Aalto University
School of Engineering

GEO – E1050
Finite Element Method in Geoengineering

Boundary element method

Wojciech Sołowski

Boundary element method

2 components:

a) some analytical solution we will use in our approximation (FUNDAMENTAL SOLUTION)

b) discretisation, so we use our fundamental solution over and over again, over some finite domain

of course… **fundamental solution must exists**

… and must be known…

… and generally should not be too complex…

Boundary element method

Method reduce the dimension of the problem by 1 so:

2D problem becomes 1D problem;

requires 2D fundamental solution

3D problem becomes 2D problem;

requires 3D fundamental solution

We discretise the boundary of the problem only – hence the name: 'boundary element method'

Boundary element method

Reducing dimensions of the problem is **HUGE**

however…

Fundamental solutions only can be analytically computed for simple cases…

- linear elasticity
- Poisson equation problems
- and similar…

Cannot be used for elasto-plasticity (at least not easily)

Boundary element method

For example, for Poisson equation,

$$\mathbf{q} = -\mathbf{D}\nabla u$$

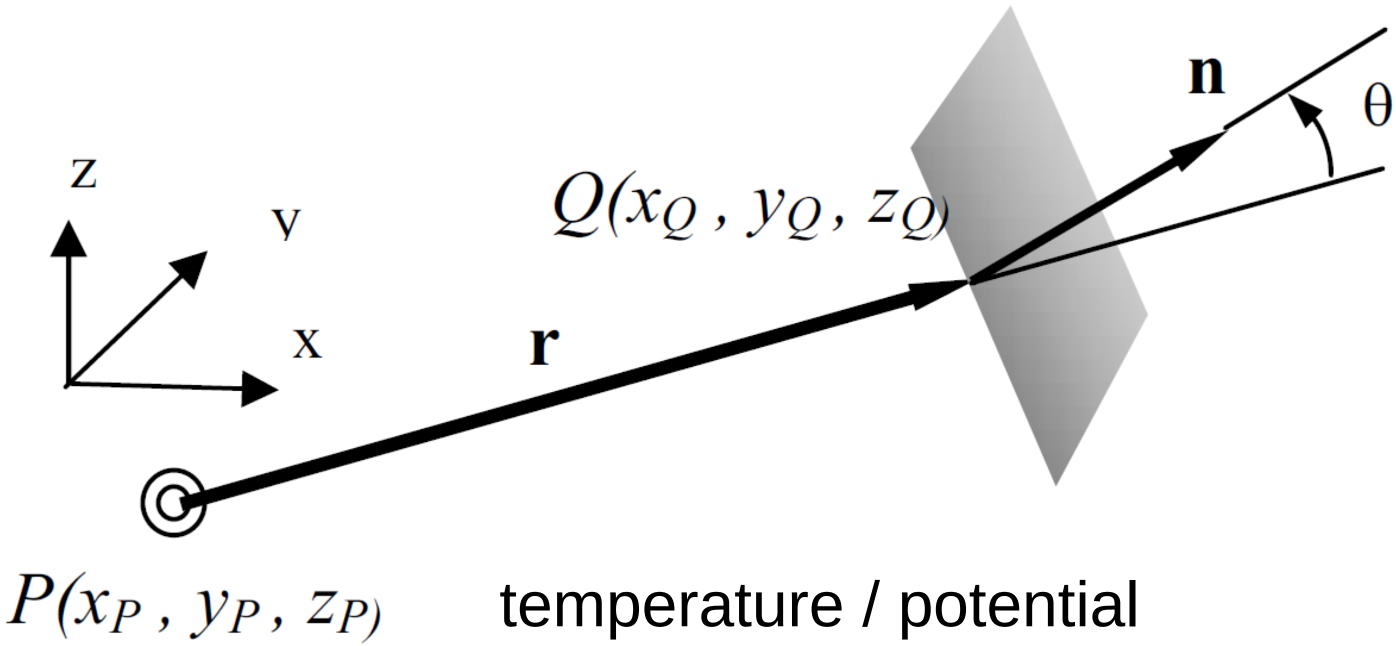
We assume some source at point $P(x_p,y_p,z_p)$ in infinite space and at some point Q the temperature / potential is:

$$U(P,Q) = \frac{1}{4\pi rk}$$

And if we assume flow in x direction, the flow at point P is:

$$T(P,Q) = \frac{\cos\theta}{4\pi r^2}$$

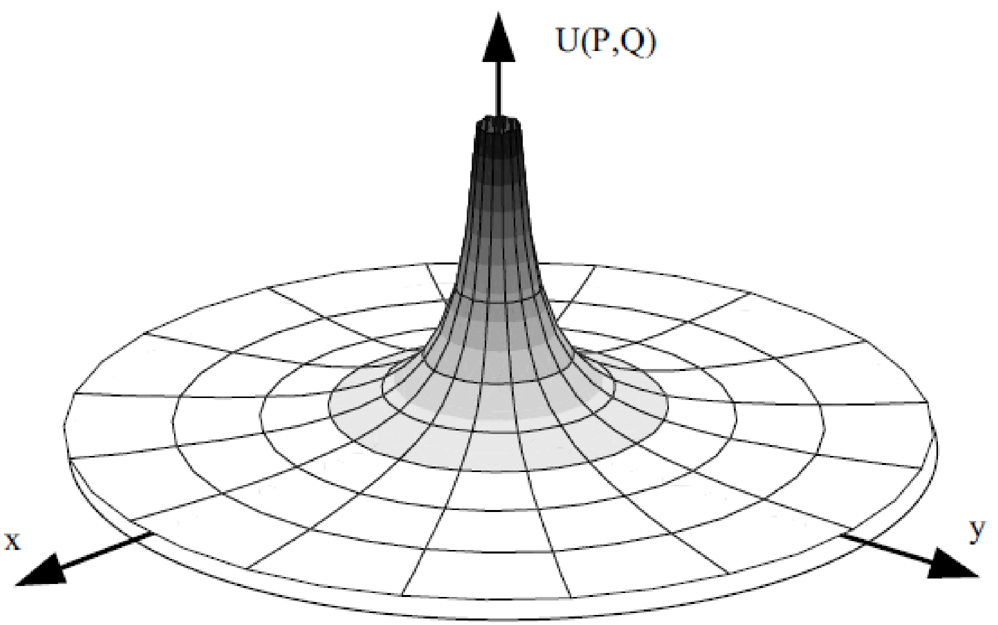
Boundary element method



temperature / potential

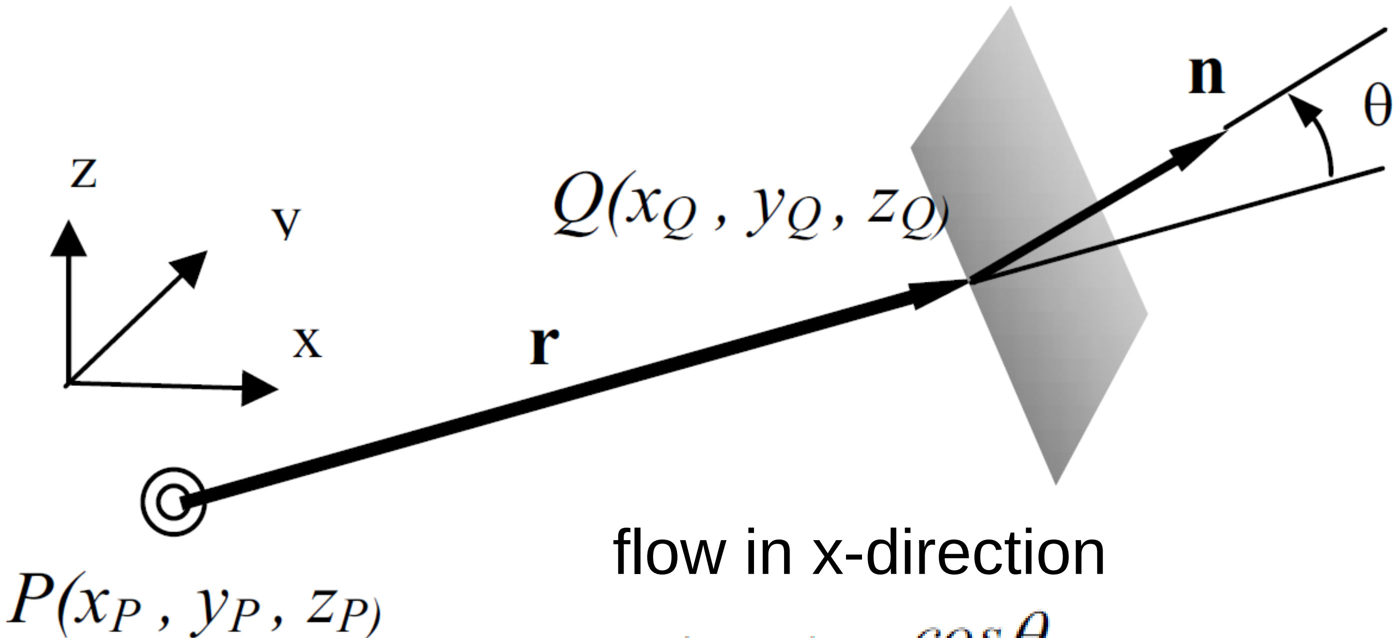
$$U(P,Q) = \frac{1}{4\pi rk}$$

Boundary element method



Variation of fundamental solution *U* (potential/temperature) in the x-y plane for 3-D potential problems (source at origin of coordinate system)

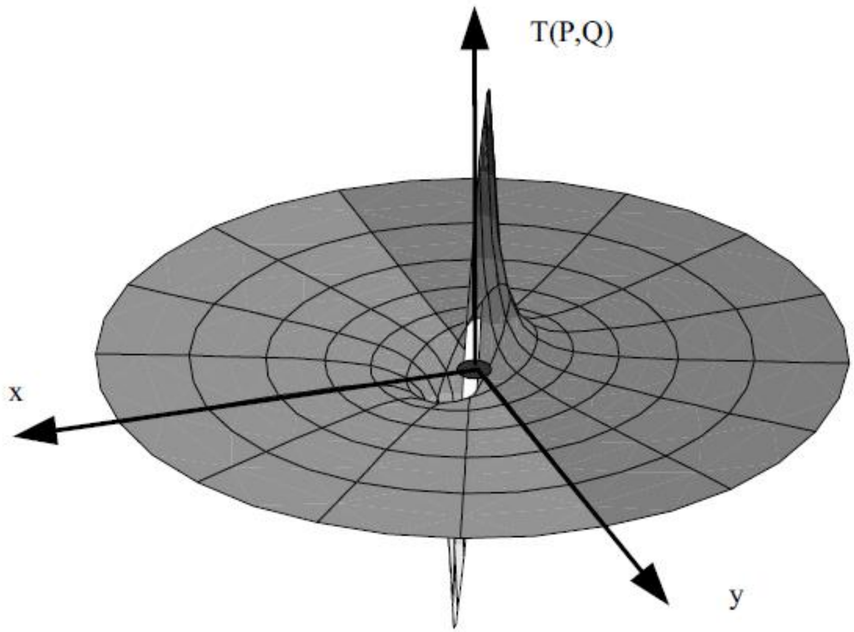
Boundary element method



flow in x-direction

$$T(P,Q) = \frac{\cos\theta}{4\pi r^2}$$

Boundary element method



Variation of fundamental solution for **n** = {1,0,0} (flow in x-direction) in x-y plane for 3-D potential problems (e.g. temperature changes if flow is happening)

Boundary element method

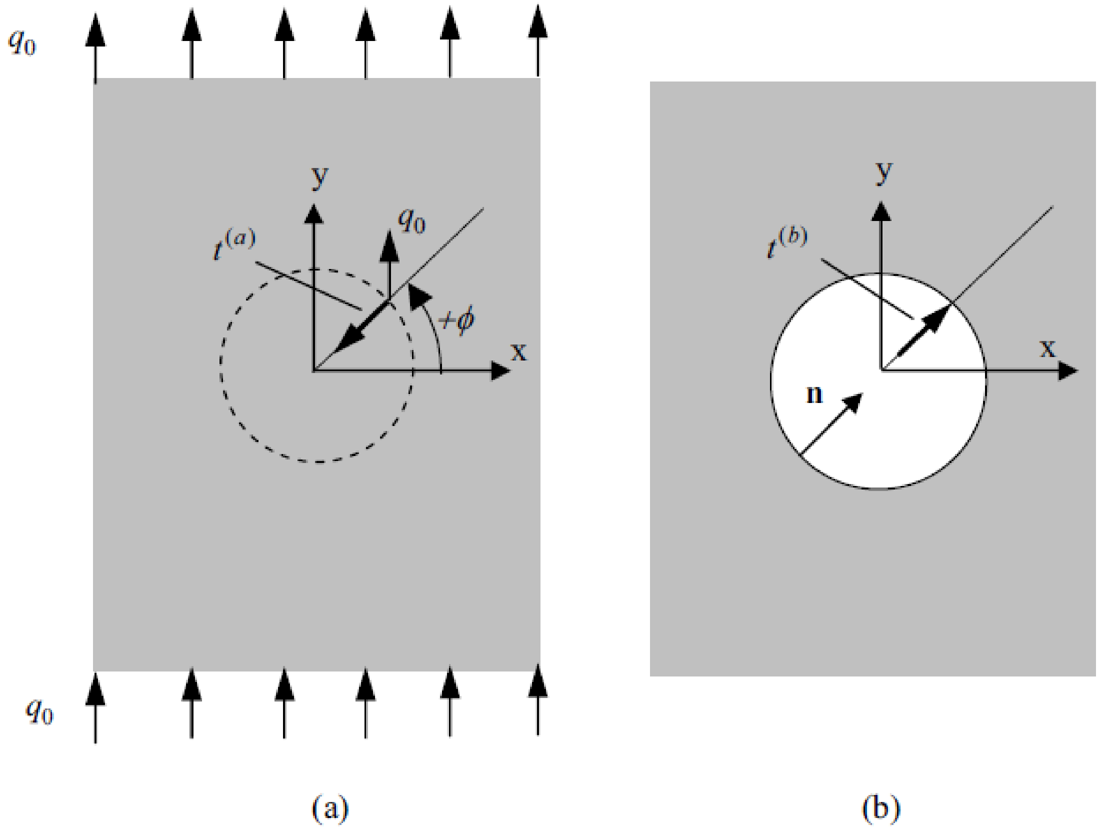


Figure 5.1 Heat flow in an infinite domain, case (a) and (b)

Boundary element method

We know the heat flow solution

We want to solve this one, with a perfect insulator inside, same boundary conditions…

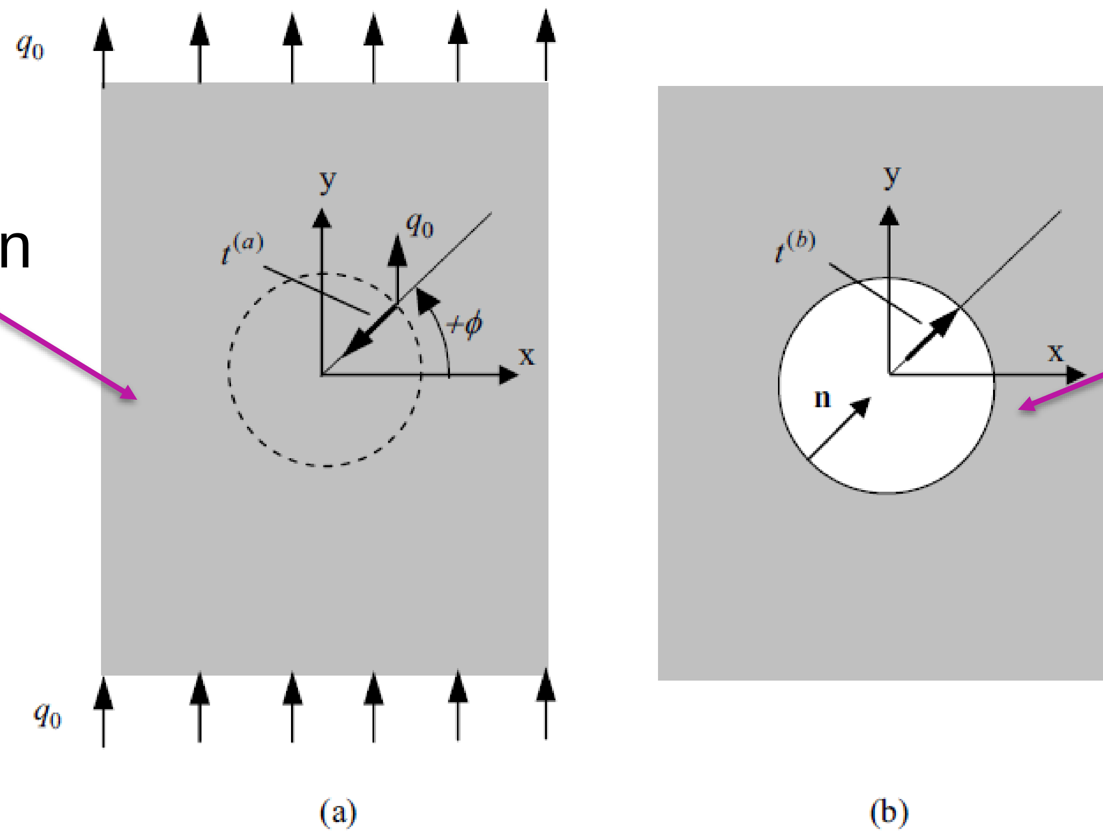
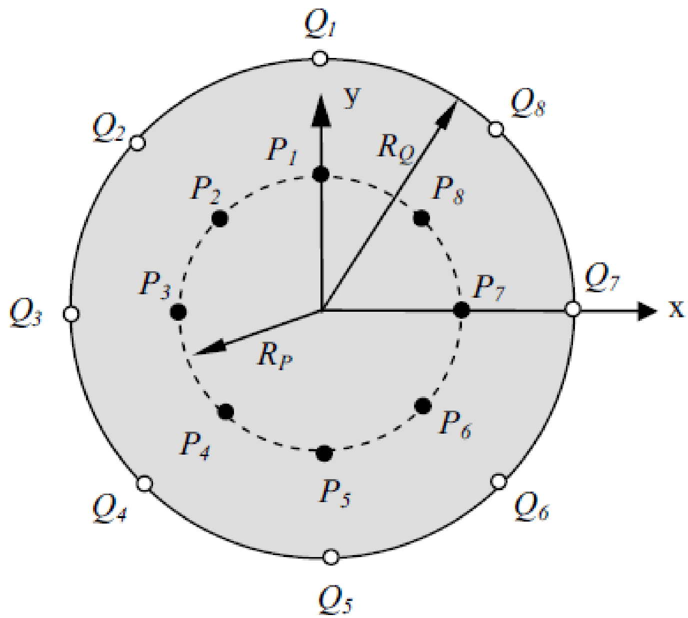


