w^3 = \left(\frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right) \approx (0.375, 0.375, 0.125, 0.125)$$

$$\min_{w \in S}[V(x^1, w, v) - V(x^2, w, v)] = -0.003 < 0$$
$$\max_{w \in S}[V(x^1, w, v) - V(x^2, w, v)] = 0.0338 > 0$$

$$\min_{w \in S}[V(x^2, w, v) - V(x^3, w, v)] = -0.045 < 0$$
$$\max_{w \in S}[V(x^2, w, v) - V(x^3, w, v)] = -0.0163 < 0$$

$$\min_{w \in S}[V(x^1, w, v) - V(x^3, w, v)] = -0.048 < 0$$
$$\max_{w \in S}[V(x^1, w, v) - V(x^3, w, v)] = 0.0175 > 0$$

Additional information

- If the information set S results leads to too many non-dominated alternatives, additional preference statements (i.e., linear constraints) can be elicited
- New information set $S' \subset S$ preserves all dominance relations and usually yields new ones → X_{ND} stays the same or becomes smaller

$$S' \subset S, ri(S) \cap S' \neq \emptyset: \begin{cases} x^k \succ_S x^j \Rightarrow x^k \succ_{S'} x^j \\ X_{ND}(S) \supseteq X_{ND}(S') \end{cases},$$

where $ri(S)$ is the relative interior of S

- $ri(S) \cap S' \neq \emptyset$: S' is not entirely on the "border" of S
- $w \in ri(S) \Leftrightarrow \exists \varepsilon > 0$ such that if $|w' - w| < \varepsilon \Rightarrow w' \in S$

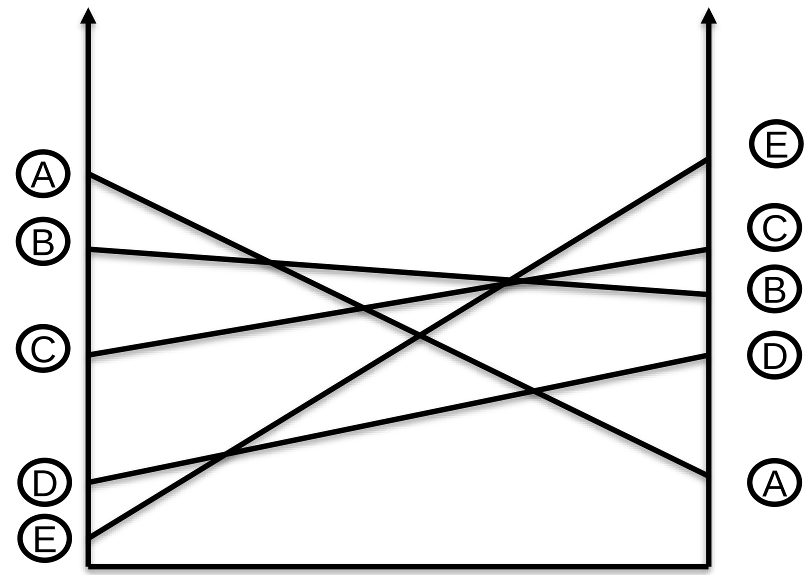
Additional information: example

- No weight information

$$S = S^0 = \left\{ w \in \mathbb{R}^2 \middle| \sum_{i=1}^{2} w_i = 1, w_i \geq 0 \right\}$$

- Dominance relations
  1. B dominates D
  2. C dominates D

- Non-dominated alternatives
  – A,B,C,E



$w_1 = 0$
$w_2 = 1$

$w_1 = 1$
$w_2 = 0$

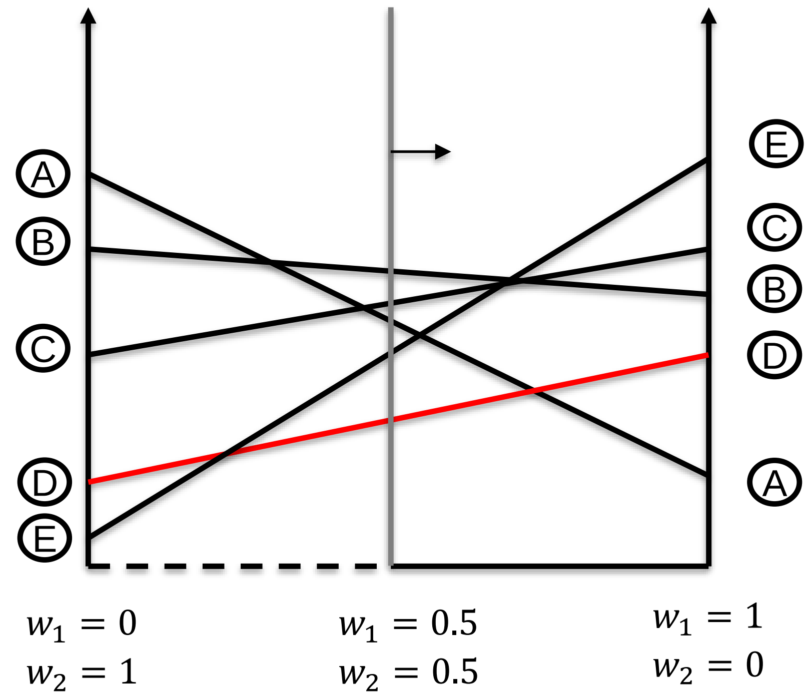
Additional information: example (2/3)

