Mathematics for Economists

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Summary of Part 1

Review - Linear Algebra

- Gauss-Jordan elimination method
- Rank of a matrix
- Matrix algebra
- Inverse matrix
- Determinant of a matrix
- Cramer's rule
- Vectors and linear independence
- Bases of $\mathbb{R}^n$

Review - Calculus of several variables

- Functions
- Sequences and their limits
- Continuous functions
- Open and closed sets
- Differentiable functions and partial derivatives
- Chain rule
- Higher order derivatives and Hessian matrix
- Implicit function theorem
- Homogeneous functions

Review - Unconstrained optimization

- Local and global extrema (minimizers or maximizers)
- Weierstrass's Theorem
- First order conditions for interior extrema
- Second order conditions for interior extrema
- Definite and semi-definite matrices
- Convex sets; concave and convex functions
- Quadratic forms
- Monotone transformations

Review - Some fundamentals

- We have seen several propositions of the form "If P, then Q"
  - In more compact form, this is often indicated as $P \implies Q$, i.e. P implies Q
  - E.g., If the matrix A is positive definite, then its diagonal entries are strictly positive
  - We say that P is a **sufficient condition** for Q; at the same time, Q is a **necessary condition** for P

- We've also seen propositions of the form "P if and only if Q"
  - In more compact form, this is often indicated as $P \iff Q$, i.e. P is equivalent to Q
  - E.g., A function f is concave if and only if its Hessian matrix is negative semi-definite
  - In this case, we say that P is both a **necessary and sufficient condition** for Q, and vice versa

Review - Some fundamentals

- When you're doing calculations and come up with expressions like

$$2 + (4 + 5 \times 2)^2 - 10/2 + 7 \times (4 - 5^2)^2,$$

  recall the order of operations:

  1. **P**arentheses
  2. **E**xponents
  3. **M**ultiplication and **D**ivision (from left to right)
  4. **A**ddition and **S**ubtraction (from left to right)

- The acronym PEMDAS is often used to indicate the above order (along with the mnemonic "Please Excuse My Dear Aunt Sally")

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Constrained Optimization

Introduction to Part II

- Lectures 12-15: Constrained Optimization (Chapters 18-19)
- Lectures 16-17: Difference equations (Chapters 23)
- Lecture 18-21: Ordinary differential equations (Chapters 24-25)
- Lecture 22: Review

Optimization in economics

- Allocation of scarce resources is at the heart of economics
- What is scarcity?

  Decision makers have constraints
- How do rational decision makers allocate resources?

  They have an objective function to maximize/minimiza
  profits, costs, social welfare, utility

Constrained Optimization

- Most optimization problems in Economics are constrained
- Given a function $f : U \to \mathbb{R}$, with $U \subseteq \mathbb{R}^n$, the *constrained maximization* of f takes the form

$$\max_{\mathbf{x}} \quad f(\mathbf{x})$$
$$\text{subject to} \quad \mathbf{x} \in C,$$

  where $C \subseteq U$ is the *constraint set*

- In words, we want to maximize f over a subset of its domain
- *Un*constrained maximization is when we have $C = U$

Classes of Optimization Problems

- The general optimization problem (for $C \subseteq \mathbb{R}^n$) is a *nonlinear programming problem*
- When the problem involves time, we have a dynamic optimization problem
  - when there is no time involved, the problem is a static optimization problem
- Nonlinear programming problems with constraints $C = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \leq 0, \dots, g_k(\mathbf{x}) \leq 0 \text{ and } h_1(\mathbf{x}) = 0, \dots, h_m(\mathbf{x}) = 0\}$, $g_i : \mathbb{R}^n \mapsto \mathbb{R}$, $h_j : \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, \dots, k$, $j = 1, \dots, m$
  - the inequalities $g_1(\mathbf{x}) \leq 0, \dots, g_k(\mathbf{x}) \leq 0$ are inequality constraints (note that $g_i(\mathbf{x}) \geq 0 \Leftrightarrow -g_i(\mathbf{x}) \leq 0$), and the constraints $h_1(\mathbf{x}) = 0, \dots, h_m(\mathbf{x}) = 0$ are equality constraints
  - if f, $g_1, \dots, g_k$, $h_1, \dots, h_m$ are linear (or affine) the problem is a linear programming problem

