31C01100 - Taloustieteen matemaattiset menetelmät - Mathematics for Economists
Aalto University - Fall 2021

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Midterm Exam, 29.10.2021

Instructions:

- The exam has 4 questions. You must answer all parts of all questions
- Please write as clearly as you can
- Explain your reasoning. You don't have to give lengthy explanations for what you do, but I should be able to see how you justify the key steps in your answers

Question 1 [26 points]

Consider the following 3×3 matrix, where $c \in \mathbb{R}$ is a constant:

$$A = \begin{pmatrix} 6 & -1 & c \\ -2 & 2 & 2 \\ 1 & 4 & 3 \end{pmatrix}.$$

1. [6 points] Find the unique value of c such that the matrix A is *not* invertible.
2. [6 points] In the case where A is *not* invertible, what is the rank of A?
3. [7 points] Let $c = -1$. Calculate the inverse matrix A^{-1} of A.
4. [7 points] Find all the solutions of the following system of linear equations:

$$\begin{pmatrix} 6 & -1 & -1 \\ -2 & 2 & 2 \\ 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 1 \end{pmatrix}.$$

   (*Hint*: Use the inverse matrix A^{-1} you obtained above.)

Question 2 [22 points]

Consider the following subset of $\mathbb{R}^3$:

$$U := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = 0 \right\}.$$

In words, U contains all the vectors of $\mathbb{R}^3$ whose second component x_2 is equal to zero. Consider also the following two vectors in U:

$$\mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

1. [6 points] Suppose a vector $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Is $\mathbf{w}$ an element of U? Explain why or why not.
2. [6 points] Suppose a vector $\mathbf{z}$ is an element of U. Can you obtain $\mathbf{z}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$? Explain why or why not.
3. [7 points] Do $\mathbf{u}$ and $\mathbf{v}$ form a basis of U? If they do, explain why. If they do not, find a basis of U.
4. [3 points] What is the dimension of U?

Question 3 [28 points]

A firm's production function is $f : \mathbb{R}^3_+ \to \mathbb{R}$ such that

$$f(x, y, z) = x^2y^3z^2.$$

1. [6 points] Is f a homogenous function? If so, determine its degree. If not, explain.
2. [5 points] Is f an injective function? Explain why or why not.
3. [5 points] Consider the point $(x_0, y_0, z_0) = (2, 1, 2)$, where $f(x_0, y_0, z_0) = 16$. Suppose you want to express y as a function of x and z around this point. Can you apply the implicit function theorem in this case? Explain why or why not.
4. [6 points] Calculate the partial derivatives $\frac{\partial y}{\partial x}(2, 2)$ and $\frac{\partial y}{\partial z}(2, 2)$.
5. [6 points] Suppose that, at $(2, 1, 2)$, the firm increases x from 2 to 2.1 and, at the same time, decreases z from 2 to 1.8 while $y = 1$ is left unchanged. Use the total differential to estimate the corresponding change in f.

Question 4 [24 points]

Consider the following function defined over $\mathbb{R}^3$:

$$f(x, y, z) = -2x^3 + y^2 + 10z^2 + 24x + 12y - 12z - 4yz.$$

1. [6 points] Find all the critical points of f.
2. [8 points] For each critical point you found, determine whether it is a local maximizer, a local minimizer or a saddle point.
3. [6 points] Is f a concave function? Is it convex? Explain. (*Hint*: You don't need to do additional calculations. Think about what you found in the previous question on local extrema.)
4. [4 points] Does f have any global maximizer or global minimizer? Explain.