- Ordinal weight information

$$S = \{w \in S^0 | w_1 \geq w_2\}$$

- Dominance relations
  1. B dominates D
  2. C dominates D
  3. E dominates D
  4. B dominates A
  5. C dominates A

- Non-dominated alternatives
  – B,C,E



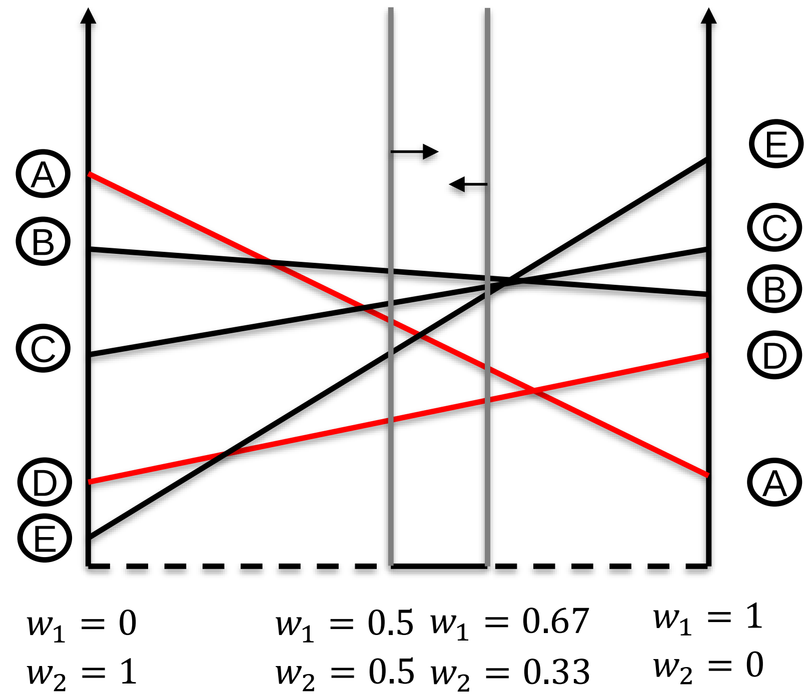
Additional information: example (3/3)

- More information

$$S = \{w \in S^0 | w_2 \leq w_1 \leq 2w_2\}$$

- Dominance relations
  1. B dominates D
  2. C dominates D
  3. E dominates D
  4. B dominates A
  5. C dominates A
  6. B dominates C
  7. B dominates E

