x) $\forall \varepsilon > 0$ sufficiently small, $x \prec_K y \Rightarrow (y-x) + \varepsilon \in K$
which leads to $\alpha(y-x) + \alpha\varepsilon \in K$ $(\forall \alpha > 0)$
By definition of cones.
Now, since $\tilde{\varepsilon} = \alpha\varepsilon > 0$ can also be made small,
we have that $\alpha(y-x) \in \text{int } K \Rightarrow \alpha x \prec_K \alpha y$.

x) $x \prec_K x \Rightarrow (x-x) = 0 \in \text{int } K$, which is false
since K is pointed and thus $0 \notin \text{int } K$.

xi) $x \prec_K y \Rightarrow (y-x) \in \text{int } K$.
For $\forall \varepsilon > 0$ sufficiently small $(y-x) + \varepsilon \in \text{int } K$.
Now, let $\varepsilon = v - u$, $\forall u$ and v small enough,
then $(y-x) + (v-u) = (y+v) - (x+u) \in \text{int } K$,
which results in $x + u \prec_K y + v$.

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(6) $S = \{(x,t) \mid \|x\|_2 \le t\}$ SOC is selfdual
Consider point $(y, r) \in S$. Thus, $\forall (x,t) \in S$,
we have

$$\begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} y \\ r \end{bmatrix} = x^T y + tr \ge -\|x\|_2 \|y\|_2 + tr \ge$$

follows from Cauchy-Swartz inequality

$$\ge -(t)(r) + tr = 0$$

follows from the fact that $(x,t), (y,r) \in S$,
Thus, as $x^T y + tr \ge 0$ $\forall (x,t) \in S$, the point $(y,r) \in S$ is also an element of the dual cone $S^* = \{y \mid x^T y \ge 0, \forall x \in S\}$.
It proves that $S \subseteq S^*$
Prove now that $S^* \subseteq S$
Consider any point $(y, r) \in S^*$

Thus, $x^T y + tr \geq 0 \quad \forall (x,t) \in S$.
without loss of generality, consider $x = -y$ and $t = \|y\|_2 = \sqrt{y^T y}$, such that

$$x^T y + tr = -y^T y + \|y\|_2 r \geq 0 \implies$$

$$\implies r \geq \|y\|_2$$

Thus, $\forall (y,r) \in S^*$ we have that $\|y\| \leq r$, which leads to $(y,r) \in S$.

(7) $S = \{(x,t) \mid \|x\|_1 \leq t\}$ $\quad S^* - ?$

Lemma: Let $\|.\|$ be a norm on $\mathbb{R}^n$. The dual cone associated with a norm-cone $S = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \leq t\}$ is the cone defined by the dual norm, i.e.,

$$S^* = \{(y,s) \in \mathbb{R}^{n+1} \mid \|y\|_* \leq s\} \text{ where}$$

$$\|y\|_* = \sup \{y^T x \mid \|x\| \leq 1\}$$

Proof: This is equivalent to prove that

$$x^T y + ts \geq 0, \quad \|x\| \leq t \iff \|y\|_* \leq s$$

$(\Leftarrow)$: Assume $(y,s) \in \mathbb{R}^{n+1}$ with $\|y\| \leq s$, and consider a point $(x,t) \in S$ such that $\|x\| \leq t$ for some $t > 0$. Given that $\|x\| \leq t$ implies $\|-x/t\| \leq 1$, we have from the definition of dual norm that

$$y^T\left(-\frac{x}{t}\right) \leq \sup \{y^T x \mid \|x\| \leq 1\} = \|y\|_* \leq s$$

Therefore $y^T(-\frac{x}{t}) \leq s$ leads to

$y^T x + ts \geq 0$, which proves that $(y, s) \in S^*$.

$(\Rightarrow)$: By contradiction! Assume that

$(y, s) \in S^*$ but $\|y\|_* > s$.

By the definition of the dual norm,

$\|y\|_* = y^T x > s$ for some $x \in R^n$ with $\|x\| = 1$.

Therefore, choosing $t = 1$ leads to

$x^T y + st \leq y^T(-x) + s < 0$, which

contradicts the assumption $(y, s) \in S^*$.

Thus, $(y, s) \in S^*$ implies that $\|y\|_* \leq s$. ∎

Now, to prove $\mathcal{G}_1^* = S_\infty$ we only

need to show that $(\|\cdot\|_1)_* = \|\cdot\|_\infty$.

Consider any two vectors $x, y \in R^n$.

We have that

$$y^T x = \sum_{i=1}^n y_i x_i \leq \sum_{i=1}^n |y_i x_i| \leq \sum_{i=1}^n |y_i||x_i| \leq$$

$$\leq \max |y_i| \sum_{i=1}^n |x_i| = \|y\|_\infty \|x\|_1$$

First inequality follows from $\sum z_i \leq \sum |z_i|$

Second follows from the multiplicativity of the absolute value.

Last one follows from $\sum |z_i| \leq \max_j |z_j|$

Considering $\|x\|_1 \leq 1$, we have $y^T x \leq \|y\|_\infty$ with

equality holding when $\|y\|_\infty = \sup y^T x$ by

definition of l_∞ norm.

Therefore, $\|y\|_\infty = \sup\{y^T x \,/\, \|x\|_1 \leq 1\}$, which is exactly the definition of the dual norm $(\|\cdot\|_1)_*$.

Finally, since $(\|\cdot\|_1)_* = \|\cdot\|_\infty$, Lemma leads to

$$S_1^* = \{(y,s) \in \mathbb{R}^{n+1} \,/\, \|y\|_\infty \leq s\} = S_\infty$$