Constrained Optimization

- **Example.** Consider the following utility maximization problem with two commodities:

$$
\begin{aligned}
\max_{x_1, x_2} \quad & u(x_1, x_2) \\
\text{s. t.} \quad & p_1 x_1 + p_2 x_2 \leq w & (1) \\
& x_1 \geq 0 & (2) \\
& x_2 \geq 0. & (3)
\end{aligned}
$$

- The utility function $u(x_1, x_2)$ is the objective function
- (1) is the budget constraint
- (2) and (3) are the non-negativity constraints on x_1 and x_2, respectively

Constrained Optimization

- **Example (cont'd).** The constraint set C is defined by the three **inequalities** (1), (2) and (3)
- More specifically,

$$
\begin{aligned}
C = &\{(x_1, x_2) \in \mathbb{R}^2 : p_1 x_1 + p_2 x_2 \leq w\} \\
&\cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\} \\
&\cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}
\end{aligned}
$$

- C is known as the consumer's *budget or opportunity set*
- Notice that C is a strict subset of $U = \mathbb{R}^2$

Constrained Optimization

- **Example.** Consider a firm that wants to find the cheapest way to produce q units of output
- The firm's cost minimization problem is

$$\begin{aligned} \min_{x_1,x_2} \quad & w_1x_1 + w_2x_2 \\ \text{s. t.} \quad & f(x_1,x_2) = q && (4) \\ & x_1 \geq 0 && (5) \\ & x_2 \geq 0. && (6) \end{aligned}$$

- The choice variables x_1 and x_2 are production inputs, and $w_1 > 0$ and $w_2 > 0$ are the corresponding unit prices
- $f(x_1, x_2)$ is the firm's production function

Constrained Optimization

- **Example (cont'd).** The constraint set C is now defined by one **equality** and two inequalities

- More specifically,

$$\begin{aligned} C = & \{(x_1, x_2) \in \mathbb{R}^2 : f(x_1, x_2) = q\} \\ & \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\} \\ & \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\} \end{aligned}$$

Constrained Optimization

- Like we did in unconstrained optimization, our goal is to find necessary and sufficient conditions for solutions

- In this lecture, we'll consider two-variable maximization problems with one equality constraint:

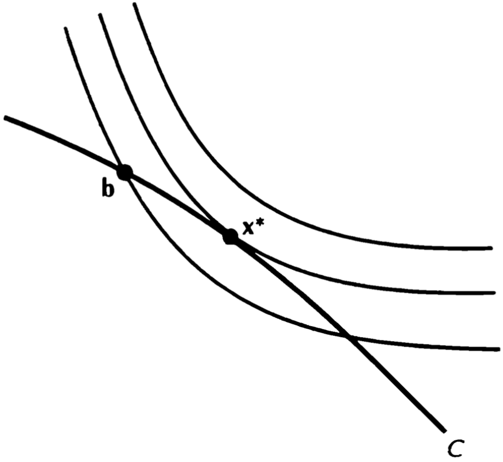
$$\max_{x_1, x_2} \quad f(x_1, x_2)$$
$$\text{s. t.} \quad h(x_1, x_2) = c,$$

  where f and h are C^1 functions on $\mathbb{R}^2$, and $c \in \mathbb{R}$

- The constraint set is $C = \{(x_1, x_2) \in \mathbb{R}^2 : h(x_1, x_2) = c\}$

Constrained Optimization

- Geometrically, the problem is to find the highest-valued level curve of f that intersects the constraint set C



At the constrained max $\mathbf{x}^*$, *the highest level curve of* f *is tangent to the constraint*

Constrained Optimization

- In the previous figure, the highest-valued level curve of f is tangent to the constraint h at the maximizer $\mathbf{x}^* = (x_1^*, x_2^*)$. This means that

$$\frac{\frac{\partial f}{\partial x_1}(\mathbf{x}^*)}{\frac{\partial f}{\partial x_2}(\mathbf{x}^*)} = \frac{\frac{\partial h}{\partial x_1}(\mathbf{x}^*)}{\frac{\partial h}{\partial x_2}(\mathbf{x}^*)}$$

  or, equivalently,

$$\frac{\frac{\partial f}{\partial x_1}(\mathbf{x}^*)}{\frac{\partial h}{\partial x_1}(\mathbf{x}^*)} = \frac{\frac{\partial f}{\partial x_2}(\mathbf{x}^*)}{\frac{\partial h}{\partial x_2}(\mathbf{x}^*)}$$