- Non-dominated alternatives: B

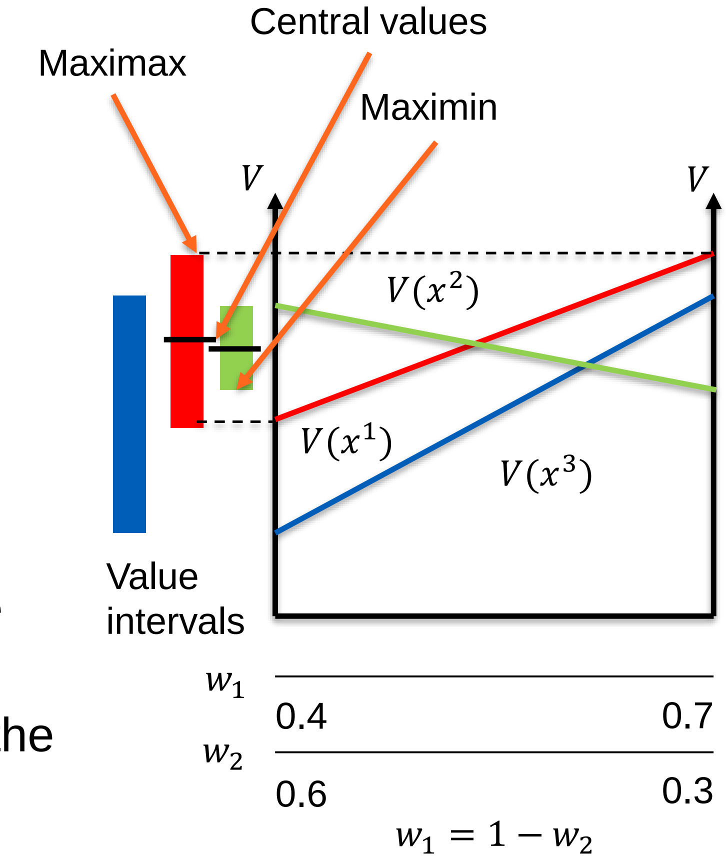


Value intervals

Can value intervals be used in deriving decision recommendations?

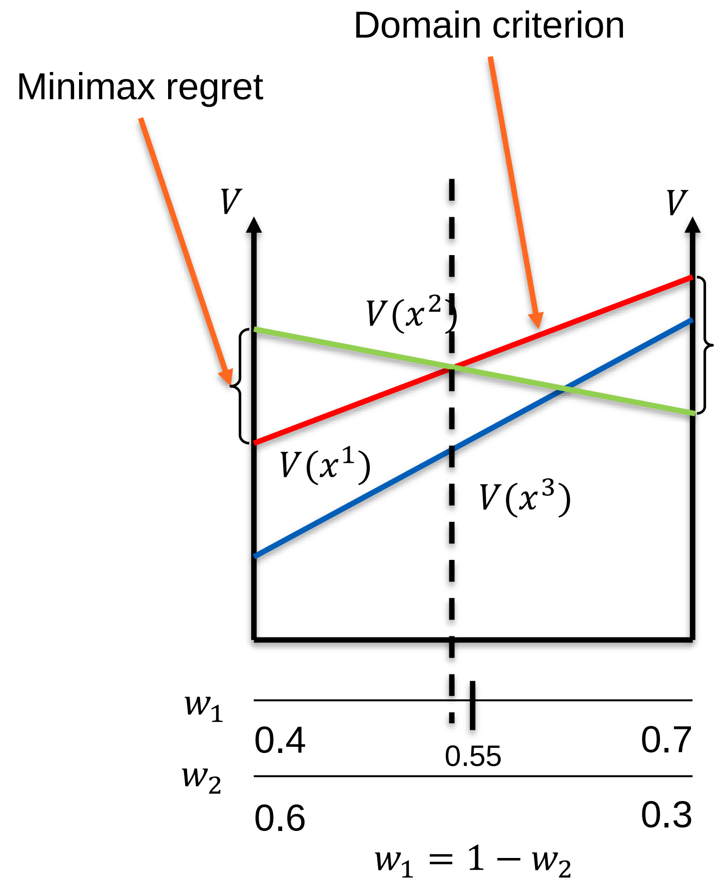
Some suggestions for “decision rules” from the literature:

- **Maximax**: choose the alternative with the highest maximum overall value over the feasible weights
- **Maximin**: choose the alternative with the highest lowest overall value over the feasible weights
- **Central values**: choose the alternative with the highest sum of the maximum and minimum values



…more decision rules

- **Minimax regret**: choose the alternative with the smallest maximum regret (= value difference compared to any other alternative)
- **Domain criterion**: choose the alternative which is favored by the largest set of weights



Example

- DM asks 2 experts to compare fruit baskets (x_1,x_2) containing apples x_1 and oranges x_2

- Linear attribute-specific value functions v_1 and v_2

- DM: (2,0) >~ (0,1) and (0,2)>~(1,0)
  - One orange is not preferred to 2 apples, one apple is not preferred to 2 oranges

- Fruit baskets (1,2) and (2,1) do not dominate each other

- What recommendations are suggested by the decision rules?

Expert 1: $x^0=(0,0)$, $x^*=(2,4)$

$$v_1^N(x_1)=\frac{x_1}{2},\ v_2^N(x_2)=\frac{x_2}{4}$$

$$V(2,0)\ge V(0,1)\Leftrightarrow$$

$$\frac{2}{2}w_1+0w_2\ge 0w_1+\frac{1}{4}w_2=\frac{1}{4}(1-w_1)\Leftrightarrow w_1\ge\frac{1}{5}$$

$$V(0,2)\ge V(1,0)\Leftrightarrow$$

$$\frac{2}{4}w_2=\frac{1}{2}(1-w_1)\ge\frac{1}{2}w_1\Leftrightarrow w_1\le\frac{1}{2}$$

$$V(x)=w_1\frac{x_1}{2}+w_2\frac{x_2}{4}=w_1\left(\frac{x_1}{2}-\frac{x_2}{4}\right)+\frac{x_2}{4}$$

