ELEC-E8101 Digital and Optimal Control
Exercise 6,
Solutions

1.

Pole placement

State feedback:

$$\mathbf{x}(k+1) = \Phi\,\mathbf{x}(k) + \Gamma\,u(k)$$
$$u(k) = -\mathbf{L}\mathbf{x}(k)$$

Closed loop system

$$\begin{aligned}\mathbf{x}(k+1) &= \Phi\,\mathbf{x}(k) + \Gamma\,u(k) = \Phi\,\mathbf{x}(k) - \Gamma\,\mathbf{L}\mathbf{x}(k)\\ &= (\Phi - \Gamma\,\mathbf{L})\mathbf{x}(k)\end{aligned}$$

If the system is controllable, the closed loop poles can be placed as desired with state feedback controller.

The system under consideration:

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\mathbf{x}(k) - \begin{bmatrix} 1 \\ 0,5 \end{bmatrix}\begin{bmatrix} l_1 & l_2 \end{bmatrix}\mathbf{x}(k)$$

$$= \begin{bmatrix} 1-l_1 & -l_2 \\ 1-0,5l_1 & 1-0,5l_2 \end{bmatrix}\mathbf{x}(k)$$

The corresponding characteristic polynomial:

$$A(s) = \det(z\mathbf{I} - \Phi + \Gamma\,\mathbf{L}) = \det\begin{bmatrix} z-1+l_1 & l_2 \\ -1+0,5l_1 & z-1+0,5l_2 \end{bmatrix}$$

$$= (z-1+l_1)(z-1+0,5l_2) - l_2(-1+0,5l_1)$$

$$= z^2 + (-2+0,5l_2+l_1)z + (1+0,5l_2-l_1)$$

In deadbeat control the poles must be placed to origin and therefore it must be set that

$$A(z) = z^2,$$

$$\Rightarrow z^2 + \left(-2 + 0,5l_2 + l_1\right) z + \left(1 + 0,5l_2 - l_1\right) = z^2$$

$$\Rightarrow \begin{cases} -2 + 0,5l_2 + l_1 = 0 \\ 1 + 0,5l_2 - l_1 = 0 \end{cases}$$

$$\Rightarrow l_1 = \frac{3}{2}, \quad l_2 = 1.$$

Usually the origin is not the best place for the poles of the system. The result systems are too snappy and too weakly robust.

Note that the above *pole placement* solves a *regulator problem*. That is, the reference value is assumed zero (or a constant value, which can always be scaled to zero in the equations). The states approach zero asymptotically.

2.

State observer:

$$\hat{\mathbf{x}}(k+1) = \Phi\, \hat{\mathbf{x}}(k) + \Gamma\, u(k) + \mathbf{K}\left(y(k) - \mathbf{C}\hat{\mathbf{x}}(k)\right)$$
$$\hat{\mathbf{x}}(k+1) = \left[\Phi - \mathbf{KC}\right] \hat{\mathbf{x}}(k) + \Gamma\, u(k) + \mathbf{K}y(k)$$

The error of the state estimation behaves in the following way:

$$\begin{aligned} \tilde{\mathbf{x}}(k+1) &= \mathbf{x}(k+1) - \hat{\mathbf{x}}(k+1) \\ &= \Phi\, \mathbf{x}(k) + \Gamma\, u(k) - \left[\Phi - \mathbf{KC}\right] \hat{\mathbf{x}}(k) - \Gamma\, u(k) - \mathbf{KCx}(k) \\ &= \left[\Phi - \mathbf{KC}\right] \tilde{\mathbf{x}}(k) \end{aligned}$$

If the system is fully observable, the poles of the observer can be placed as desired.

In this problem:

$$\Phi - \mathbf{KC} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -k_1 \\ 1 & 1 - k_2 \end{bmatrix}$$

The characteristic polynomial:

$$\det\left(z\mathbf{I} - \Phi + \mathbf{KC}\right) = \begin{bmatrix} z-1 & k_1 \\ -1 & z-1+k_2 \end{bmatrix} = z^2 + \left(-2 + k_2\right) z + 1 - k_2 + k_1$$

The deadbeat observer places the poles into origin, so

$$k_1 = 1$$
$$k_2 = 2.$$

The state observer is:

$$\hat{\mathbf{x}}(k+1)=\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\hat{\mathbf{x}}(k)+\begin{bmatrix}1\\ 0,5\end{bmatrix}u(k)+\begin{bmatrix}1\\ 2\end{bmatrix}y(k).$$

Placing the poles into origin is not usually sensible. The observer is too fast and it responses too much to the noise disturbances and it is not robust enough.

3.