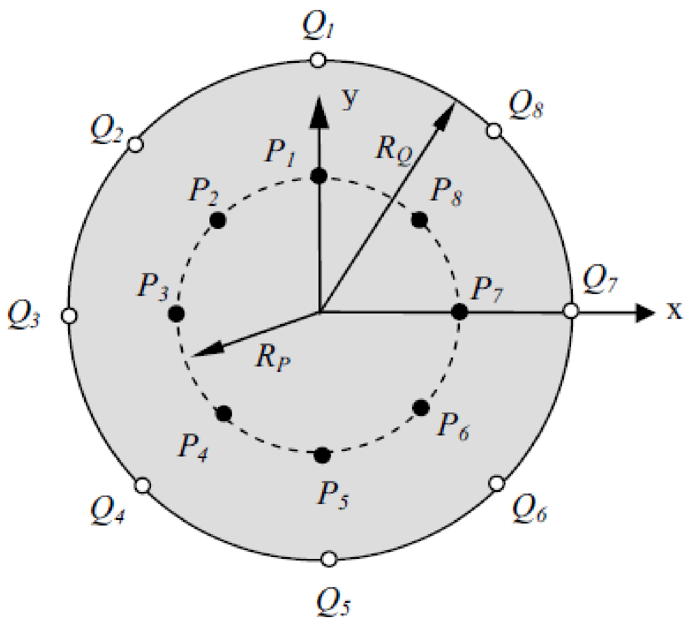
Figure 5.1 Heat flow in an infinite domain, case (a) and (b)

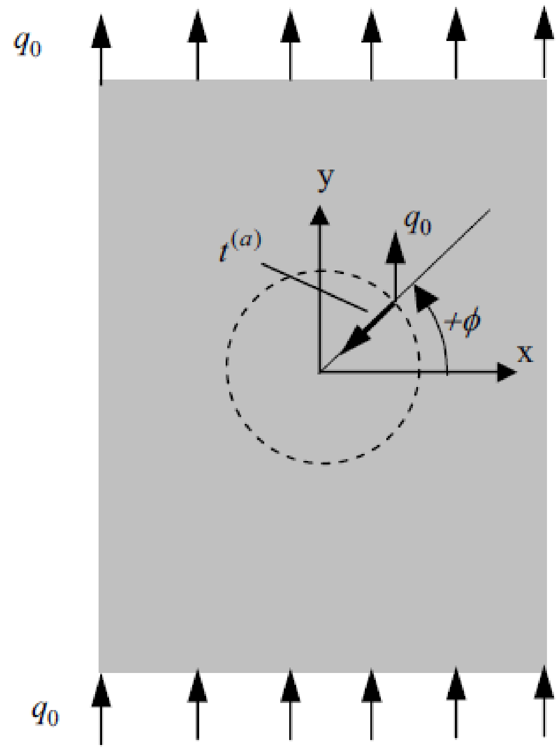
Boundary element method



We want to approximate the known solution on the outside by the sources (we have the fundamental solution for those)

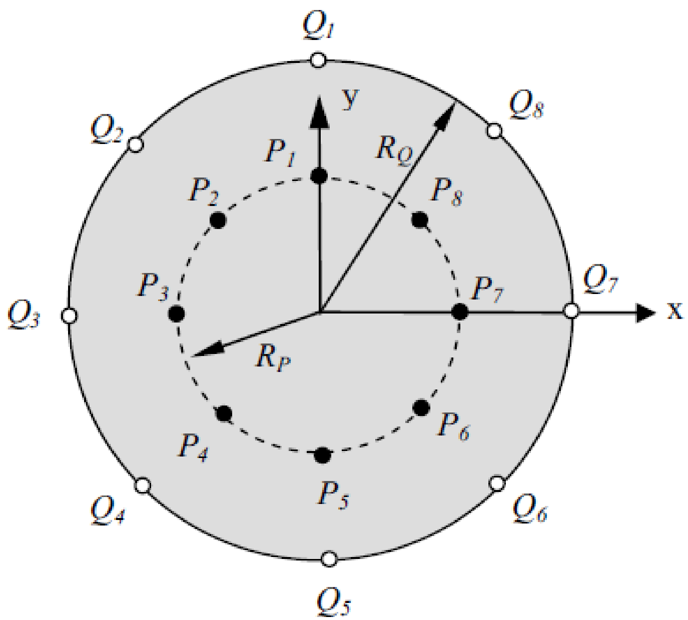
Boundary element method

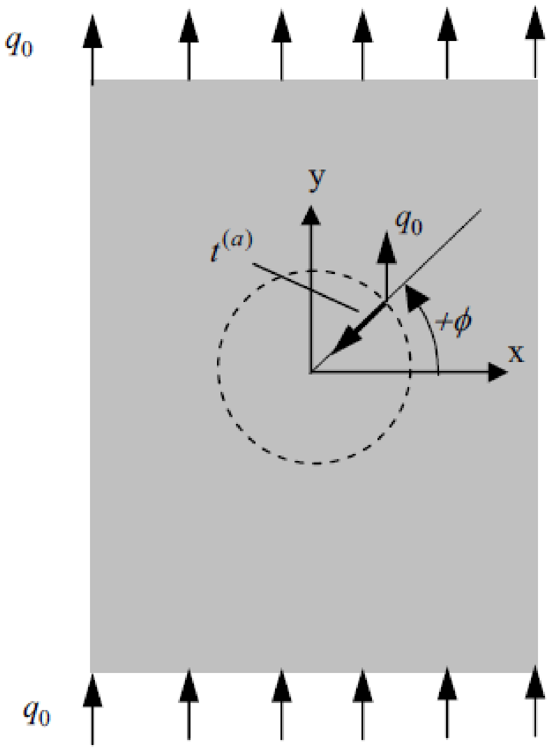




The sources should be such, that on the boundary we have exactly same solution as the one without the insulator…

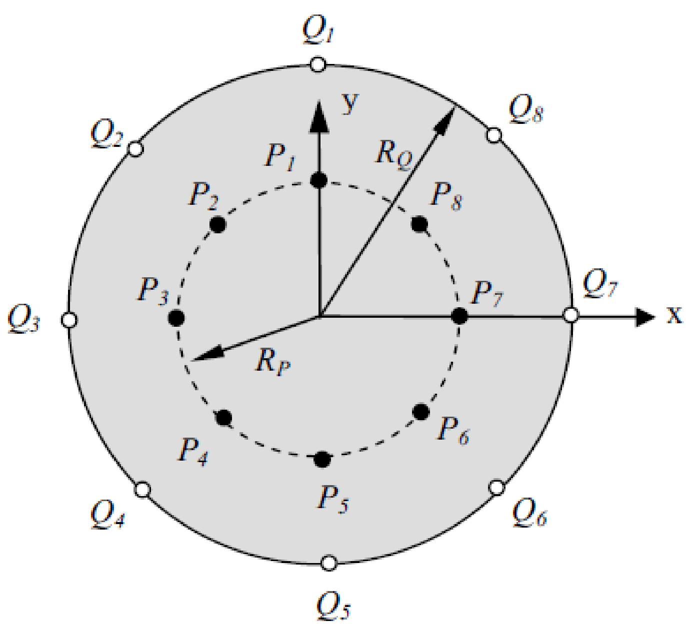
Boundary element method

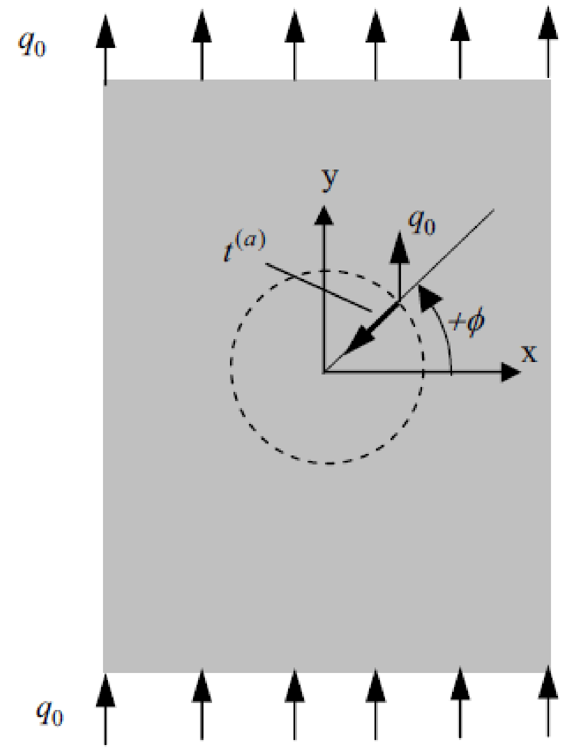




Now, to get the solution with the insulator, we use the superposition – from the known solution we subtract the one obtained with the sources... **And we are done…**

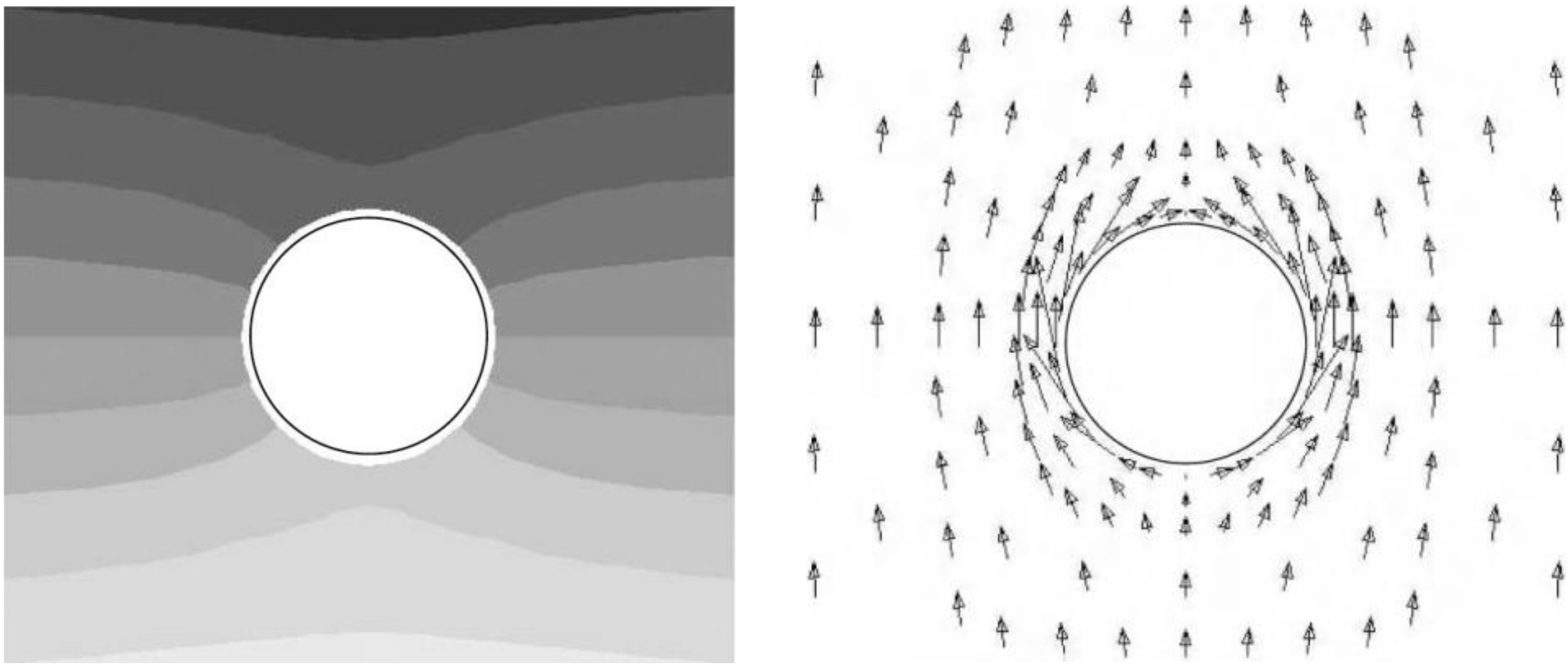
Boundary element method





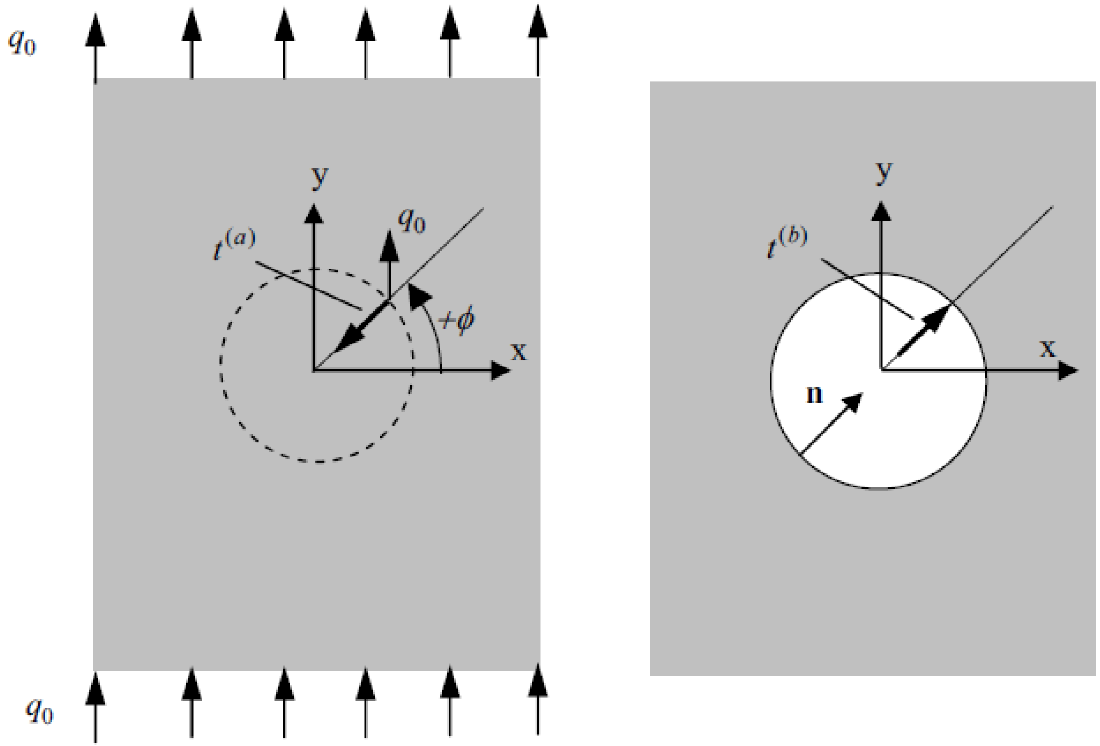
The better we discretize the boundary of the insulator, the more sources we can add, the better the solution…

Boundary element method



Temperature and flow vectors for the solved problem

Boundary element method



How to solve the problem on the right, where we have void and the boundary condition on the void is T=const.