Expert 2: $x^0=(0,0)$, $x^*=(4,2)$

$$v_1^N(x_1)=\frac{x_1}{4},\ v_2^N(x_2)=\frac{x_2}{2}$$

$$V(2,0)\ge V(0,1)\Leftrightarrow$$

$$\frac{2}{4}w_1\ge\frac{1}{2}w_2=\frac{1}{2}(1-w_1)\Leftrightarrow w_1\ge\frac{1}{2}$$

$$V(0,2)\ge V(1,0)\Leftrightarrow$$

$$w_2=1-w_1\ge\frac{1}{4}w_1\Leftrightarrow w_1\le\frac{4}{5}$$

$$V(x)=w_1\left(\frac{x_1}{4}-\frac{x_2}{2}\right)+\frac{x_2}{2}$$

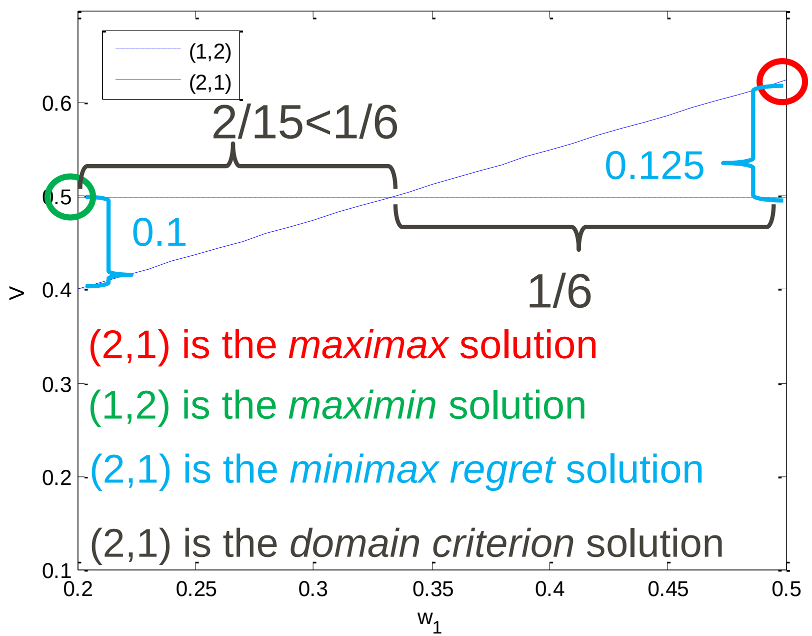
$$V(x) = w_1\left(\frac{x_1}{2} - \frac{x_2}{4}\right) + \frac{x_2}{4}$$

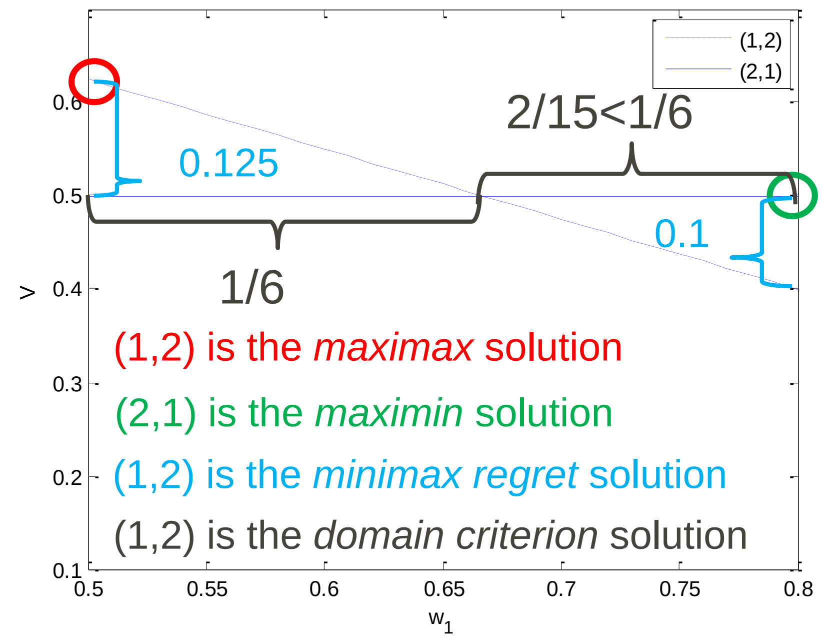
$$V(1,2) = w_1\left(\frac{1}{2} - \frac{2}{4}\right) + \frac{2}{4} \equiv \frac{1}{2}$$