First a result from the realization theory. A given pulse transfer function, where all possible pole-zero cancellations have been done, has always state-space representations in *controllable canonical form* and *observable canonical form* as follows

$$H(z)=\frac{b_0 z^{n_b}+b_1 z^{n_b-1}+\cdots+b_{n_b}}{z^{n_a}+a_1 z^{n_a-1}+\cdots+a_{n_a}}$$

Controllable canonical form:

$$\mathbf{x}(k+1)=\begin{bmatrix}-a_1 & -a_2 & \cdots & -a_{n_a-1} & -a_{n_a}\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}1\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix}\mathbf{u}(k)$$

$$\mathbf{y}(k)=\begin{bmatrix}b_{n_b-n_a+1} & b_{n_b-n_a+2} & \cdots & b_{n_b-1} & b_{n_b}\end{bmatrix}\mathbf{x}(k)$$

and *observable canonical form*

$$\mathbf{x}(k+1)=\begin{bmatrix}-a_1 & 1 & 0 & \cdots & 0\\ -a_2 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -a_{n_a-1} & 0 & 0 & \cdots & 1\\ -a_{n_a} & 0 & 0 & \cdots & 0\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}b_{n_b-n_a+1}\\ b_{n_b-n_a+2}\\ \vdots\\ b_{n_b-1}\\ b_{n_b}\end{bmatrix}\mathbf{u}(k)$$

$$\mathbf{y}(k)=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\end{bmatrix}\mathbf{x}(k)$$

See also lecture slides Chapter 3, Canonical forms. The controllable canonical form is always reachable, the observable canonical form is always observable. Forming a state feedback to a controllable canonical form or an observer to observable canonical form result in easy calculations.

a. The considered pulse transfer operator is:

$$H(q)=\frac{0.030q+0.026}{q^2-1.65q+0.68}.$$

A corresponding state-space representation is (controllable canonical form):

$$\mathbf{x}(12(k+1))=\begin{bmatrix}1.65 & -0.68\\ 1 & 0\end{bmatrix}\mathbf{x}(12k)+\begin{bmatrix}1\\0\end{bmatrix}u(12k)$$
$$y(12k)=\begin{bmatrix}0.030 & 0.026\end{bmatrix}\mathbf{x}(12k)$$

Let us find the state-feedback controller leading to a characteristic equation:

$z^2 - 1.55z + 0.64 = 0.$

Write $u(12k)$ = - $\begin{bmatrix}l_1 & l_2\end{bmatrix}\mathbf{x}(12k)+$ $r(12k)$, where r is a reference signal. (We use the reference signal here only to make simulations of a *servo problem* possible. Servo problem means that the reference signal may change.)

By the substitution of the above we get:

$$\mathbf{x}(12(k+1))=\begin{bmatrix}1.65-l_1 & -0.68-l_2\\ 1 & 0\end{bmatrix}\mathbf{x}(12k)+\begin{bmatrix}1\\0\end{bmatrix}r(12k)$$
$$y(12k)=\begin{bmatrix}0.030 & 0.026\end{bmatrix}\mathbf{x}(12k)$$

The characteristic polynomial of this system is:

$$\det(z\mathbf{I}-\mathbf{\Phi})=\det\left(\begin{bmatrix}z-1.65+l_1 & 0.68+l_2\\ -1 & z\end{bmatrix}\right)=z(z-1.65+l_1)+0.68+l_2=$$
$$=z^2+(l_1-1.65)z+0.68+l_2\equiv z^2-1.55z+0.64$$
$$\Rightarrow\begin{cases}l_1-1.65=-1.55\\ l_2+0.68=0.64\end{cases}\Rightarrow\begin{cases}l_1=0.10\\ l_2=-0.04\end{cases}$$

Hence the overall system is:

$$\mathbf{x}(12(k+1))=\begin{bmatrix}1.55 & -0.64\\ 1 & 0\end{bmatrix}\mathbf{x}(12k)+\begin{bmatrix}1\\0\end{bmatrix}r(12k)$$
$$y(12k)=\begin{bmatrix}0.030 & 0.026\end{bmatrix}\mathbf{x}(12k)$$

Simulation by Matlab:

```
» [num,den]=ss2tf([1.55 -0.64;1 0],[1 0]',[0.03 0.026],0)
num =
        0    0.0300    0.0260
den =
    1.0000   -1.5500    0.6400
» G=tf(num,den,12);   % Note the discrete time, 12 is the sampling interval.
» step(G)
```

» a=dcgain(G); % command *dcgain* gives the static gain of a dynamic system

zeta=0.7;omega=0.027;