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Finite Element Method in Geoengineering

Lecture 11-12. Other numerical methods

To learn today & next time…

The lectures should give you **overview of other numerical methods**

1. Discrete element method (DEM, also distinct element method)
   - assumptions, solutions, problmes & accuracy

2. Smoothed particle hydrodynamics (SPH)

3. Material Point Method (MPM)

4. Particle Finite Element Method in Geoengineering (PFEM)

5. XFEM – eXtended Finite Element Method in Geoengineering (XFEM)

Other existing methods, not covered today:

6. ALE , CLE – Coupled Lagrangian – Eurlerian FEM

7. Meshfree methods

Methods on continuous – discontinuous scale

FEM

CLE, ALE
Coupled Lagrangian – Eurlerian FEM

MPM

Smoothed Particle Hydrodynamics (SPH)

Discrete element method (DEM)

Continuous

Classical Meshfree methods

PFEM

XFEM

Discontinuous

Based on continuum mechanics

Not based on continuum mechanics

Methods on continuous – discontinuous scale

FEM

CLE, ALE: Coupled / Arbitrary Lagrangian – Eurlerian FEM

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Smoothed Particle Hydrodynamics (SPH)

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PFEM

XFEM

Discontinuous

Based on continuum mechanics

Not based on continuum mechanics

Discrete Element Method (DEM)

Also known as distinct element method

Idea: we model each grain of soil separately

We need to model all the contacts and contact behaviour

Each time step – we evaluate forces and velocities of all particles

Contact & contact forces are essential

Normally used for granular materials and atoms

Discrete Element Method (DEM)

Also known as distinct element method

Idea: we model each grain of soil separately

We need to model all the contacts and contact behaviour

Each time step – we evaluate forces and velocities of all particles. Method is time-step dependent

Contact & contact forces are essential

Due to simplified shapes of particles and simplified contact, method is known to be **problem and size dependent** (i.e. requires different parameters for different problems with same material)

Discrete Element Method (DEM)

Also known as distinct element method

Few simulations: Hannover, group of Prof. Wriggers

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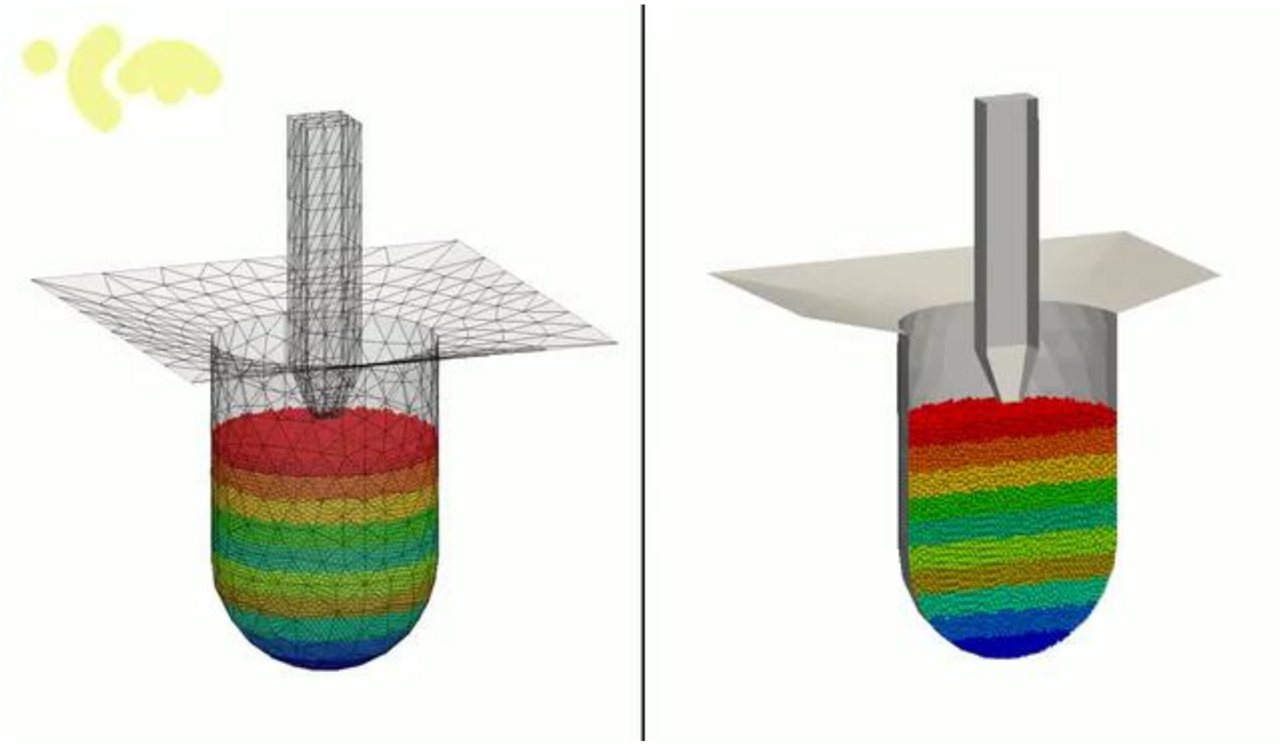

Few simulations: Hannover, group of Prof. Wriggers

Discrete Element Method (DEM)

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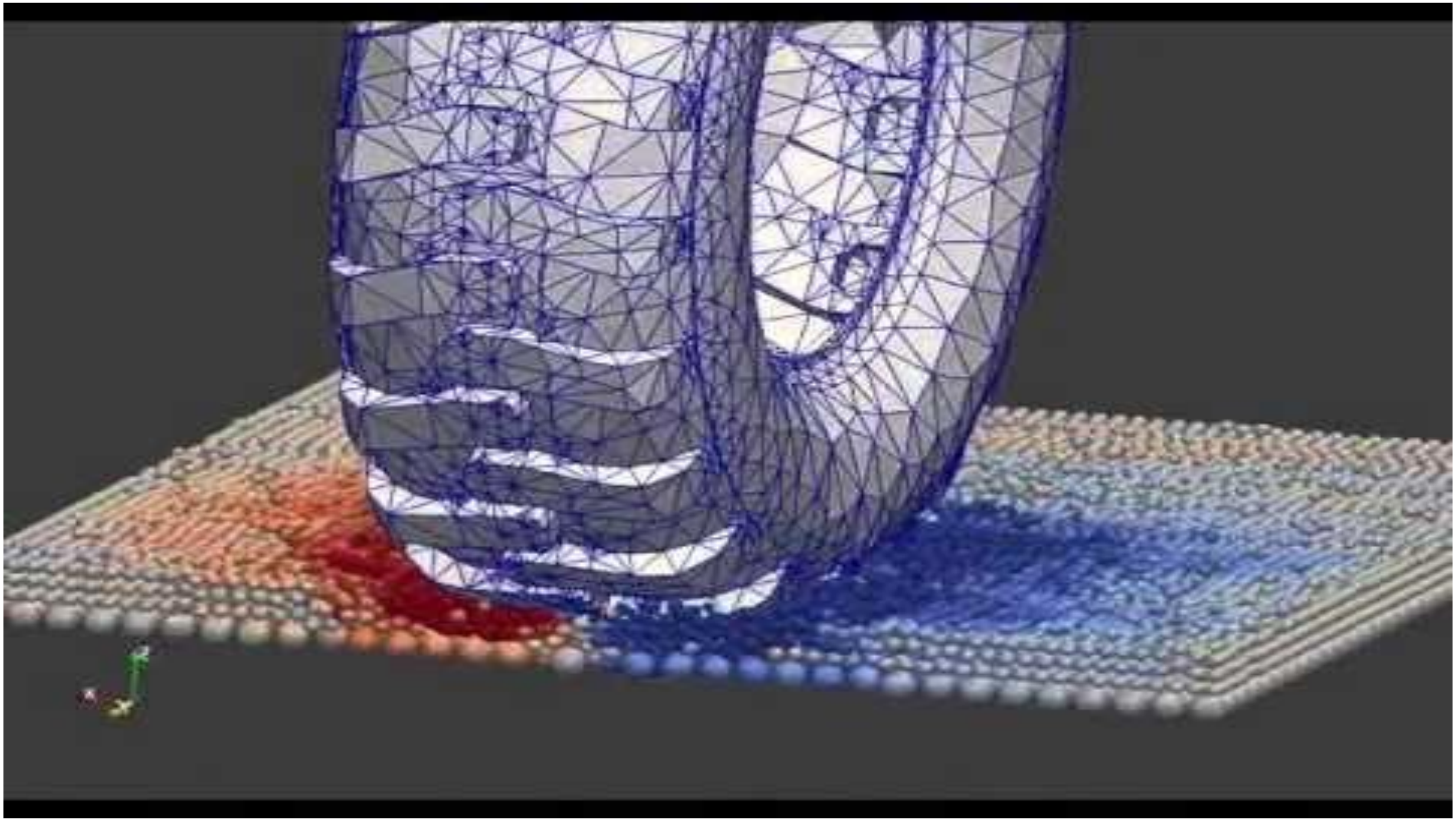


DEM FEM coupling

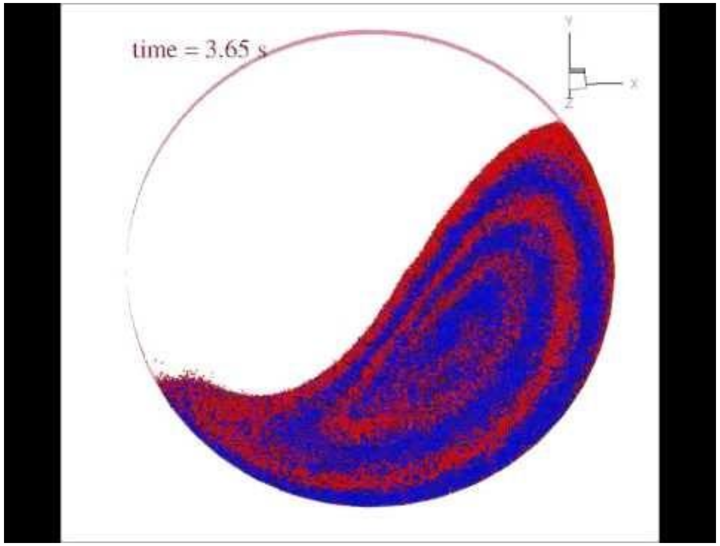
Discrete Element Method (DEM)

Also known as distinct element method

Convergence:

UoM presentation - DEM convergence.pdf

More simulations:



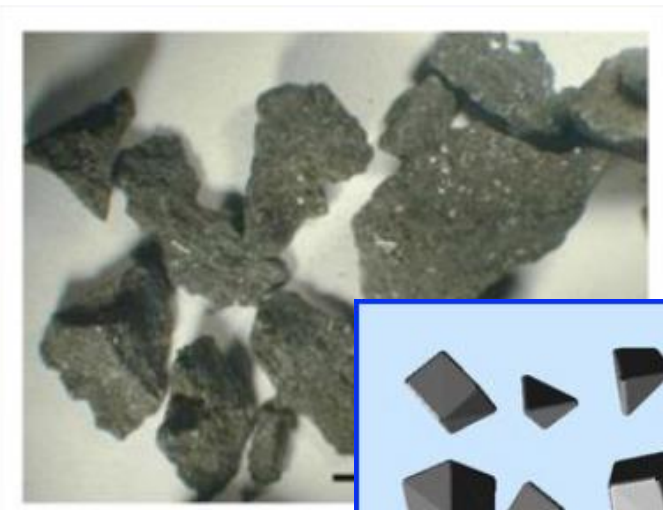
In short, 2D quite does not work and 3D is very expensive… and even then it may not work…

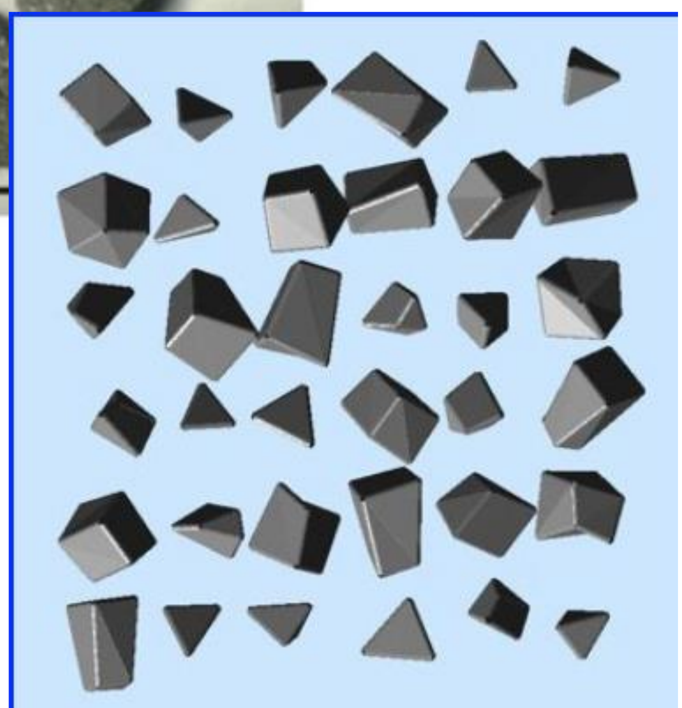
Discrete Element Method (DEM)

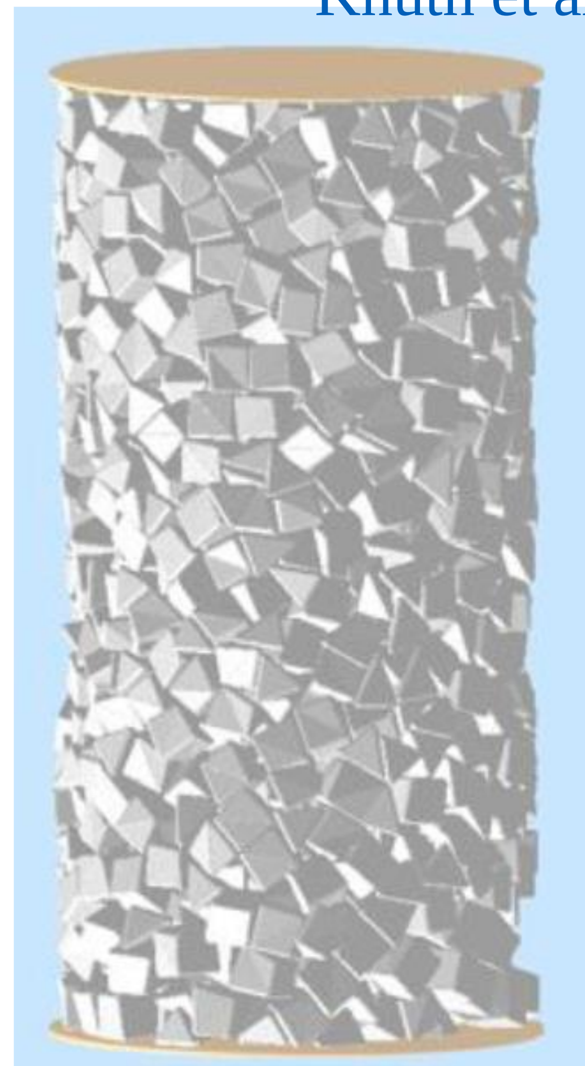
Also known as distinct element method

Knuth et al. 2010

Shape effects…







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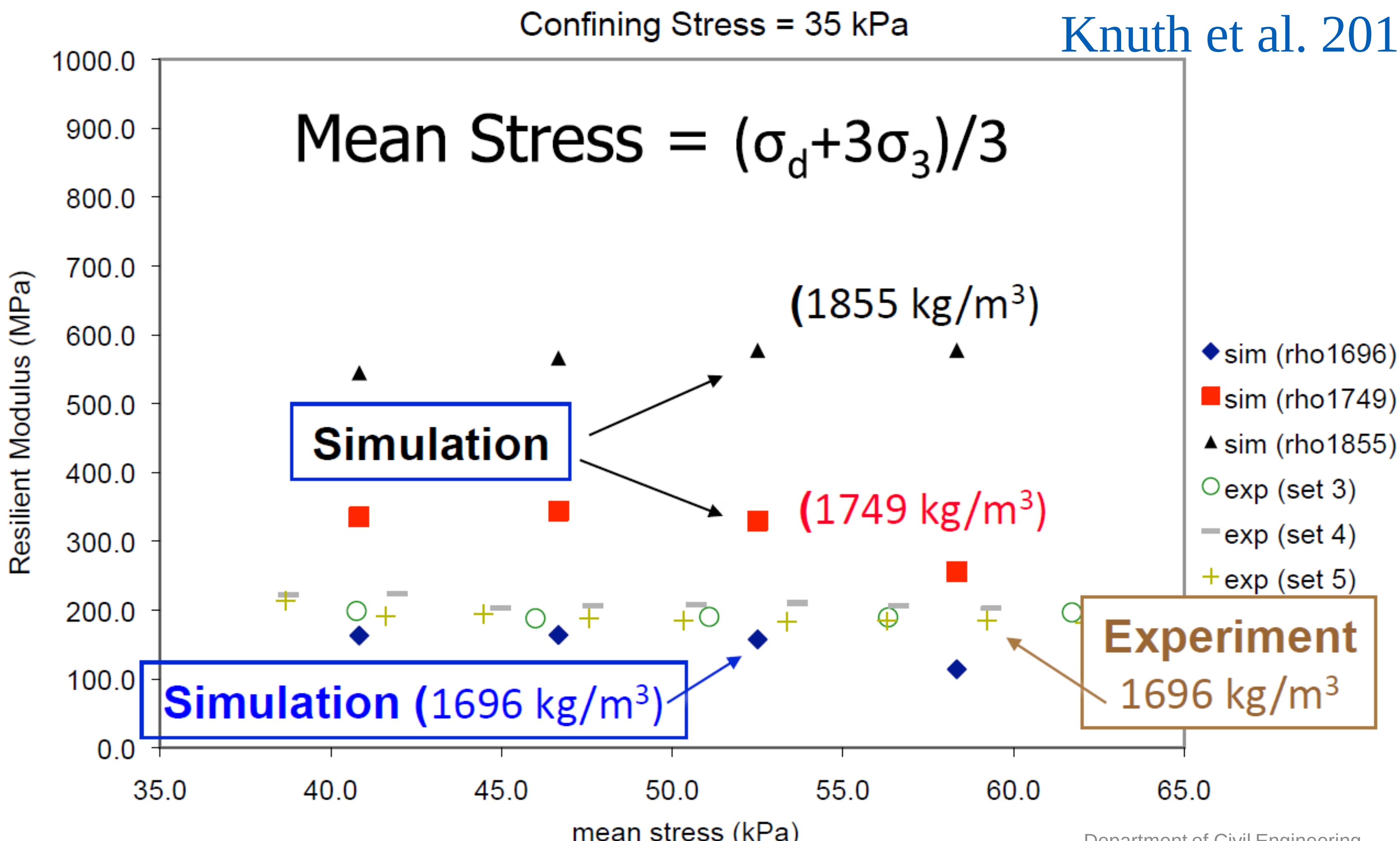


Size effects…

Discrete Element Method (DEM)

Also known as distinct element method

Software:

YADE (free & open source)

- generally non-cohesive materials, or materials with some cohesion

Quite a lot of other software…

3DEC – by ITASCA, now popular in mining industry

Discrete Element Method (DEM)

Also known as distinct element method

Wait, using DEM for non-granular materials???

