Mathematics for Economists

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Constrained Optimization

Exercise (from Lecture 13)

Study the definiteness of the quadratic form

$$Q(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 + x_3^2 + x_4^2 + 4x_2x_3 - 2x_1x_4$$

on the following constraint set:

$$\begin{aligned} x_2 + x_3 + x_4 &= 0 \\ x_1 - 9x_2 + x_4 &= 0. \end{aligned}$$

Exercise (from Lecture 13)

- We can write Q in matrix form $Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$, where the coefficient matrix is

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

- The set of linear constraints is

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -9 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Exercise (from Lecture 13)

- We can form the following bordered matrix

$$H_6 = \begin{pmatrix} 0 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & -\mathbf{9} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & 1 & 0 & 0 & -1 \\ \mathbf{1} & -\mathbf{9} & 0 & -1 & 2 & 0 \\ \mathbf{1} & \mathbf{0} & 0 & 2 & 1 & 0 \\ \mathbf{1} & \mathbf{1} & -1 & 0 & 0 & 1 \end{pmatrix}$$

- The problem has $n = 4$ variables and $m = 2$ constraints. Hence we need to check the last $n - m = 2$ leading principal minors

Exercise (from Lecture 13)

- The last leading principal minor is the determinant of H_6 itself, which is $\det H_6 = 24$

- The second-to-last leading principal minor is the determinant of

$$H_5 = \begin{pmatrix} 0 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & -\mathbf{9} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & 1 & 0 & 0 \\ \mathbf{1} & -\mathbf{9} & 0 & -1 & 2 \\ \mathbf{1} & \mathbf{0} & 0 & 2 & 1 \end{pmatrix},$$

  which is $\det H_5 = 77$.

- Finally, since $(-1)^m = (-1)^2 = +1$, $\det H_6 > 0$ and $\det H_5 > 0$, we can conclude that our constrained quadratic form is positive definite

Constrained Optimization

- Suppose we want to solve the following optimization problem with one **inequality constraint**:

$$\begin{aligned} \max_{x,y} \quad & f(x,y) \\ \text{s.t.} \quad & g(x,y) \leq b \end{aligned}$$

- Let $\nabla f(\mathbf{p})$ and $\nabla g(\mathbf{p})$ be the *gradient vectors* of f and g at $\mathbf{p}$, respectively

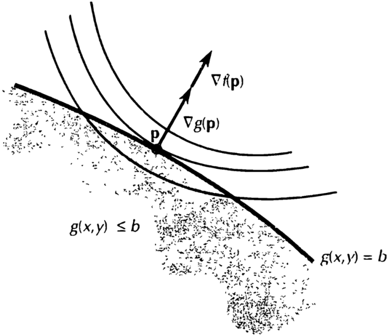
- For a given function F of n variables, the gradient of F at $\mathbf{x}$ is the vector

$$\nabla F(\mathbf{x}) = \begin{pmatrix} \frac{\partial F}{\partial x_1}(\mathbf{x}) \\ \frac{\partial F}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial F}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

- The gradient vector points in the direction in which the value of F increases most rapidly

Constrained Optimization

- Suppose that **p** solves our optimization problem and that it lies on the boundary of the constraint set where $g(x, y) = b$. We say that the constraint g is binding or active at **p**
- We have $\nabla f(\mathbf{p}) = \mu \nabla g(\mathbf{p})$, where μ is a Lagrange multiplier



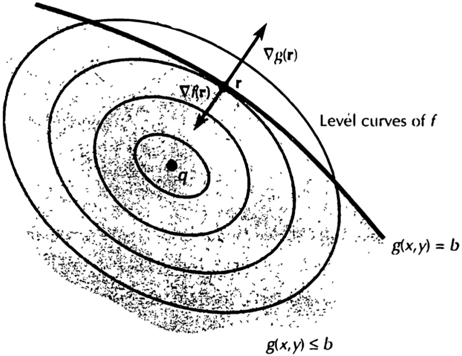
∇f and ∇g point in the same direction at the maximizer **p**.

Constrained Optimization

- In $\nabla f(\mathbf{p}) = \mu \nabla g(\mathbf{p})$, μ must be non-negative
  - $\nabla g(\mathbf{p})$ points to the region where $g(x, y) \geq b$, and not to the constraint set $g(x, y) \leq b$
  - Since $\mathbf{p}$ maximizes f on the set $g(x, y) \leq b$, $\nabla f(\mathbf{p})$ cannot point to the constraint set $g(x, y) \leq b$
  - Thus $\nabla f(\mathbf{p})$ and $\nabla g(\mathbf{p})$ must point in the same direction, i.e. $\mu \geq 0$

Constrained Optimization

- Now suppose that the solution to the constrained maximization problem is $\mathbf{q}$
- At $\mathbf{q}$, the constraint is not binding (or inactive), i.e. $g(\mathbf{q}) < b$



The situation in which the constraint is not binding.