» Gcont=tf([a*omega^2],[1 2*omega*zeta omega^2]) %Desired closed loop in continuous %time

```
 Transfer function:
       0.0004536
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s^2 + 0.0378 s + 0.000729
```
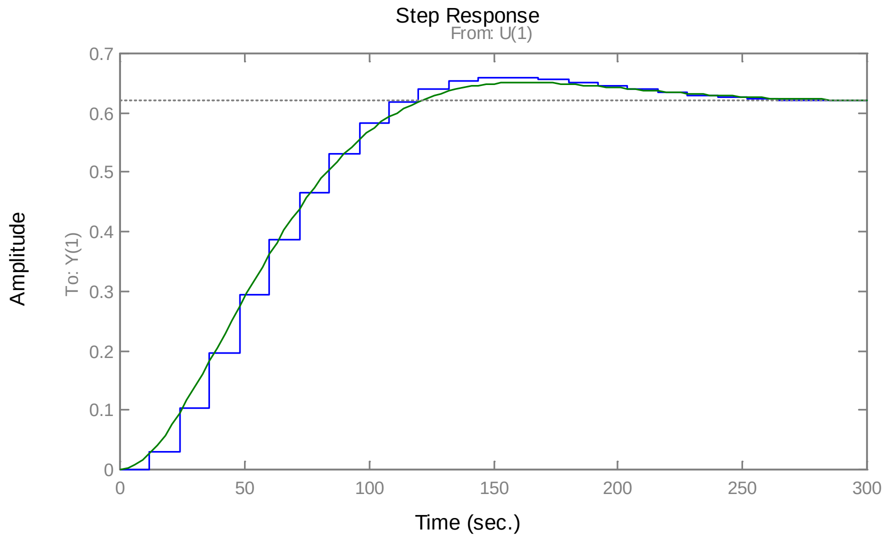

» hold on;step(Gcont,300)



Note that in the above example the command *ss2tf* was used. Another, maybe more common way today would be to read the numerical values into matrix objects Phi, Gamma, C, D, and then write something like Gss=ss(Phi,Gamma,C,D,12); G=tf(Gss).

The static gain of the closed loop was not 1, so there will be a permanent error between the reference and system output.

b. Let's add an integrator to remove the steady state error by augmenting the integrator to the state-space model. See lecture slides Chapter 4 and the Harmonic oscillator example.

$$\Delta x_{n+1}(k) = x_{n+1}(k) - x_{n+1}(k-1) = h\left[y(k-1) - r(k-1)\right]$$
$$x_{n+1}(k) = hCx(k-1) - hr(k-1) + x_{n+1}(k-1)$$

Form the augmented state-space model and design s state feedback control law for it (by matrices Φ_{aug} and Γ_{aug})

$$\begin{bmatrix} x\left(12(k+1)\right) \\ x_{n+1}\left(12(k+1)\right) \end{bmatrix} = \underbrace{\begin{bmatrix} \Phi & 0 \\ hC & 1 \end{bmatrix}}_{\Phi_{aug}} \begin{bmatrix} x(12k) \\ x_{n+1}(12k) \end{bmatrix} + \underbrace{\begin{bmatrix} \Gamma \\ 0 \end{bmatrix}}_{\Gamma_{aug}} u(12k) + \begin{bmatrix} \Gamma \\ -h \end{bmatrix} r(12k)$$

$$u(12k) = -[L \quad l_{n+1}] x_{aug}(12k) \quad \Rightarrow$$

$$\begin{bmatrix} x\left(12(k+1)\right) \\ x_{n+1}\left(12(k+1)\right) \end{bmatrix} = \underbrace{\begin{bmatrix} \Phi - \Gamma L & -\Gamma l_{n+1} \\ hC & 1 \end{bmatrix}}_{\Phi_{cl}} \begin{bmatrix} x(12k) \\ x_{n+1}(12k) \end{bmatrix} + \underbrace{\begin{bmatrix} \Gamma \\ -h \end{bmatrix}}_{\Gamma_{cl}} r(12k)$$

$$\begin{bmatrix} x_1\left(12(k+1)\right) \\ x_2\left(12(k+1)\right) \\ x_3\left(12(k+1)\right) \end{bmatrix} = \begin{bmatrix} 1.65 - l_1 & -0.68 - l_2 & -l_3 \\ 1 & 0 & 0 \\ 0.36 & 0.312 & 1 \end{bmatrix} \begin{bmatrix} x_1(12k) \\ x_2(12k) \\ x_3(12k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -12 \end{bmatrix} r(12k)$$