$$V(2,1) = w_1\left(\frac{2}{2} - \frac{1}{4}\right) + \frac{1}{4} = \frac{3}{4}w_1 + \frac{1}{4}$$

$$V(x) = w_1\left(\frac{x_1}{4} - \frac{x_2}{2}\right) + \frac{x_2}{2}$$

$$V(1,2) = -\frac{3}{4}w_1 + 1$$



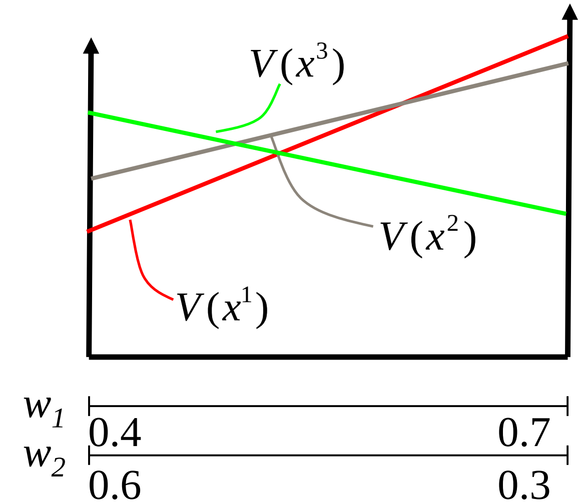


On decision rules

- A common concern in all of the above decision rules: changing the measurement scales $[x_i^0, x_i^*]$ can change the recommendations
- Different attribute weightings w and w* represent value functions V and V* – they cannot be compared
  - If V represents the DM's preferences, so do all its positive affine transformations, too
  - How to choose one of the value functions which all represent the same preferences?
- **Avoid using measures which compare overall values across different value functions (i.e. attribute weightings)**

Rank (sensitivity) analysis

- For any weights, the alternatives can be ranked based on their overall values
  - This ranking is **<u>not</u>** influenced by normalization (i.e., positive affine transformations of V)



- How do the rankings of alternatives change when attribute weights vary?

ranks	x^1	x^2	x^3
minimum	1	1	1
maximum	3	2	3

Computation of rank intervals

The minimum ranking of x^k is

$$r_S^-(x^k) = 1 + \min_{(w,v)\in S} |\{x^j \in X \,|\, V(x^j,w,v) > V(x^k,w,v)\}|$$

which is obtained as a solution to the mixed integer LP

$$\min_{\substack{(w,v)\in S \\ y^j\in\{0,1\}}} \sum_{j=1}^{m} y^j$$

$$V(x^j,w,v) \le V(x^k,w,v) + y^j M \qquad j = 1,\ldots,m$$

$$y^k = 1$$

Maximum rankings with a similar model

Rank analysis – example (1/5)

- **Academic ranking of world universities 2007**
  - 508 universities

- **Additive multi-attribute model**
  - 6 attributes
  - Attribute weights (denoted by w^*) and scores
  - Universities ranked based on overall values

Rank analysis – example (2/5)

Criteria	Indicator	Code	Weight
Quality of Education	Alumni of an institution winning Nobel Prizes and Fields Medals	Alumni	10%
Quality of Faculty	Staff of an institution winning Nobel Prizes and Fields Medals	Award	20%
	Highly cited researchers in 21 broad subject categories	HiCi	20%
Research Output	Articles published in Nature and Science*	N&S	20%
	Articles in Science Citation Index-expanded, Social Science Citation Index	SCI	20%
Size of Institution	Academic performance with respect to the size of an institution	Size	10%
Total			100%

Rank analysis – example (3/5)

Scores (some of them)