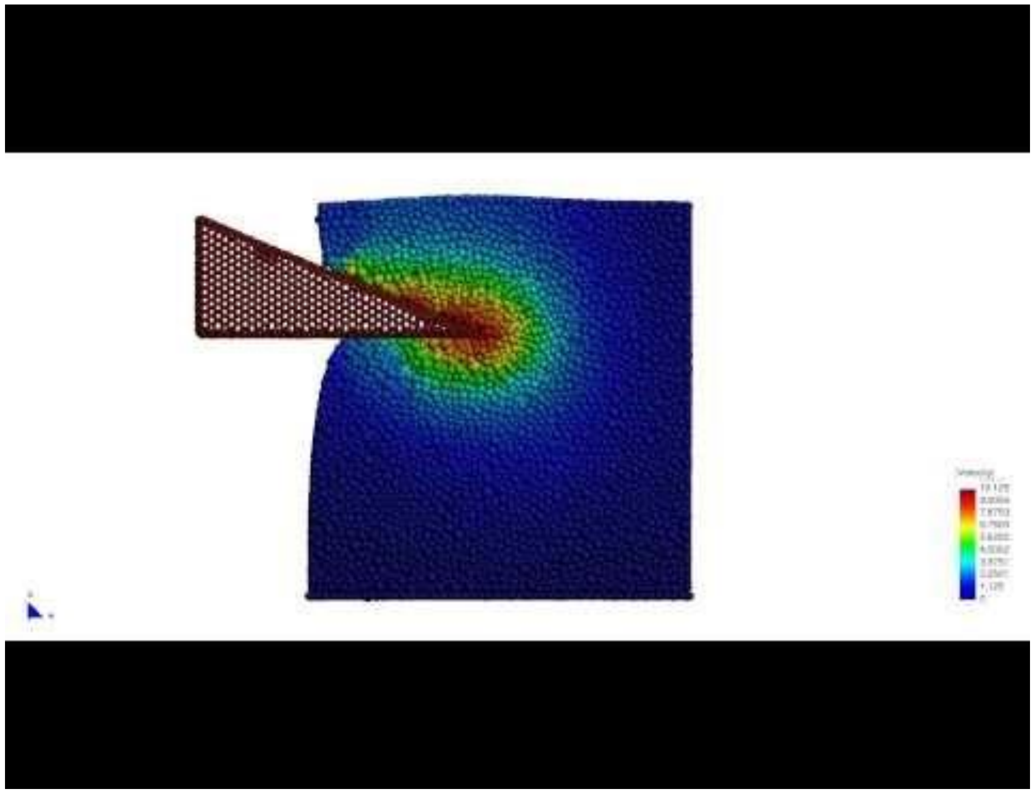
Of course leads to lots of problems and issues:

- in reality to get decent results cohesion between grains should be scale dependent
- contact between assemblies of non – smooth particles (i.e. when crack is formed) is again problematic (generally, to get real surface with required roughness, **MANY** particles are needed, and other solutions do not work well
- currently used rather for flow than anything else…

Discrete Element Method (DEM)

Also known as distinct element method

Wait, using DEM for non-granular materials???



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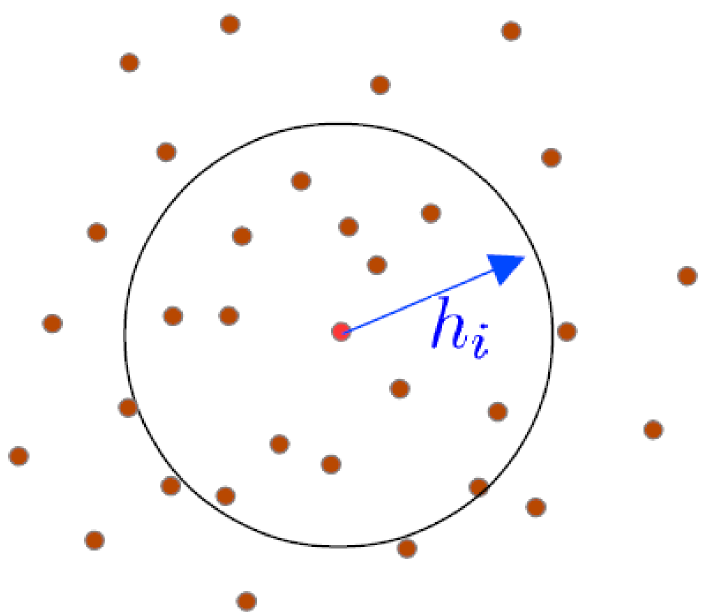
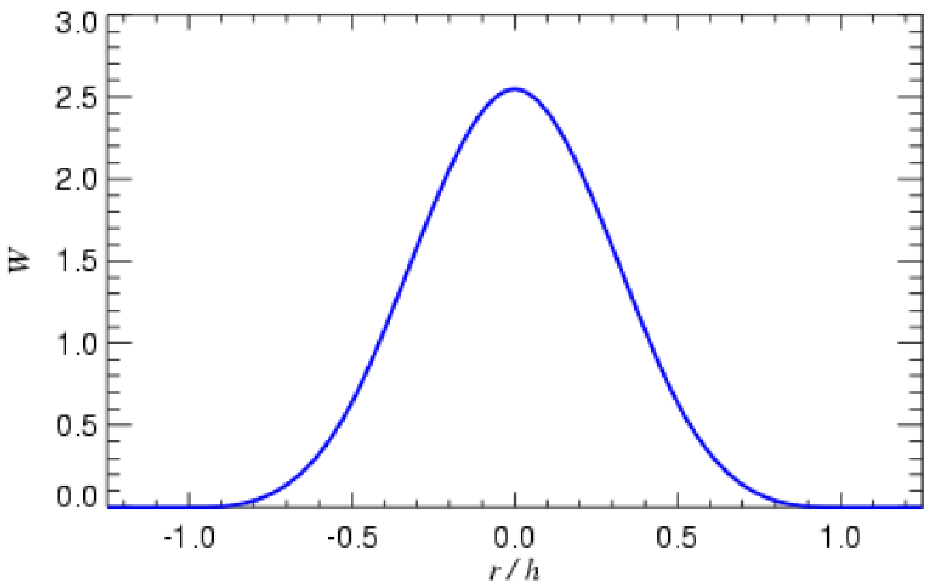
Smoothed particle hydrodynamics (SPH)

Smoothed Particle Hydrodynamics (SPH)



Density computed via weighted sum over neighbouring particles…



Smoothed Particle Hydrodynamics (SPH)

Problems:

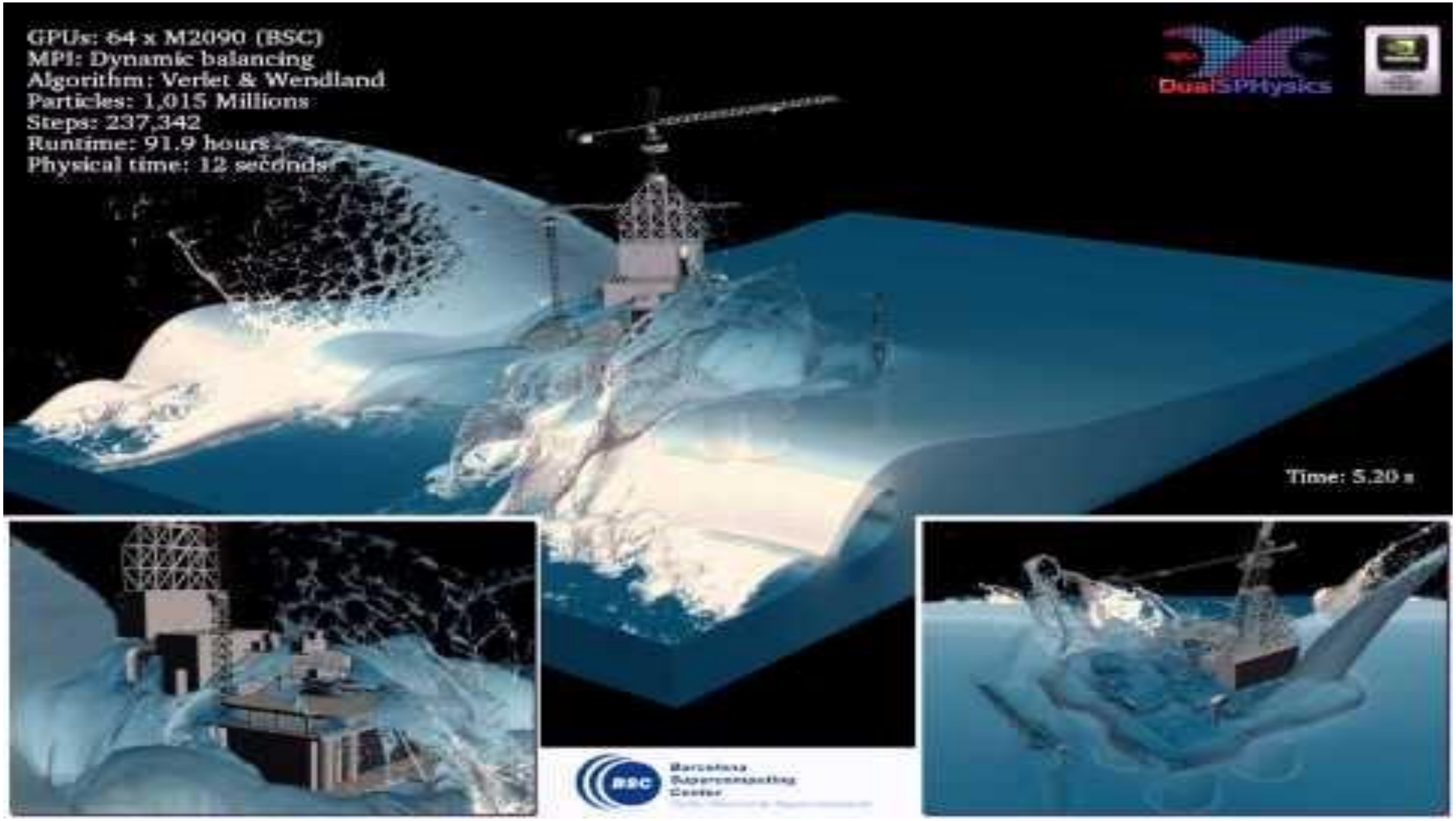
- boundary conditions
- numerical noises
  - sometimes – velocity noise of few percent of local sound speed…
  - instabilities over contact discontinuities

- requires high artificial viscosity (to mute errors), giving high viscosity of the system, leading to errors

Benefits:

- versatile, simple, good conservation properties
- quite robust

Smoothed Particle Hydrodynamics (SPH)



Barcelona supercomputing centre

Smoothed Particle Hydrodynamics (SPH)

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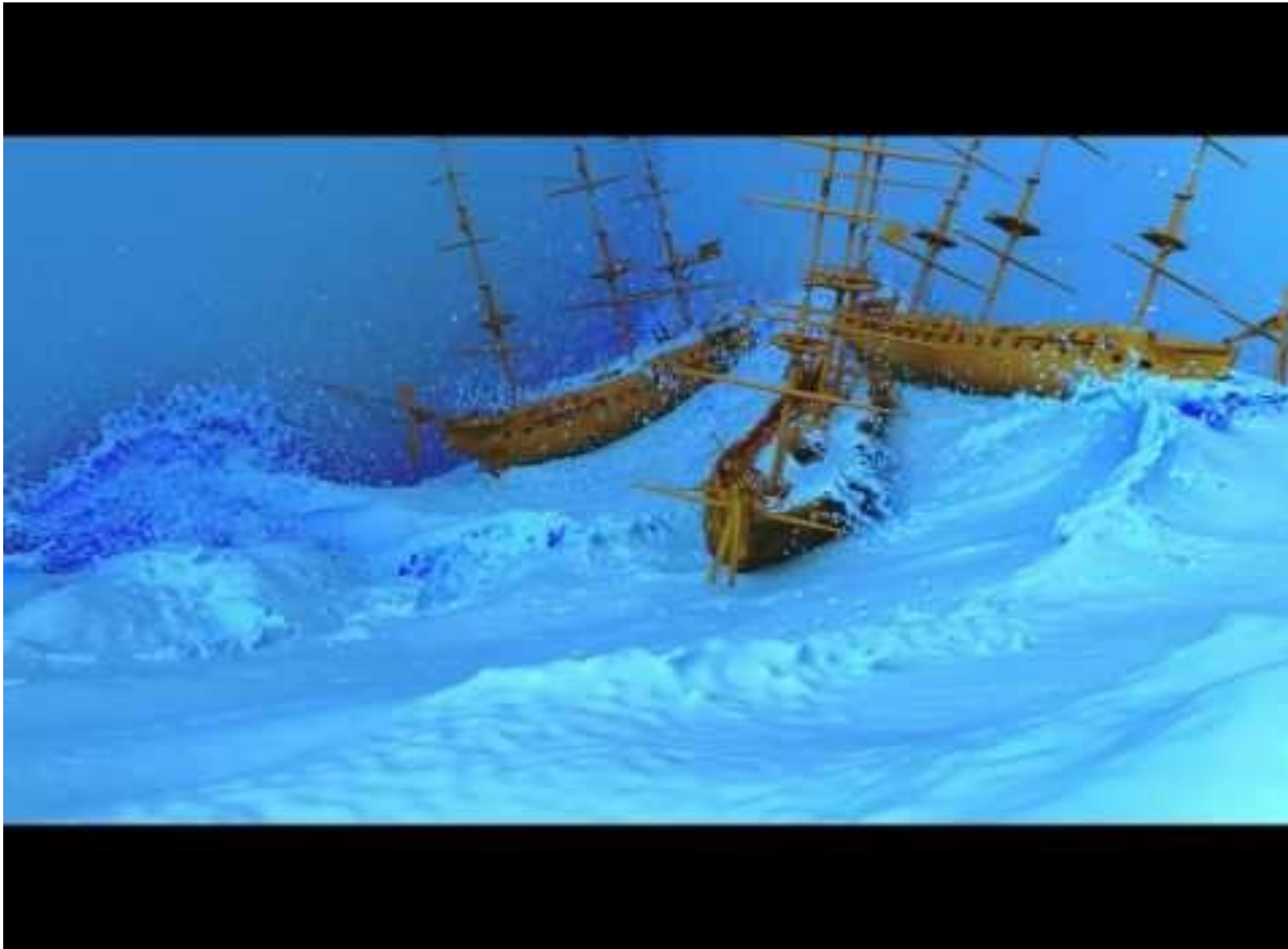