- (Recall how we derived the Marginal Rate of Substitution, i.e. the slope of indifference curves, in Lecture 6)

Constrained Optimization

- Assume that both $\frac{\partial h}{\partial x_1}(\mathbf{x}^*)$ and $\frac{\partial h}{\partial x_2}(\mathbf{x}^*)$ are different from zero

- Introduce a new variable λ such that

$$\lambda = \frac{\frac{\partial f}{\partial x_1}(\mathbf{x}^*)}{\frac{\partial h}{\partial x_1}(\mathbf{x}^*)} = \frac{\frac{\partial f}{\partial x_2}(\mathbf{x}^*)}{\frac{\partial h}{\partial x_2}(\mathbf{x}^*)} \tag{7}$$

- We can rewrite (7) as the two equations:

$$\frac{\partial f}{\partial x_1}(\mathbf{x}^*) - \lambda \frac{\partial h}{\partial x_1}(\mathbf{x}^*) = 0$$
$$\frac{\partial f}{\partial x_2}(\mathbf{x}^*) - \lambda \frac{\partial h}{\partial x_2}(\mathbf{x}^*) = 0$$

Lagrange Function

- Thus at a solution $\mathbf{x}^*$, the following must be true:

$$\frac{\partial f}{\partial x_1}(\mathbf{x}^*) - \lambda \frac{\partial h}{\partial x_1}(\mathbf{x}^*) = 0 \quad (8)$$

$$\frac{\partial f}{\partial x_2}(\mathbf{x}^*) - \lambda \frac{\partial h}{\partial x_2}(\mathbf{x}^*) = 0 \quad (9)$$

$$h(x_1^*, x_2^*) = c \quad (10)$$

- The crucial observation here is that the system of three equations (8)-(10) identifies the *critical points* of the following function of three variables:

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\left(h(x_1, x_2) - c\right)$$

- L is called the **Lagrangian function**
- λ is called the **Lagrange multiplier**

Critical Points of Lagrange Function

- The idea is to solve a constrained optimization problem by studying the critical points of an auxiliary function
- We can use the Lagrangian function provided that $\frac{\partial h}{\partial x_1}(\mathbf{x}^*) \neq 0$ or $\frac{\partial h}{\partial x_2}(\mathbf{x}^*) \neq 0$ (or both)
- The latter condition on the partial derivatives of h is called **constraint qualification**

First Order Necessary Conditions

Proposition

Let f and h be C^1 functions defined over $\mathbb{R}^2$. Suppose that:

1. (x_1^*, x_2^*) *is a solution of* $\max_{x_1,x_2} f(x_1, x_2)$ *subject to* $h(x_1, x_2) = c$ *or a solution of* $\min_{x_1,x_2} f(x_1, x_2)$ *subject to* $h(x_1, x_2) = c$;
2. (x_1^*, x_2^*) *is not a critical point of* h.

Then, there exists a real number λ^* *such that* $(x_1^*, x_2^*, \lambda^*)$ *is a critical point of the following Lagrangian function:*

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda\left(h(x_1, x_2) - c\right).$$

- The proposition does not say that a solution exists. It says that, if a solution exists, and if the constraint qualification is satisfied, the solution must be a critical point of the Lagrangian

Constrained Optimization

- **Example.** Consider the following maximization problem:

$$\max_{x_1, x_2} \quad x_1 x_2$$
$$\text{s. t.} \quad x_1 + 4x_2 = 16$$

- Since $\frac{\partial h}{\partial x_1} = 1$ and $\frac{\partial h}{\partial x_2} = 4$, the constraint qualification is satisfied

- The Lagrangian is

$$L(x_1, x_2, \lambda) = x_1 x_2 - \lambda \left(x_1 + 4x_2 - 16 \right)$$

Constrained Optimization

- **Example (cont'd).** Critical points of the Lagrangian are found by the following three conditions:

$$\frac{\partial L}{\partial x_1} = x_2 - \lambda = 0 \tag{11}$$

$$\frac{\partial L}{\partial x_2} = x_1 - 4\lambda = 0 \tag{12}$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + 4x_2 - 16) = 0 \tag{13}$$

- The unique solution of the system (11)-(13) is $x_1 = 8$, $x_2 = 2$, and $\lambda = 2$

- Can we use the proposition and conclude that $(x_1, x_2) = (8, 2)$ is a solution of our constrained maximization problem?