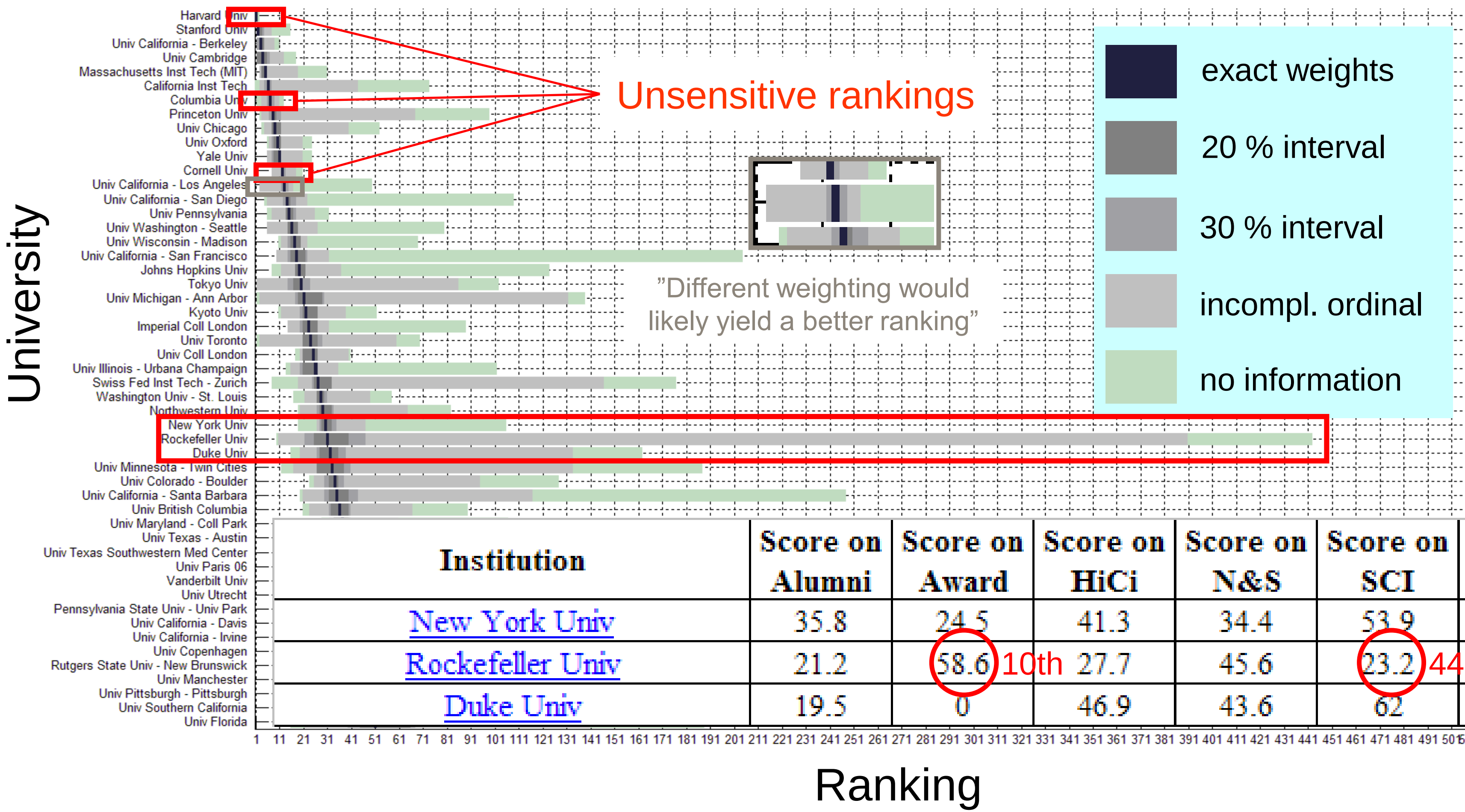
World Rank	Institution	Score on Alumni	Score on Award	Score on HiCi	Score on N&S	Score on SCI	Score on Size	Total Score
1	Harvard Univ	100	100	100	100	100	73	100
2	Stanford Univ	42	78.7	86.1	69.6	70.3	65.7	73.7
3	Univ California - Berkeley	72.5	77.1	67.9	72.9	69.2	52.6	71.9
4	Univ Cambridge	93.6	91.5	54	58.2	65.4	65.1	71.6
5	Massachusetts Inst Tech (MIT)	74.6	80.6	65.9	68.4	61.7	53.4	70.0
6	California Inst Tech	55.5	69.1	58.4	67.6	50.3	100	66.4
7	Columbia Univ	76	65.7	56.5	54.3	69.6	46.4	63.2
8	Princeton Univ	62.3	80.4	59.3	42.9	46.5	58.9	59.5
9	Univ Chicago	70.8	80.2	50.8	42.8	54.1	41.3	58.4
10	Univ Oxford	60.3	57.9	46.3	52.3	65.4	44.7	56.4
11	Yale Univ	50.9	43.6	57.9	57.2	63.2	48.9	55.9
12	Cornell Univ	43.6	51.3	54.5	51.4	65.1	39.9	54.3
13	Univ California - Los Angeles	25.6	42.8	57.4	49.1	75.9	35.5	52.6
14	Univ California - San Diego	16.6	34	59.3	55.5	64.6	46.6	50.4
15	Univ Pennsylvania	33.3	34.4	56.9	40.3	70.8	38.7	49.0
16	Univ Washington - Seattle	27	31.8	52.4	49	74.1	27.4	48.2
17	Univ Wiscon http://www.washingto	40.3	35.5	52.9	43.1	67.2	28.6	48.0
18	Univ California follow. Click and hold t	0	36.8	54	53.7	59.8	46.7	46.8
19	Johns Hopkins Univ	48.1	27.8	41.3	50.9	67.9	24.7	46.1
20	Tokyo Univ	33.8	14.1	41.9	52.7	80.9	34	45.9
21	Univ Michigan - Ann Arbor	40.3	0	60.7	40.8	77.1	30.7	44.0
22	Kyoto Univ	37.2	33.4	38.5	35.1	68.6	30.6	43.1
23	Imperial Coll London	19.5	37.4	40.6	39.7	62.2	39.4	43.0
23	Univ Toronto	26.3	19.3	39.2	37.7	77.6	44.4	43.0
25	Univ Coll London	28.8	32.2	38.5	42.9	63.2	33.8	42.8
26	Univ Illinois - Urbana Champaign	39	36.6	44.5	36.4	57.6	26.2	42.7
27	Swiss Fed Inst Tech - Zurich	37.7	36.3	35.5	39.9	38.4	50.5	39.9
28	Washington Univ - St. Louis	23.5	26	39.2	43.2	53.4	39.3	39.7
29	Northwestern Univ	20.4	18.9	46.9	34.2	57	36.9	38.2
30	New York Univ	35.8	24.5	41.3	34.4	53.9	25.9	38.0
30	Rockefeller Univ	21.2	58.6	27.7	45.6	23.2	37.8	38.0
32	Duke Univ	19.5	0	46.9	43.6	62	39.2	37.4
33	Univ Minnesota - Twin Cities	33.8	0	48.6	35.9	67	23.5	37.0
34	Univ Colorado - Boulder	15.6	30.8	39.9	38.8	45.7	30	36.6
35	Univ California - Santa Barbara	0	35.3	42.6	36.2	42.7	35.1	35.8
36	Univ British Columbia	19.5	18.9	31.4	31	63.1	36.3	35.4
37	Univ Maryland - Coll Park	24.3	20	40.6	31.2	53.3	25.9	35.0
38	Univ Texas - Austin	20.4	16.7	46.9	28	54.8	21.3	34.4
39	Univ Texas Southwestern Med Center	22.8	33.2	30.6	35.5	38	31.9	33.8