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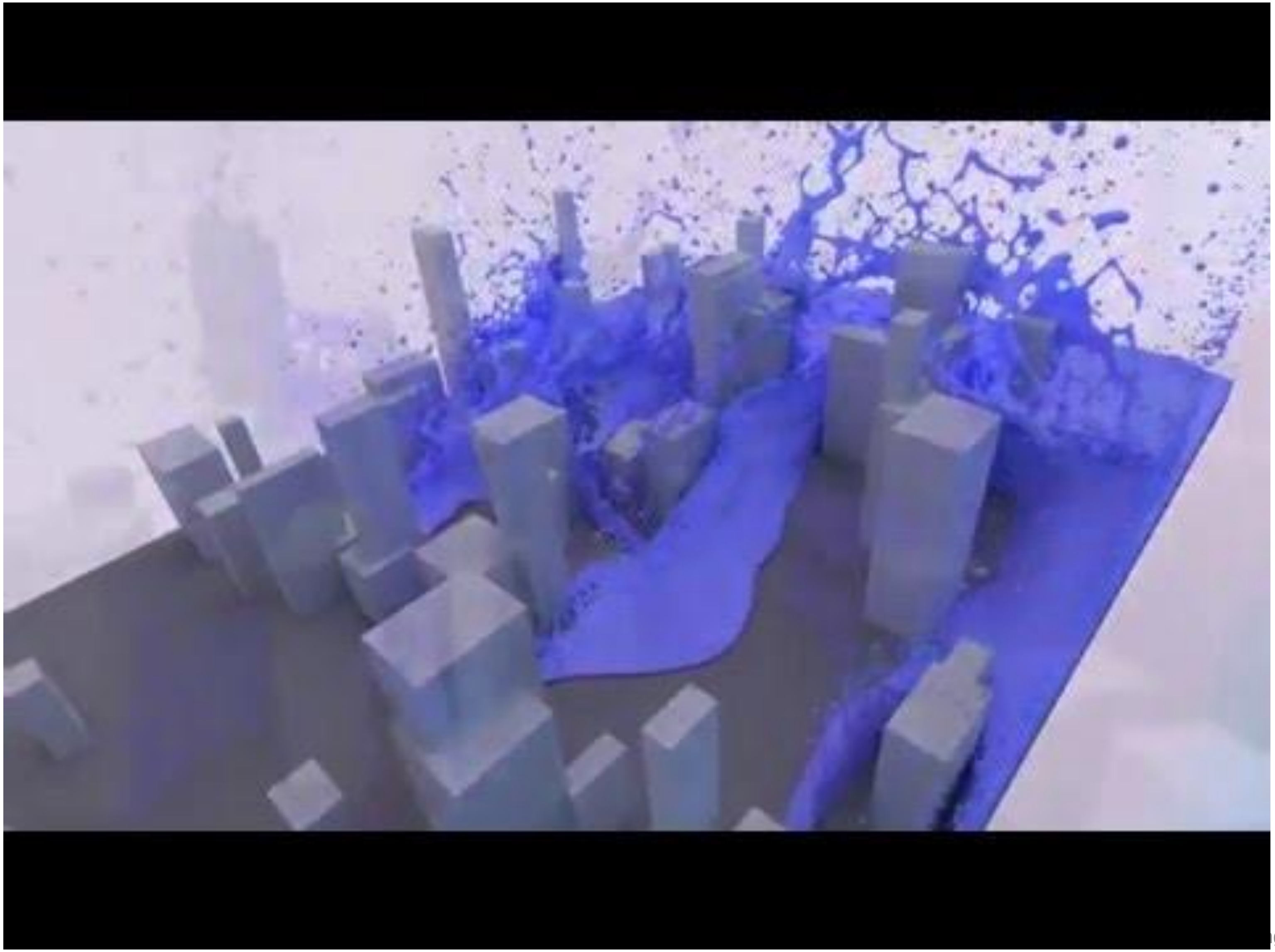
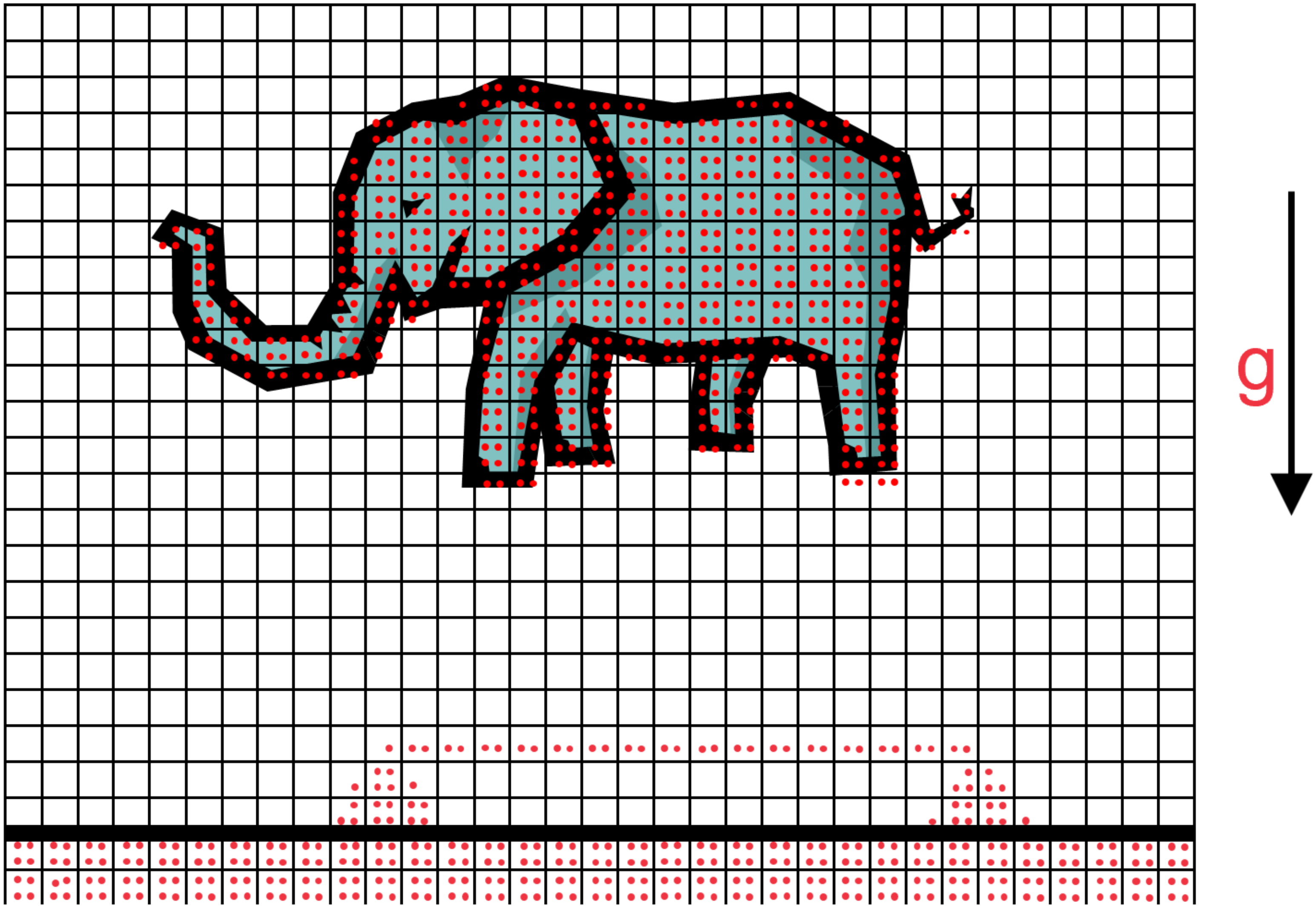


Material Point Method MPM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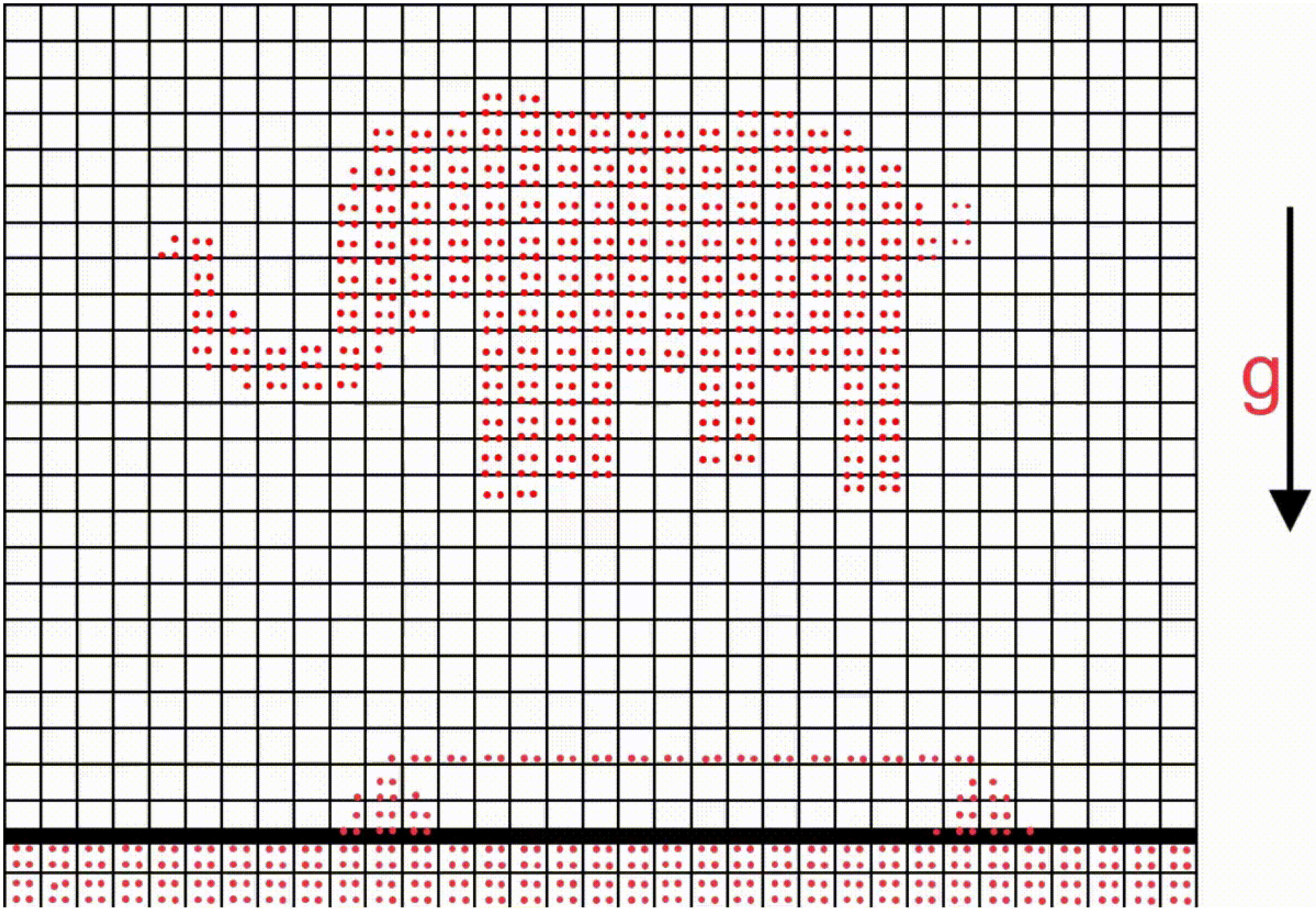
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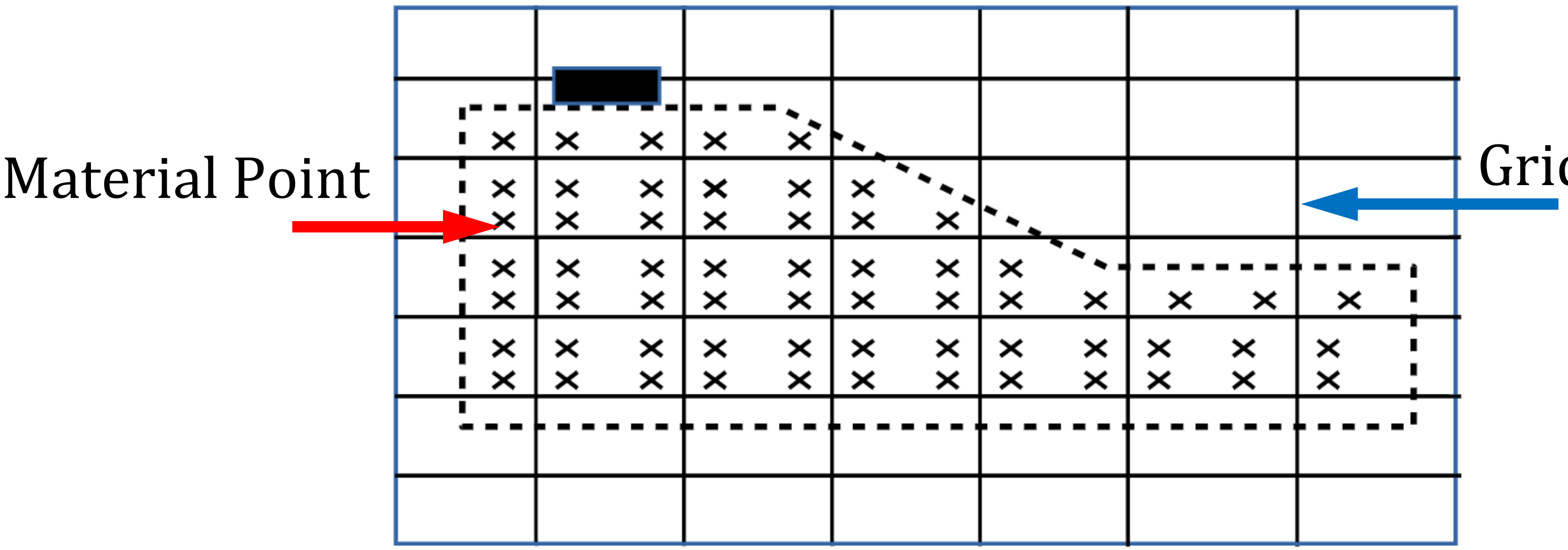
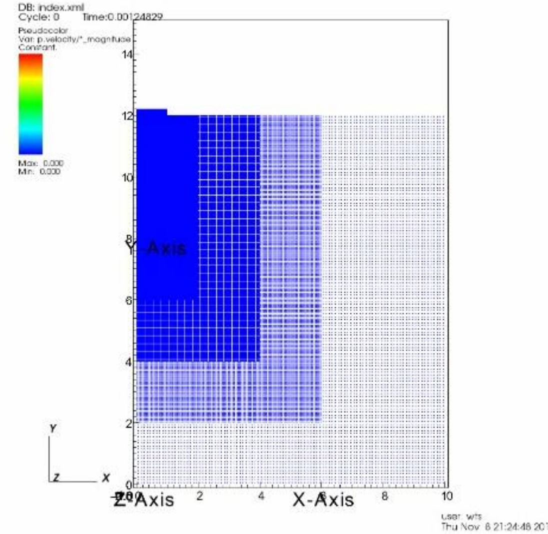


Based on C.E. Augarde et al., 2021

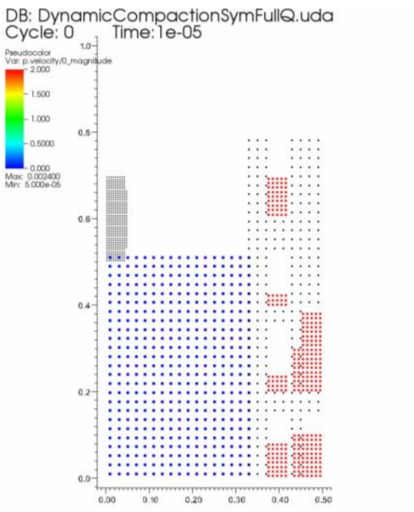
Some more movies ;)

Material Point Method:

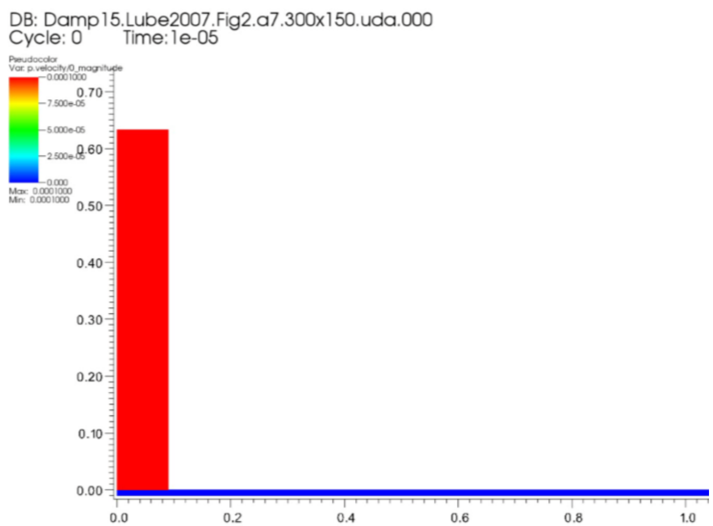
- good for very large deformations
- explicit
- dynamics
- continuum method (like FEM)



Sołowski & Sloan (2015)



Sołowski et al. (2013)



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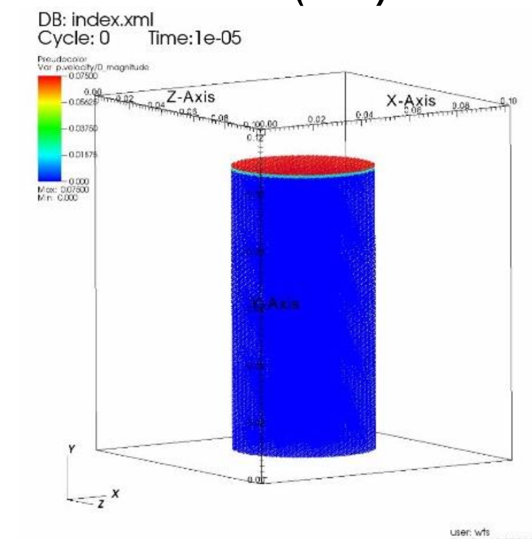