- No! We can use the proposition only to conclude that, if our problem has a solution, then it must be $(8, 2)$

Constrained Optimization

- First order conditions can be applied as follows:

1. Check the constraint qualification by finding the solutions of $\frac{\partial h}{\partial x_1} = 0$ and $\frac{\partial h}{\partial x_2} = 0$
2. Find the critical points of the Lagrangian function
3. If the critical points of h are *not* included in the constraint set C, the constraint qualification is satisfied. Therefore, the critical points of the Lagrangian are the only candidates for a solution to the original constrained optimization problem
4. If some of the critical points of h are included in the constraint set C, then the candidates for a solution to the original optimization problem are both *i)* the critical points of the Lagrangian and *ii)* the critical points of h included in the constraint set C

Constrained Optimization

- **Example.** Let $f(x, y) = y$ and $g(x, y) = y^3 - x^2$ be defined over $\mathbb{R}^2$. Consider the following constrained problem:

$$\begin{aligned} \min_{x,y} \quad & f(x, y) \\ \text{s.t.} \quad & g(x, y) = 0. \end{aligned}$$

- The Lagrangian is
$$L(x, y, \lambda) = y - \lambda(y^3 - x^2).$$
- The critical points of L are found by solving

$$2\lambda x = 0 \tag{14}$$
$$1 - 3\lambda y^2 = 0 \tag{15}$$
$$y^3 - x^2 = 0. \tag{16}$$

Constrained Optimization

- **Example (cont'd).** From (14) we have either $\lambda = 0$ or $x = 0$. If $\lambda = 0$, (15) cannot hold. If $x = 0$, $y = 0$ by (16) and, consequently, (15) cannot hold. Thus the system (14)-(16) does not admit any solution. This means that L does not have critical points

- The constraint qualification fails when $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$. That is, $3y^2 = 2x = 0$, which holds only at the point $(x, y) = (0, 0)$. Notice that $(0, 0)$ belongs to the constraint set.

- The only candidate for a solution is $(0, 0)$. To show that this point is actually a solution, we can argue as follows. The constraint requires $y^3 = x^2$. Since $x^2 \geq 0$ for every x, this implies that $y \geq 0$. Since we want to minimize f, the lowest possible value that f can take on is when $y = 0$, which requires $x = 0$. Thus $(0, 0)$ is the unique global constrained minimizer.

Sufficient Optimality Conditions

Proposition (Sufficient condition for the existence of a solution)

Let f and h be C^1 functions defined over an open and convex set $U \subseteq \mathbb{R}^n$. Suppose there exists a real number λ^* and an interior point $(x_1^*, x_2^*) \in U$ such that $(x_1^*, x_2^*, \lambda^*)$ is a stationary point of the Lagrangian function:

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda \left(h(x_1, x_2) - c \right).$$

- *If L is concave in (x_1, x_2) given λ^*–in particular, if f is concave and $\lambda^* h$ is convex–then (x_1^*, x_2^*) is a solution to $\max_{x_1, x_2} f(x_1, x_2)$ subject to $h(x_1, x_2) = c$*
- *If L is convex in (x_1, x_2) given λ^*–in particular, if f is convex and $\lambda^* h$ is concave–then (x_1^*, x_2^*) is a solution to $\min_{x_1, x_2} f(x_1, x_2)$ subject to $h(x_1, x_2) = c$*

Constrained Optimization

- A useful corollary of the proposition in the previous page applies to cases where the constraint function h is linear. Recall that a linear function is both concave and convex, and so is $\lambda^* h$ for any value of λ^*

- Therefore, when h is *linear*, the proposition in the previous page implies that:

    - If f is concave, then any critical point of the Lagrangian is a solution to $\max_{x_1,x_2} f(x_1, x_2)$ subject to $h(x_1, x_2) = c$