Rank analysis – example (4/5)

Incomplete weight information

- Relative intervals: $w \in \{w \in S_w^0 \mid (1-\alpha)w_i^* \leq w_i \leq (1+\alpha)w_i^*\}$
  - *For α=0.1, 0.2, 0.3*
  - *e.g. α=0.2 , w_i*=0.20:* $0.16 \leq w_i \leq 0.24$
- Incomplete ordinal: $w \in \{w \in S_w^0 \mid w_i \geq w_k \geq 0.02 \, \forall \, i \in \{2,3,4,5\}, k \in \{1,6\}\}$
  - *Consistent with initial weights and lower bound b = 0.02*
- Only lower bound: $w \in \{w \in S_w^0 \mid w_i \geq 0.02 \forall i = 1,\ldots,6\}$
- No weight information: $w \in S_w^0$

Rank analysis – example (5/5)



Institution	Score on Alumni	Score on Award	Score on HiCi	Score on N&S	Score on SCI	Score on Size
New York Univ	35.8	24.5	41.3	34.4	53.9	25.9
Rockefeller Univ	21.2	58.6	27.7	45.6	23.2	47.8
Duke Univ	19.5	0	46.9	43.6	62	39.2

Example: Prioritization of innovation ideas*

- 28 “innovation ideas” evaluated by several experts on a scale from 1 – 7 with regard to *novelty*, *feasibility* and *relevance*
  - For each attribute, each idea assessed by the average of reported evaluations
- No preference information about the relative values of the attributes
- “Which 10 innovation ideas should be selected for further development?”
  - Sets of ideas called *portfolios*
- The value of a portfolio is the sum of its constituent projects

* Könnölä et al. (Technological Forecasting & Social Change, 2007)

