CS–E4500 Advanced Course in Algorithms

Week 06 – Tutorial

1. The events A_1, A_2, A_3 are pairwise independent if, for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than mutual independence, which requires the additional condition that $\mathrm{P}(A_1, A_2, A_3) = \mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3)$. Construct an example where three events are pairwise independent but not mutually independent.

   Solution. One example would concern two independent coin tosses. Let

   - A_1 : first toss is Head
   - A_2 : second toss is Head
   - A_3 : both tosses are the same

   Clearly $\mathrm{P}(A_i \mid A_j) = \mathrm{P}(A_i)$ for all $i, j \in \{1, 2, 3\}$ such that $i \neq j$. However, since events A_1 and A_2 determine event A_3, they are not mutually independent.

2. Let $X = \sum_{i=1}^{n} X_i$, where the X_i are pairwise independent random variables. Prove that

   $$\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(X_i)\ .$$

   This equality allows us to apply Chebyshev's inequality even when the random variables are only pairwise independent.

   Solution. By a well known relation,

   $$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2\sum_{i<j} \mathrm{Cov}(X_i, X_j)\ ,$$

   where

   $$\mathrm{Cov}(X_i, X_j) = \mathrm{E}((X_i - \mathrm{E}(X_i))(X_j - \mathrm{E}(X_j))) = \mathrm{E}(X_i X_j) - \mathrm{E}(X_i)\mathrm{E}(X_j)\ .$$

   Since $X_1, X_2, \ldots, X_n$ are pairwise independent, it is clear that for any $i \neq j$ we have

   $$\mathrm{E}(X_i X_j) - \mathrm{E}(X_i)\mathrm{E}(X_j) = 0\ ,$$

   and therefore

   $$\mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(X_i)\ .$$