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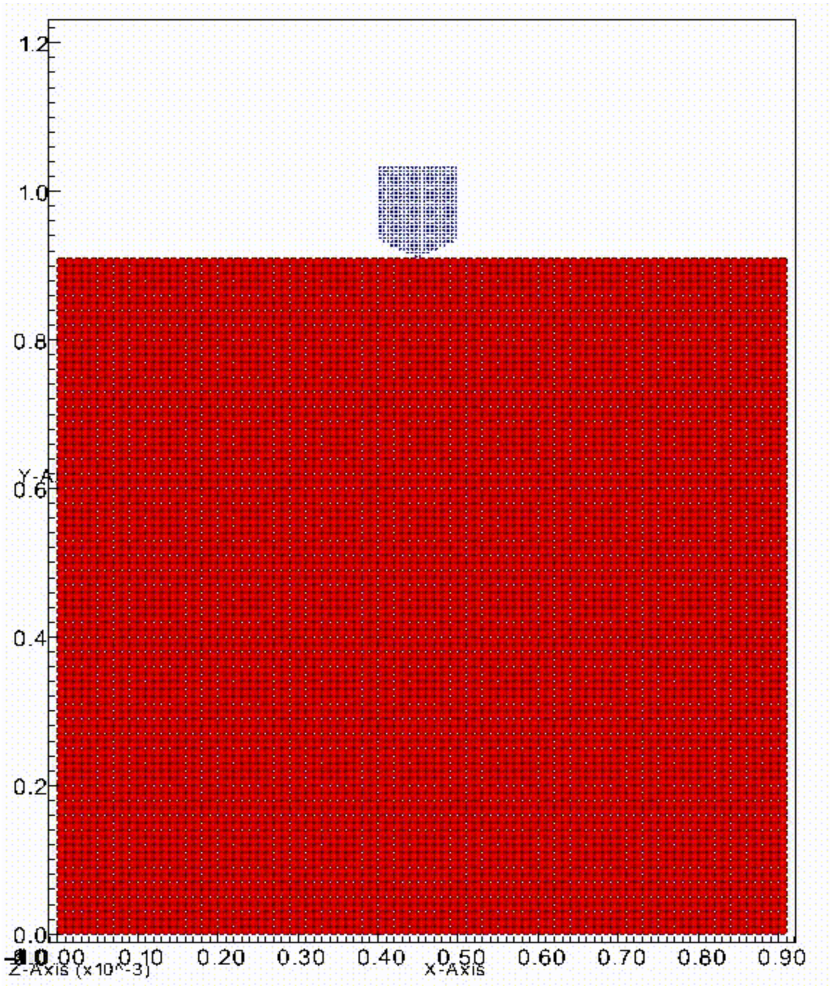
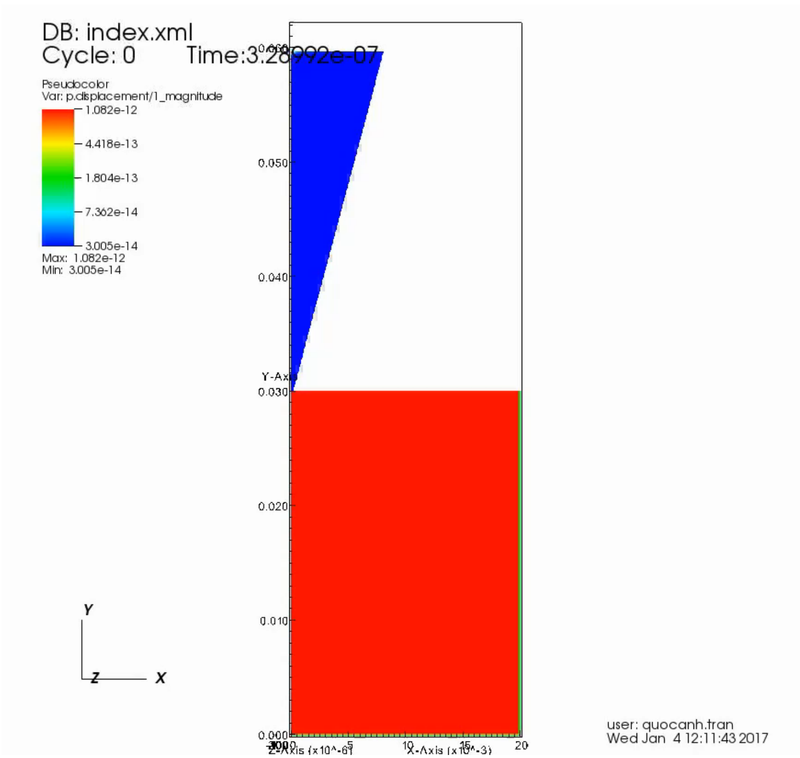
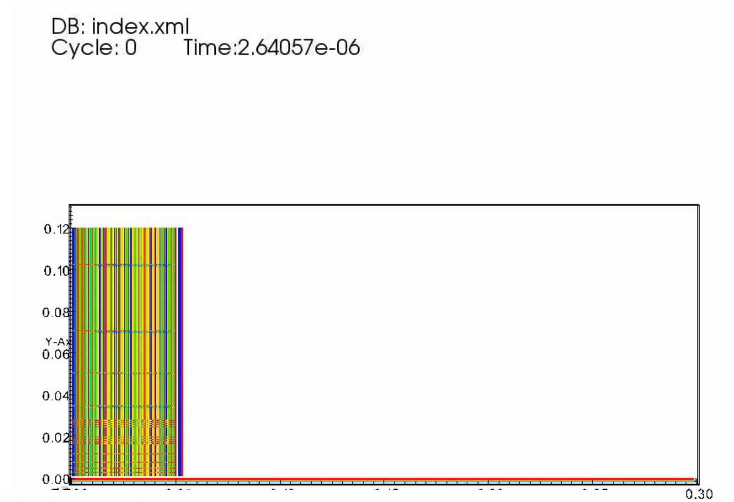
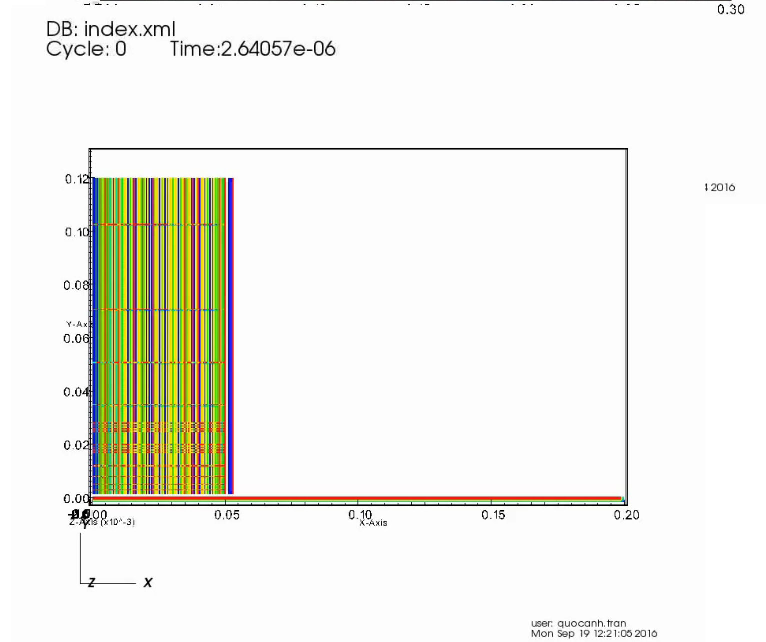


Seyedan and Sołowski (2022?)

Some more movies ;)

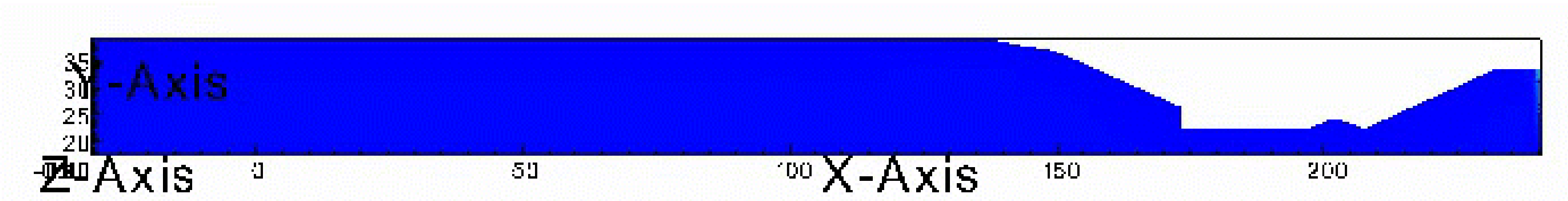


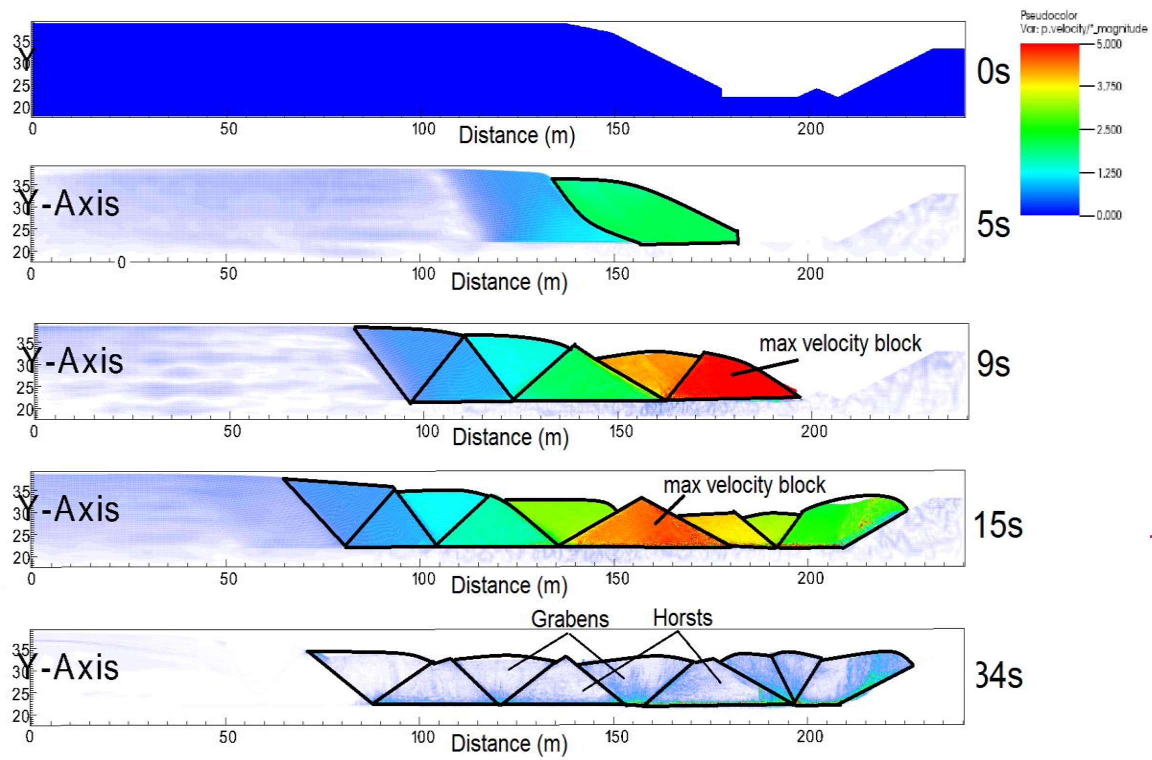




Tran et al. (2017)

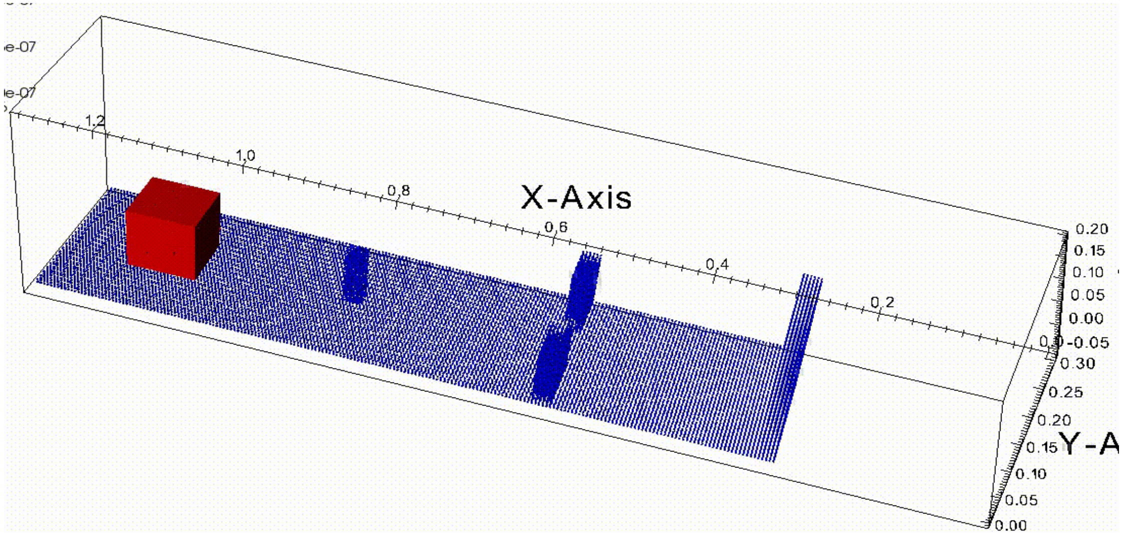
Some more movies ;)

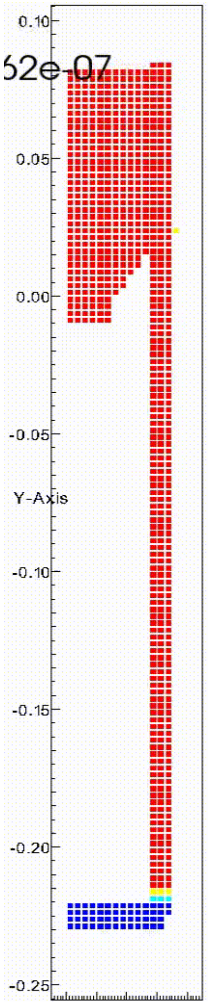




Tran & Sołowski (2019)

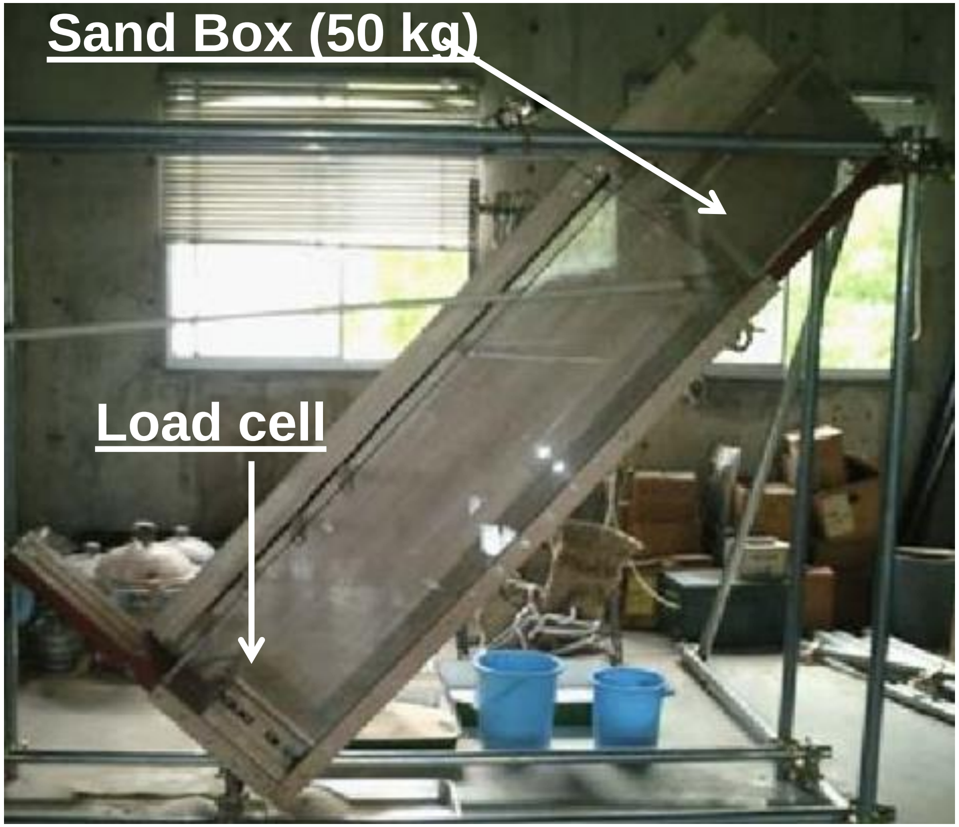
Some more movies ;)

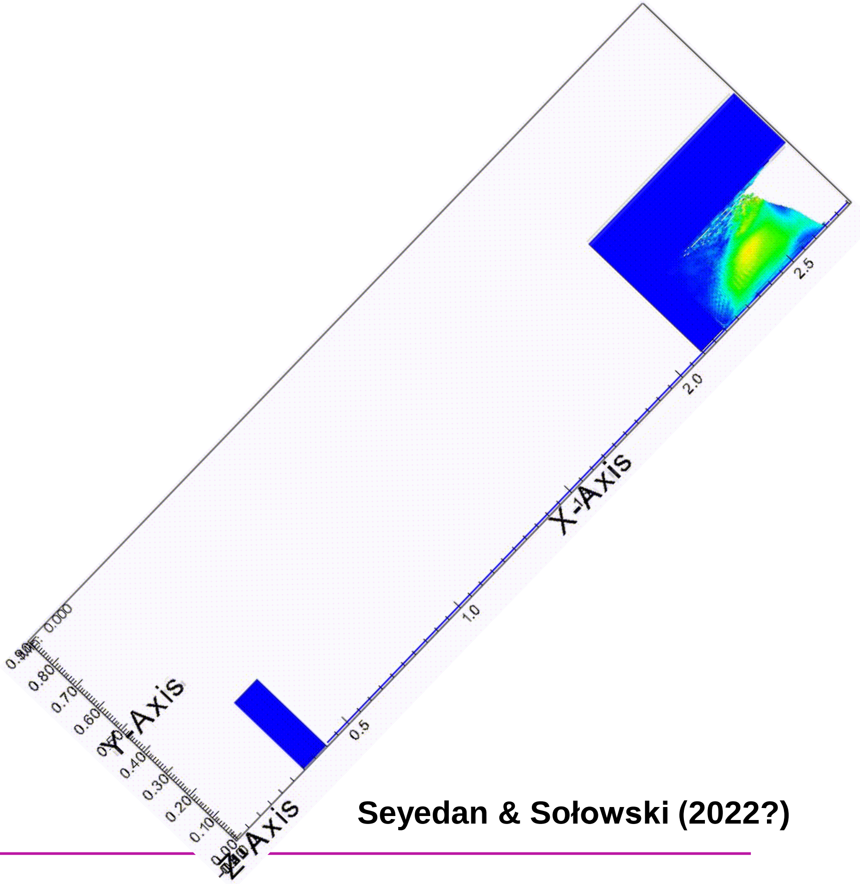




Seyedan & Sołowski (2022?)