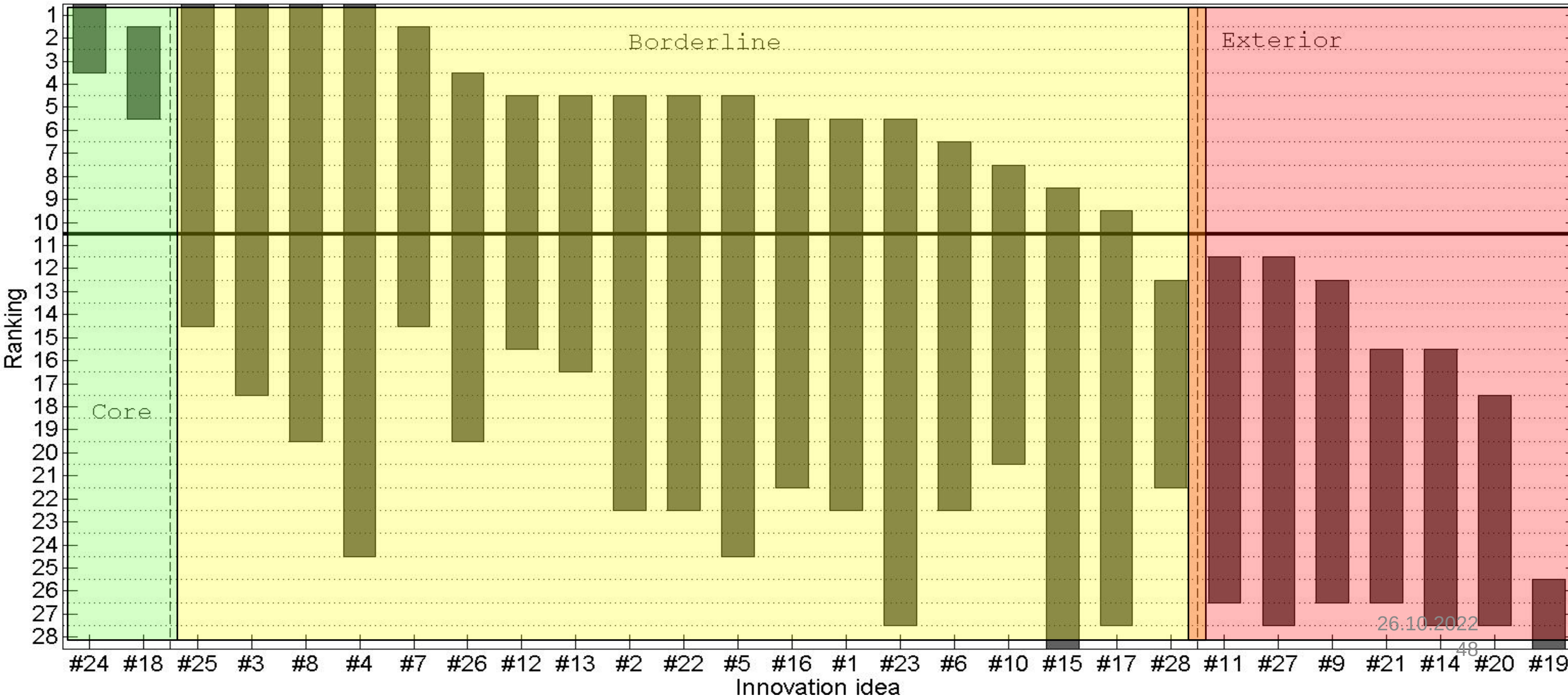
Example: prioritization of innovation ideas

- Robust Portfolio Modeling* was used to compute *non-dominated portfolios* of 10 ideas and discriminate between
  - *Core* ideas that belong to all non-dominated portfolios
  - *Borderline* ideas that belong to some non-dominated portfolios
  - *Exterior* ideas that do not belong to any non-dominated portfolio

- How do ranking intervals relate to this classification?
  - If the ranking of an idea cannot be worse than 10, is it a core project?
  - If the ranking of an idea cannot be better than 11, is it an exterior project?

* Liesiö, Mild, Salo (European Journal of Operational Research, 2007)

Ranking intervals vs. core, borderline and exterior ideas



Ranking intervals divide the innovation ideas into core, borderline and exterior ideas among *potentially optimal* portfolios

Rationales for using incomplete information

- Often only limited time and effort can be devoted to preference elicitation
- Complete preference specification may not be needed to reach a decision
- DM’s preferences may evolve during the analysis $\rightarrow$ iteration supports learning
- Experts / stakeholders may have conflicting preferences
- Take-it-or-leave-it solutions may be resented in group decision settings $\rightarrow$ results based on incomplete information leave room for negotiation

Summary

- Complete specification of attribute weights is often difficult
  - Trade-off methods take time and effort
  - SWING and SMARTS are prone to biases

- Incomplete preference statements can be modeled by linear inequalities on the weights → alternatives' overall values become intervals

- Preference over interval-valued alternatives can be established through dominance relations
  - Non-dominated alternatives are defensible decision recommendations