    - If f is convex, then any critical point of the Lagrangian is a solution to $\min_{x_1,x_2} f(x_1, x_2)$ subject to $h(x_1, x_2) = c$

Constrained Optimization

- **Example.** Consider the constrained optimization problem:

$$\max_{x_1, x_2} \quad x_1^{0.5} x_2^{0.5}$$
$$\text{s. t.} \quad x_1 + 4x_2 = 16$$

where both f and h are defined over $\mathbb{R}^2_{++}$

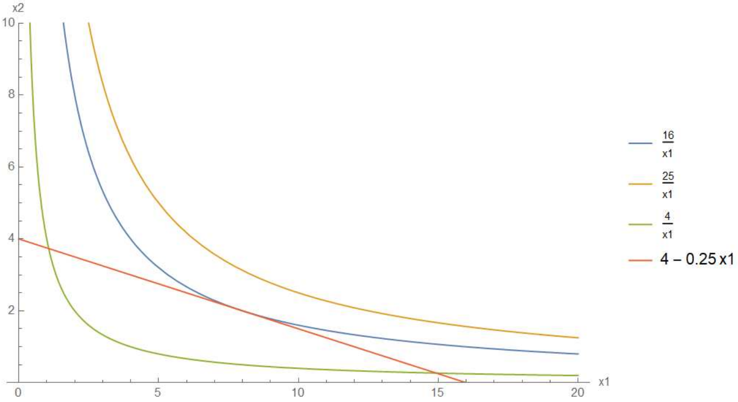
- Here we have that $f(x_1, x_2)$ is concave (see the slides from Lecture 8), and $h(x_1, x_2)$ is linear

- You can verify that the Lagrangian has a unique critical point $(x_1^*, x_2^*, \lambda^*) = \left(8, 2, \frac{1}{4}\right)$

- Thus we can conclude that $(8, 2)$ is a solution to this constrained maximization problem

Constrained Optimization

- **Example (cont'd).** A graphical representation of $\max_{x_1, x_2} x_1^{0.5} x_2^{0.5}$ subject to $x_1 + 4x_2 = 16$



Constrained Optimization

- **Example.** Consider again the firm's cost minimization problem, but without the non-negativity constraints:

$$\min_{x_1,x_2} \quad w_1x_1 + w_2x_2$$
$$\text{s. t.} \quad f(x_1, x_2) = q$$

- Suppose that the production function f is concave and that marginal products are positive, i.e $\frac{\partial f}{\partial x_1} > 0$ and $\frac{\partial f}{\partial x_2} > 0$ for any (x_1, x_2)

- Notice that the objective function is convex (and concave too)

- So we can conclude that the critical points of the Lagrangian are solutions to this constrained minimization problem

Constrained Optimization

- **Example (cont'd).** Here the Lagrangian is:

$$L(x_1, x_2, \lambda) = w_1 x_1 + w_2 x_2 - \lambda \left(f(x_1, x_2) - q \right)$$

- Critical points of L are the solutions to the system:

$$\frac{\partial L}{\partial x_1} = w_1 - \lambda \frac{\partial f}{\partial x_1}(x_1, x_2) = 0$$
$$\frac{\partial L}{\partial x_2} = w_2 - \lambda \frac{\partial f}{\partial x_2}(x_1, x_2) = 0$$
$$\frac{\partial L}{\partial \lambda} = -(f(x_1, x_2) - q) = 0$$

Constrained Optimization

- **Example (cont'd).** At a solution $(x_1^*, x_2^*, \lambda^*)$, we have that:

$$\lambda^* = \frac{w_1}{\frac{\partial f}{\partial x_1}(x_1^*, x_2^*)} = \frac{w_2}{\frac{\partial f}{\partial x_2}(x_1^*, x_2^*)} > 0$$

- What is the economic interpretation of the above expression?

Exercise

Consider the following constrained maximization problem:

$$\max_{x_1, x_2} \quad x_1^2 x_2$$
$$\text{s. t.} \quad 2x_1^2 + x_2^2 = 3$$

1. What can you say about the existence of a solution? (Think about Weierstrass's Theorem)
2. Solve this optimization problem