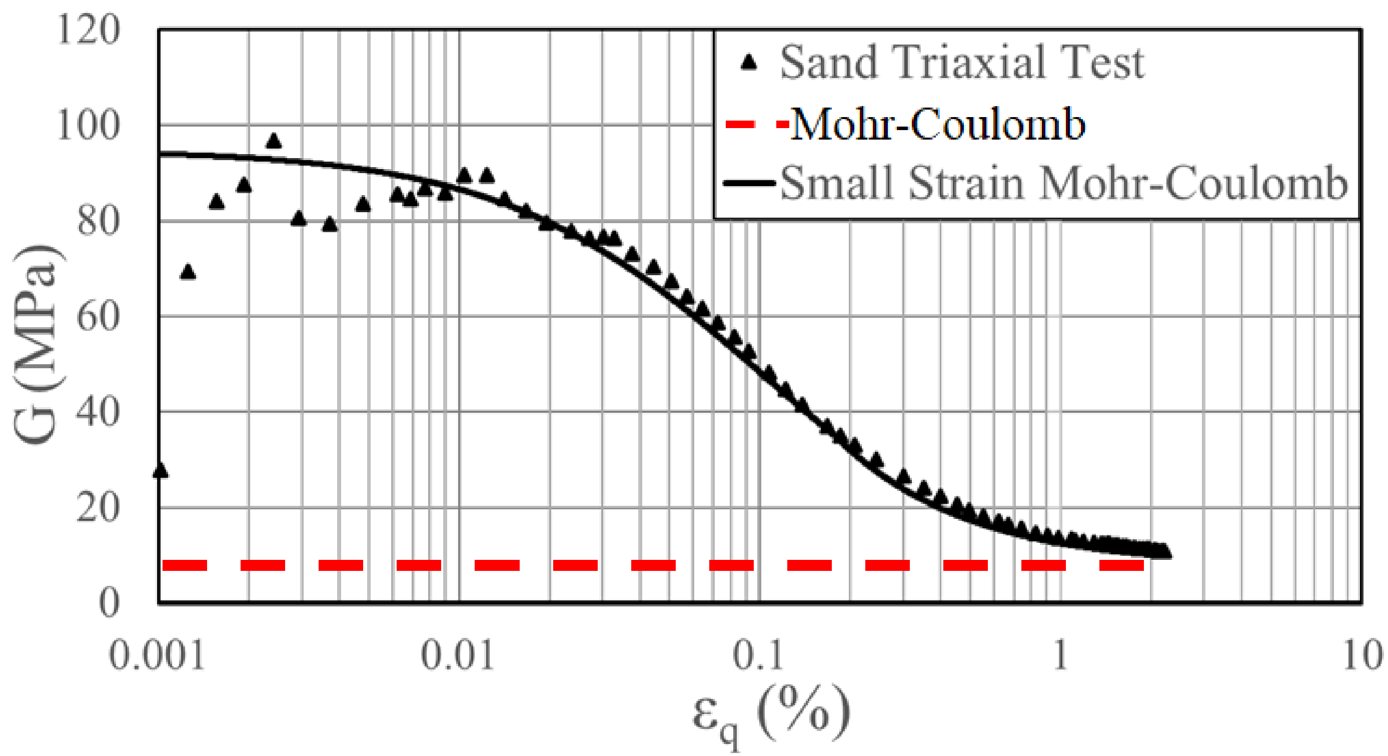
Some more movies ;)



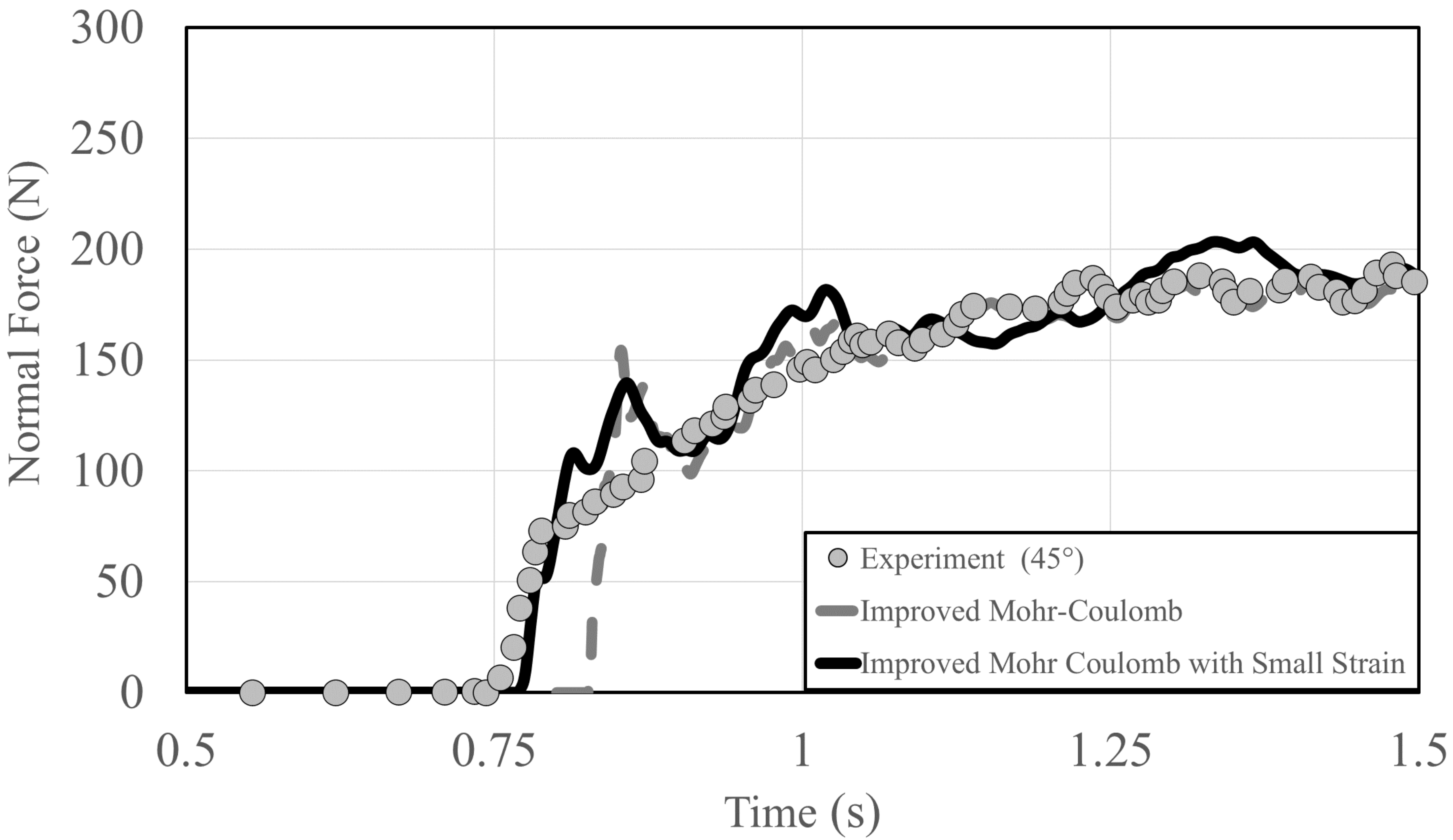


Seyedan & Sołowski (2022?)

A yield surface to capture small strain behavior



Seyedan & Sołowski (2022?)



300
250
200
150
100
50
0
Normal Force (N)
0.5
0.75
1
1.25
1.5
Time (s)
Experiment (45°)
Improved Mohr-Coulomb
Improved Mohr Coulomb with Small Strain

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XFEM

with thanks to K. Agathos (Aristotle U. of Thessaloniki) and E. Chatzi, (IBK, D-BAUG, ETH Zurich)

XFEM – eXtended Finite Element Method

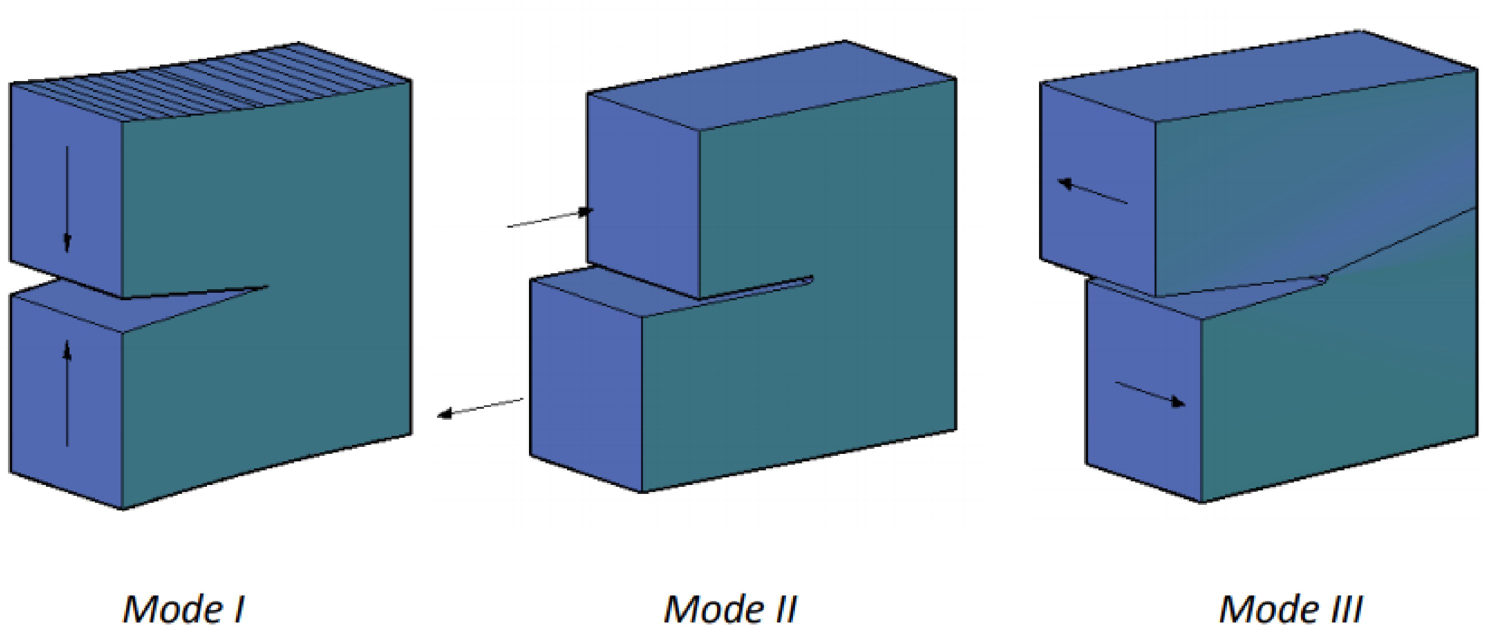
Aim: to introduce discontinuities into continuous FEM

- Strong discontinuity: crack - jump in displacements
- Weak discontinuity – jump in strains

Used to determine displacement, strain and stress fields in structures with cracks and small holes. Allows for discontinuous displacements and strain fields

XFEM – eXtended Finite Element Method

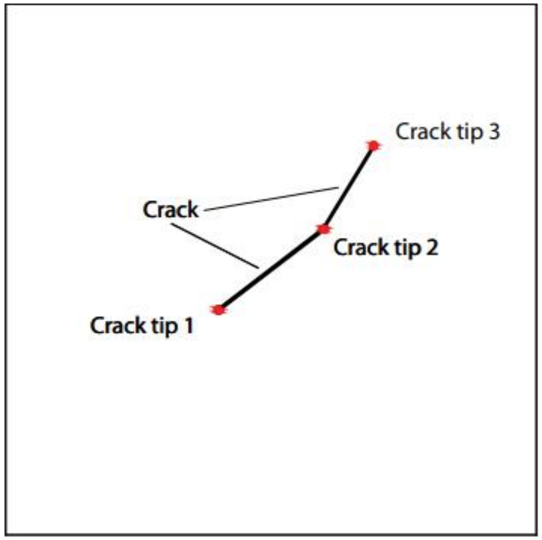
Aim: to introduce discontinuities into continuous FEM

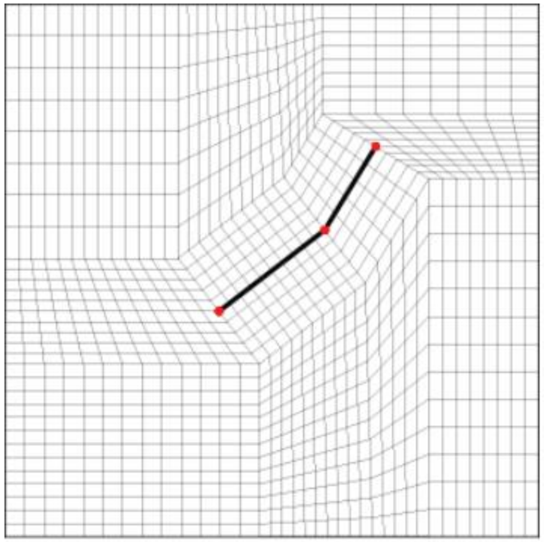


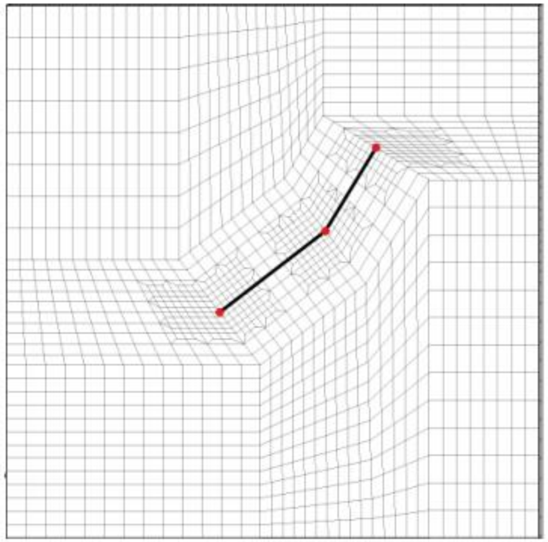
© Agathos and Chatzi

XFEM – eXtended Finite Element Method

To model the crack, we need nodes placed across the crack and on the crack tips







© Agathos and Chatzi

XFEM – Jump enrichment

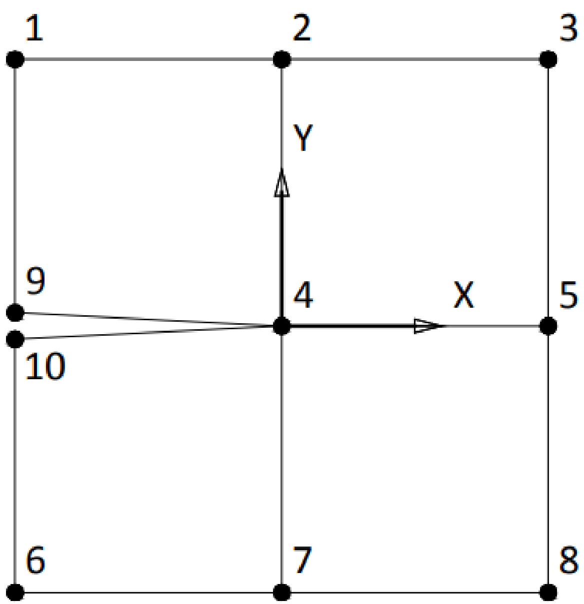
When we have a crack, we have jump in displacements. However, we want to describe it with a continuous mesh, i.e. without physically modelling crack width.

For that, we enrich the element nodes with jump function for displacements. At one side of the node, it has a different value than at the other side of the node.

