Mathematics for Economists

Problem Set 5
Due date: Wednesday 2.11 at 12.15

Exercise 1

Let $u(x_1, x_2) = \ln x_1 + \ln x_2$ be a consumer's utility function. Use the Lagrangian to solve the following utility maximization problem:

$$\max_{x_1, x_2} \quad u(x_1, x_2)$$
$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 = w$$

where $p_1 > 0$ and $p_2 > 0$ are commodity prices, and $w > 0$ is income. You may assume that the solution satisfies $x_1, x_2 > 0$.

Exercise 2

Find the solutions of the first order conditions of $\max x_1 x_2$ subject to $x_1^2 + x_2^2 = 1$. Do the second order sufficient optimlaity conditions hold? How can you deduce the global optimality of a critical point of the Lagrange function?

Exercise 3

Let $U(c_1, c_2) = u(c_0) + \delta u(c_1)$ be a consumer's utility function over consumption in two periods; c_0 is the money spent on consumption in period 0 and c_1 is the money spent on consumption in period 1. Assume that all the consumer's income is earned in the first period, let I_0 be the income.
a) Formulate an intertemopral budget constraint for the consumer when the interest rate on savings is r (i.e., amount s invested in period 0 yields $(1 + r)s$ in period 1). Note the intertemporal budget constraint can be written in the form $p_0 c_0 + p_1 c_1 = p_0 I_0$ with a suitable choice of paramteres p_0 and p_1.

b) Use the Lagrangian show that the consumer's utility maximization problem over the intertemporal budget constraint satisfies:

$$MRS(c_0, c_1) = p_0/p_1$$
$$p_0 c_0 + p_1 c_1 = p_0 I_0$$

where $p_0 > 0$ and $p_1 > 0$ are parameters you obtain in your formulation of the intertemporal budget constraint, and $MRS(c_0, c_1) = [\partial U(c_0, c_1)/\partial c_0]/[\partial U(c_0, c_1)/\partial c_1]$ (marginal rate of substitution).

Exercise 4

Let $f(x, y, z) = x + 2z$ be a function defined over $\mathbb{R}^3$. In addition, let $g_1(x, y, z) = x + y + z$ and $g_2(x, y, z) = x^2 + y^2 + z$ be two additional functions defined over $\mathbb{R}^3$. Consider the constrained maximization problem:

$$\begin{aligned} \max_{x,y,z} \quad & f(x, y, z) \\ \text{s.t.} \quad & g_1(x, y, z) = 1 \\ & g_2(x, y, z) = \frac{7}{4}. \end{aligned}$$

(a) Check if the NDCQ is satisfied.

(b) Solve the maximization problem.

Exercise 5

Consider the following constrained minimization problem:

$$\begin{aligned} \min_{x,y,z} \quad & f(x, y, z) = x^2 + y^2 + z^2 \\ \text{s.t.} \quad & g_1(x, y, z) = x + 2y + z = 30 \\ & g_2(x, y, z) = 2x - y - 3z = 10. \end{aligned}$$

(a) Find the unique point that satisfies the first order conditions for optimality.

(b) Argue why the point you found in (a) is a local minimizer of f subject to the constraints.