The L-polynomial can be calculated by hand:

$$\det\left(z\mathbf{I} - \mathbf{\Phi}_{cl}\right) = \det\left(\begin{bmatrix} z - 1.65 + l_1 & 0.68 + l_2 & +l_3 \\ -1 & z & 0 \\ -0.36 & -0.312 & z - 1 \end{bmatrix}\right)$$

$$= z^3 + \left(l_1 - 2.65\right) z^2 + \left(2.33 - l_1 + l_2 + 0.36 l_3\right) z + \left(-0.68 - l_2 + 0.312 l_3\right)$$

$$\equiv z\left(z^2 - 1.55z + 0.64\right)$$

$$\Rightarrow \begin{cases} l_1 = 1.10 \\ l_2 = -0.638 \\ l_3 = 0.134 \end{cases}$$

The same with Matlab:
\>> fii = [1.65 -0.68 0; 1 0 0; 0.36 0.312 1]
\>> gamma = [1; 0; 0]

\>> p1 = (1.55+sqrt(1.55^2-4*0.64))/2
p1 = 0.7750 + 0.1984i

\>> p2 = (1.55-sqrt(1.55^2-4*0.64))/2
p2 = 0.7750 - 0.1984i

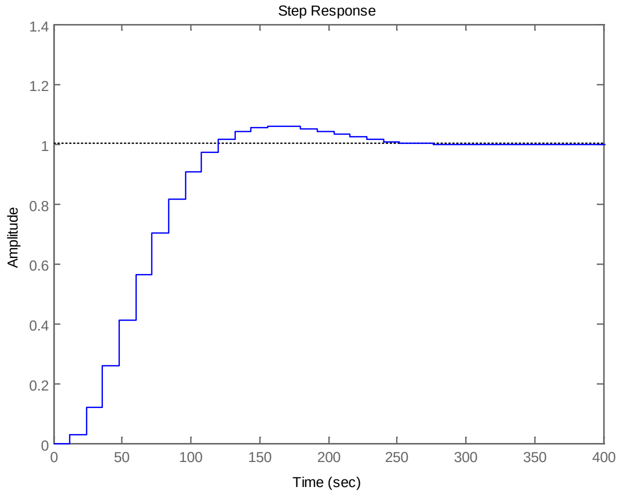
\>> Laug = place(fii, gamma, [p1 p2 0])
Laug =1.1000 -0.6382 0.1339

A closed-loop system is received:

$$\begin{cases} x\left(12(k+1)\right) = \begin{bmatrix} 0.55 & -0.042 & -0.134 \\ 1 & 0 & 0 \\ 0.36 & 0.312 & 1 \end{bmatrix} x\left(12k\right) + \begin{bmatrix} 1 \\ 0 \\ -12 \end{bmatrix} r(12k), \\ y\left(12k\right) = \begin{bmatrix} 0.030 & 0.026 & 0 \end{bmatrix} x\left(12k\right) \end{cases}$$

fiicl=[0.55 -0.042 -0.134; 1 0 0; 0.36 0.312 1]

```
gammacl=[1; 0; -12]
ccl = [0.030 0.026 0]
sys = ss(fiicl, gammacl, ccl, 0, 12)
step(sys)
```



The steady state error is removed when the integrator was added to the controller.

Note that from the above one might get the wrong impression that we were changing the process somehow! No, the process remains exactly the same, of course. We used the augmented process model only to the purpose of designing the controller with integrator.

This can more clearly be seen, if the simulation is done in Simulink. See again Chapter 4 in the lectures and the Harmonic Oscillator example there.

4. See lecture slides Chapter 4, section Output feedback. The resulting equation for closed loop state and observer error was

$$\begin{cases} \begin{bmatrix} \mathbf{x}(k+1) \\ \hline \tilde{\mathbf{x}}(k+1) \end{bmatrix} = \left[\begin{array}{c|c} \Phi - \Gamma\mathbf{L} & \Gamma\mathbf{L} \\ \hline \mathbf{0} & \Phi - \mathbf{KC} \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \hline \tilde{\mathbf{x}}(k) \end{bmatrix} \\ \mathbf{y}(k) = \left[\begin{array}{c|c} \mathbf{C} & \mathbf{0} \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \hline \tilde{\mathbf{x}}(k) \end{bmatrix} \end{cases}$$

The system matrix is a block-diagonal matrix, where the “elements” are square matrices. The eigenvalues (poles of the closed-loop system) are then (by known result of matrix analysis)

$$\det\left(zI - \Phi + \Gamma\mathbf{L}\right) \cdot \det\left(zI - \Phi + \mathbf{KC}\right) = 0$$

The poles are clearly a combination of state feedback design and observer design. That means that these can be designed separately without taking the other one into account at all. That is known as the *certainty equivalence principle*, which holds in the case of linear systems.