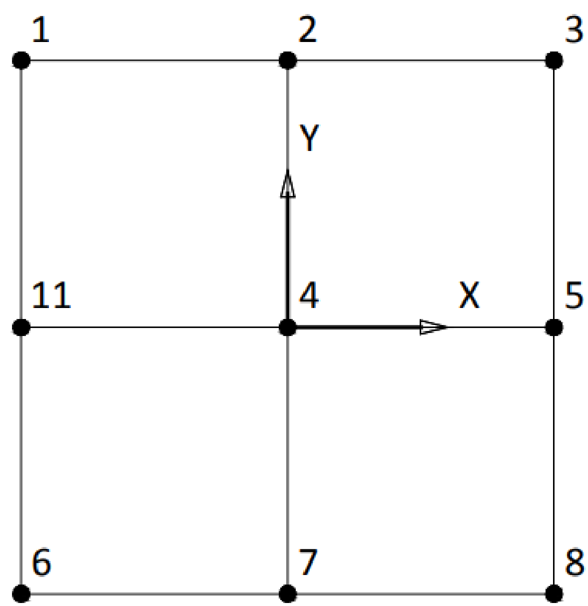
Technically we use Heaviside function $H(x)$ for that…

XFEM – Jump enrichment

In other words, we want to represent the situation in Mesh 1 (physical crack), with Mesh 2



Mesh 1



Mesh 2

XFEM – Jump enrichment

The displacements at any point (and in particular in nodes 9 and 10) are:

$$\underbrace{\Delta \mathbf{u}}_{\text{displacements}} = \sum_{i=1}^{10} \underbrace{\mathbf{N}}_{\text{Shape functions values}} \underbrace{\Delta \mathbf{d}}_{\text{Vector containing increments of displacements of element nodes}}$$

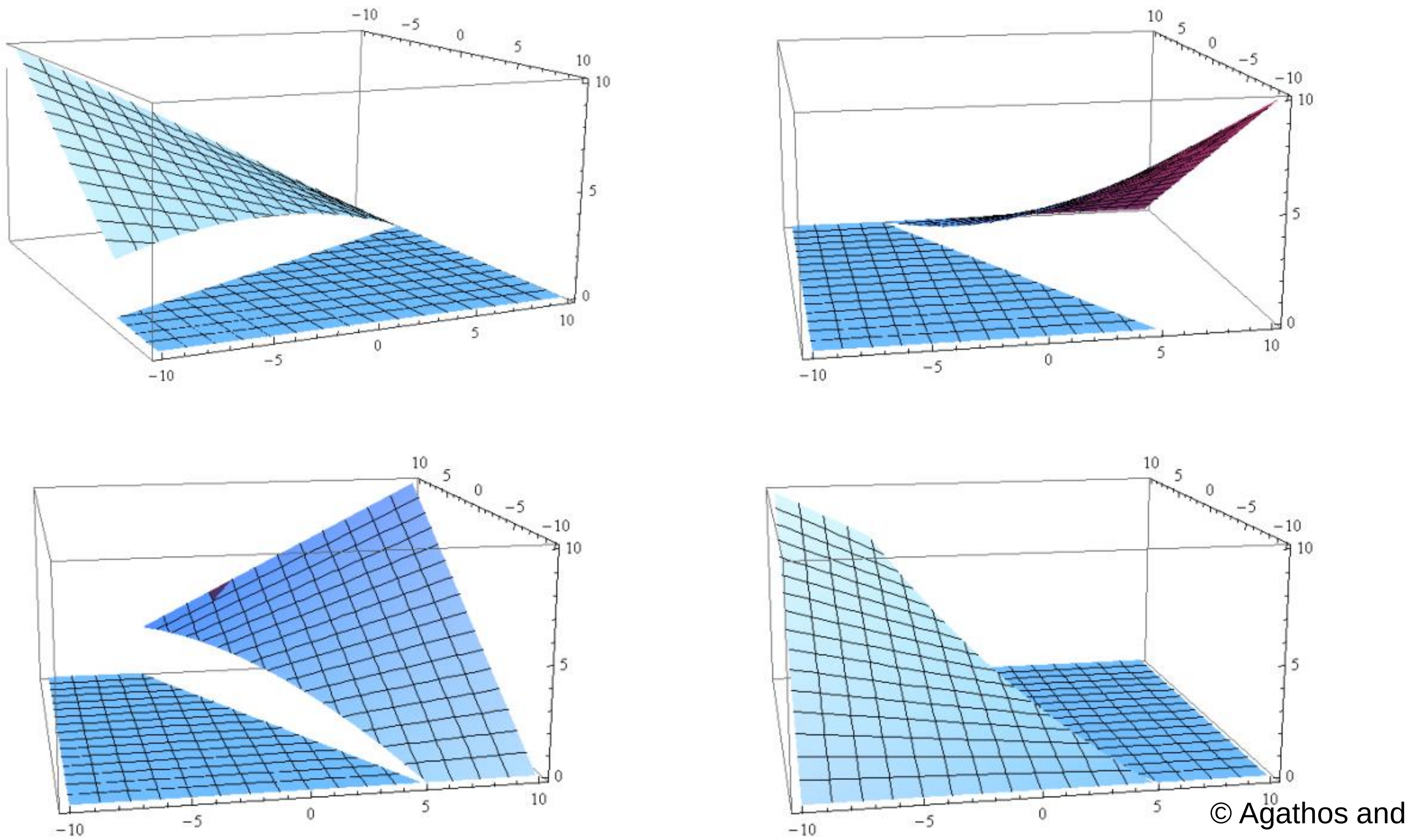
Defining **a**=0,5 ($\mathbf{d}_9$+$\mathbf{d}_{10}$) and **b**=0,5 ($\mathbf{d}_9$-$\mathbf{d}_{10}$) we get

$$\underbrace{\Delta \mathbf{u}}_{\text{displacements}} = \sum_{i=1}^{8} \underbrace{\mathbf{N}}_{\text{Shape functions values}} \underbrace{\Delta \mathbf{d}}_{\text{Vector containing increments of displacements of element nodes}} + \mathbf{a}(N_9 + N_{10}) + \mathbf{b}(N_9 + N_{10}) H(x) =$$

$$= \sum_{i=1}^{8} \underbrace{\mathbf{N}}_{\text{Shape functions values}} \underbrace{\hat{\Delta \mathbf{d}}}_{\text{Vector containing increments of displacements of element nodes}} + u_{11}(N_{11}) + \mathbf{b}(N_{11}) H(x)$$

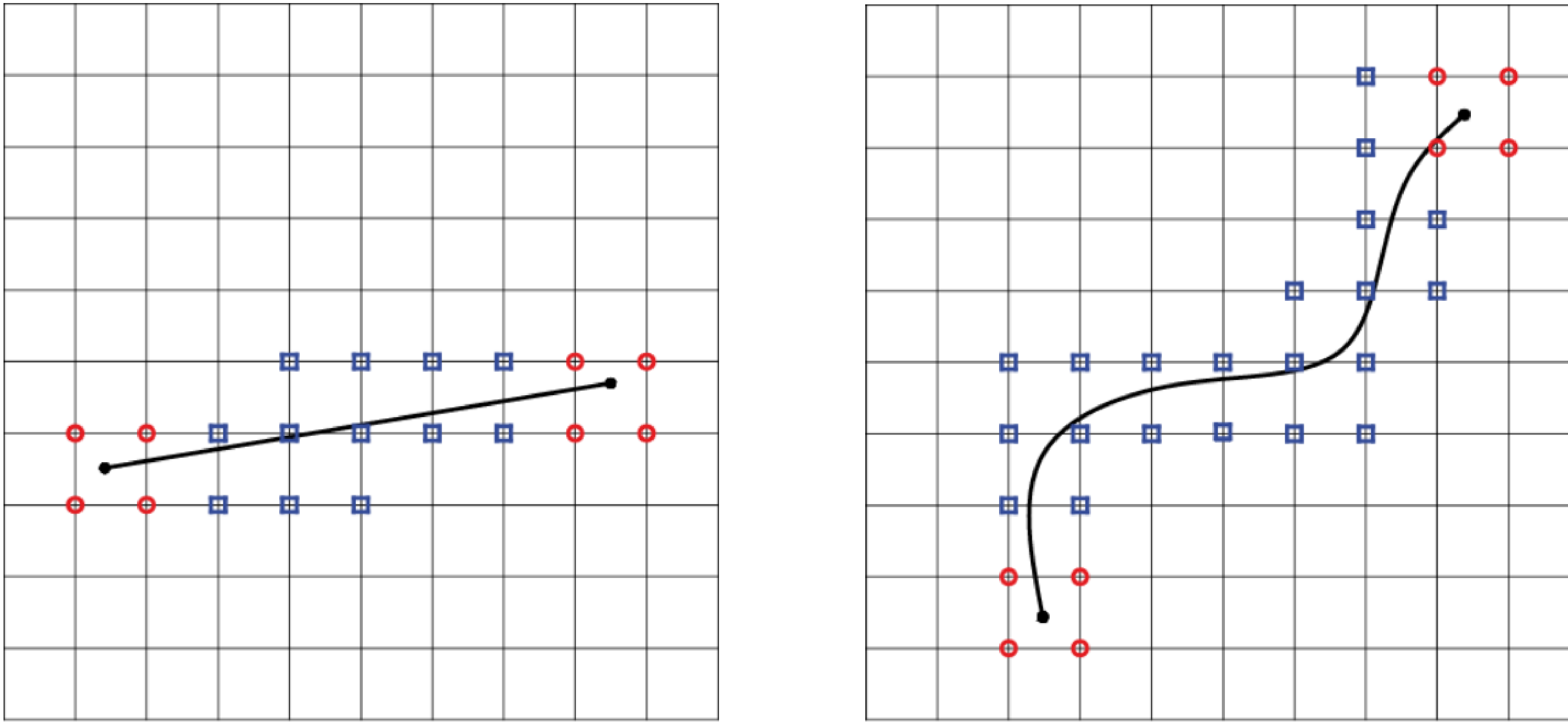
H(x)=1 for y>0 and -1 for y<0

XFEM – Jump enrichment



Jump enrichment in action ☺

XFEM – Jump enrichment



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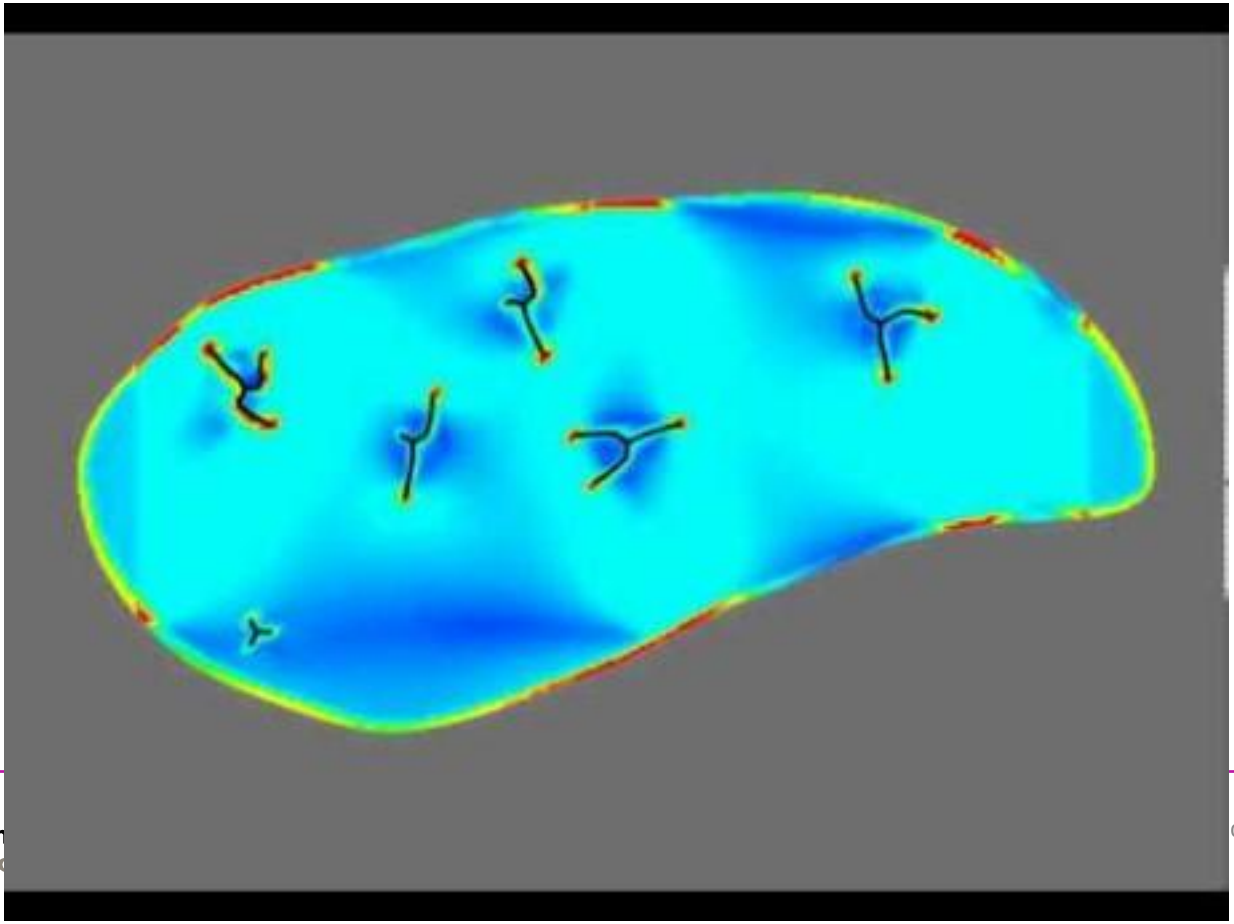
tip enrichment

jump enrichment

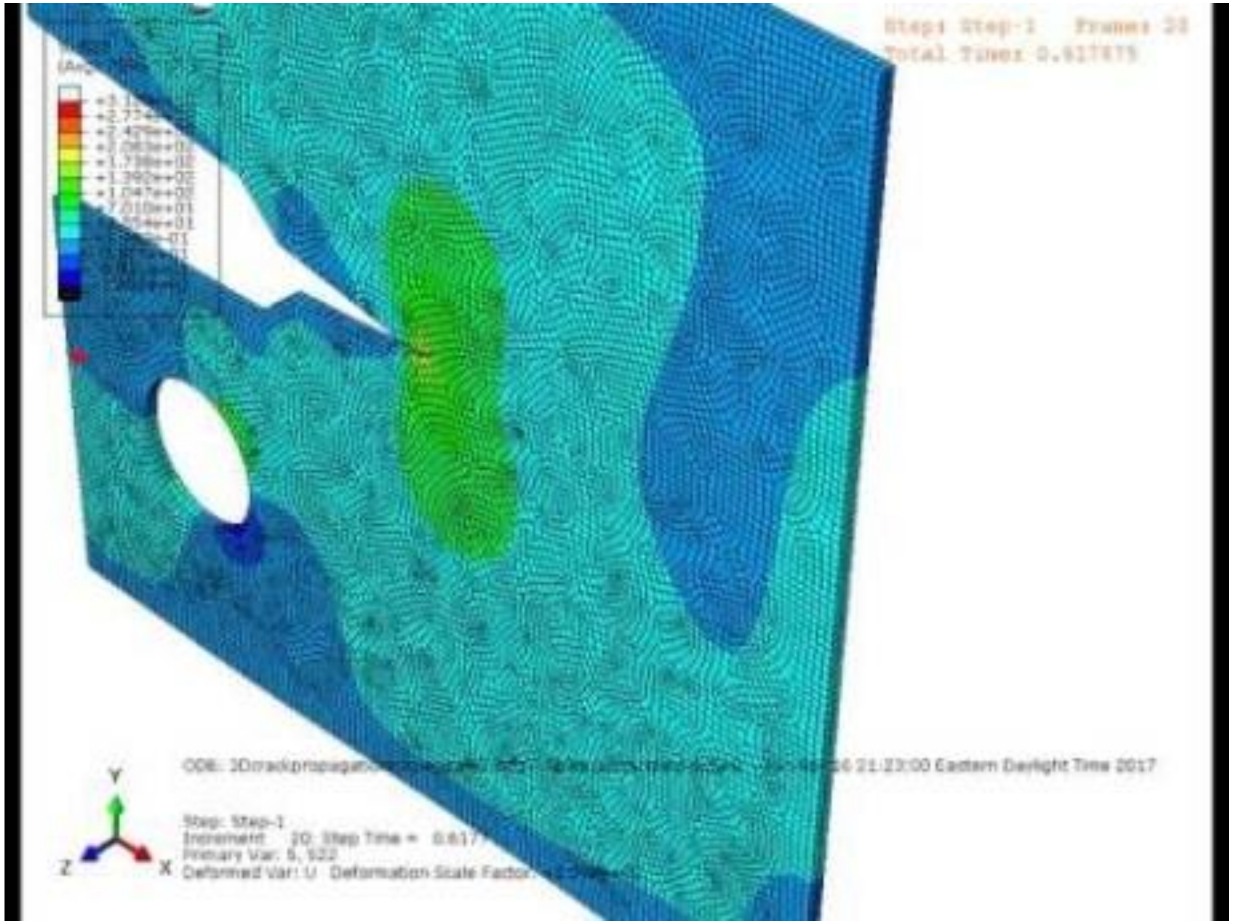
XFEM – abilities

The method – with extensions – can deal with crack propagation, crack branching and intersecting etc.
Also can be used with plasticity and in dynamic problems



https://youtu.be/q6VY8qHTYco

XFEM – abilities



https://youtu.be/eKhrRpwxOq0

Thank you