Constrained Optimization

- $\mathbf{q}$ is also a local unconstrained maximizer of f

- From the first order conditions for unconstrained maximization,

$$\frac{\partial f}{\partial x}(\mathbf{q}) = 0 = \frac{\partial f}{\partial y}(\mathbf{q})$$

- In the constrained problem, we can still form the Lagrangian

$$L(x, y, \mu) = f(x, y) - \mu\left(g(x, y) - b\right)$$

- When the constraint $g(x, y) \leq b$ is inactive, we require $\mu = 0$

Constrained Optimization

- In summary:
  - Either the constraint is binding, $g(x, y) = b$, in which case the multiplier must be $\mu \geq 0$;
  - or the constraint is inactive, $g(x, y) < b$, in which case the multiplier must be $\mu = 0$
- The two cases above are summarized in the following **complementary slackness condition**:

$$\mu \left[g(x, y) - b \right] = 0$$

Constrained Optimization

Proposition (First order necessary conditions, 2 variables, 1 inequality constraint)

Let f and g be C^1 functions defined over $\mathbb{R}^2$. Suppose that:

1. (x^*, y^*) **maximizes** f *on the constraint set* $g(x, y) \leq b$;
2. *if* $g(x^*, y^*) = b$, *then* $\frac{\partial g}{\partial x}(x^*, y^*) \neq 0$ *or* $\frac{\partial g}{\partial y}(x^*, y^*) \neq 0$.

Let the Lagrangian function be:

$$L(x, y, \mu) = f(x, y) - \mu \left[g(x, y) - b \right].$$

Then, there exists a number μ^* such that

1. $\frac{\partial L}{\partial x}(x^*, y^*) = 0$ *and* $\frac{\partial L}{\partial y}(x^*, y^*) = 0$
2. $\mu^* \left[g(x^*, y^*) - b \right] = 0$
3. $\mu^* \geq 0$
4. $g(x^*, y^*) \leq b$

Constrained Optimization

Equality vs. Inequality constraints:

- Both use the Lagrangian and both require that the partial derivatives of L with respect to the x_i's be zero
- The condition $\frac{\partial L}{\partial \lambda} = 0$ for equality constraints may no longer hold for inequality constraints. It is replaced by

$$\mu \left[g(x,y) - b \right] = 0 \quad \text{and} \quad \frac{\partial L}{\partial \mu} = g(x,y) - b \leq 0$$

- The constraint qualification for inequality constraints must be checked only for those constraints that are binding
- With equality constraints, there are no restrictions on the sign of the Lagrange multipliers; with inequality constraints, the multiplier must be non-negative
- The first order conditions for equality constraints work both for maximization and minimization problems; the first order conditions for inequality constraints in the previous slide work only for maximization problems

Constrained Optimization

- **Example.** Consider the utility maximization problem

$$\begin{aligned} \max_{x_1, x_2} \quad & u(x_1, x_2) \\ \text{s.t.} \quad & p_1 x_1 + p_2 x_2 \leq w, \end{aligned}$$

  where p_1 and p_2 are prices, and w is income or wealth

- Suppose that preferences are monotone. That is, at every (x_1, x_2), the marginal utility of each commodity is positive:

$$\frac{\partial u}{\partial x_1}(x_1, x_2) > 0 \quad \text{and} \quad \frac{\partial u}{\partial x_2}(x_1, x_2) > 0$$

Constrained Optimization

- The Lagrangian is

$$L(x_1, x_2, \mu) = u(x_1, x_2) - \mu(p_1 x_1 + p_2 x_2 - w)$$

- At a solution (x_1^*, x_2^*), we have

$$\frac{\partial u}{\partial x_1}(x_1^*, x_2^*) - \mu p_1 = 0, \quad \frac{\partial u}{\partial x_2}(x_1^*, x_2^*) - \mu p_2 = 0 \tag{1}$$

$$\mu^* \geq 0, \quad p_1 x_1^* + p_2 x_2^* \leq w, \quad \mu^* \left(p_1 x_1^* + p_2 x_2^* - w\right) = 0$$

- Notice that μ^* cannot be equal to zero, otherwise both equations in (1) would be inconsistent with positive marginal utilities

- Since $\mu^* > 0$, the complementary slackness condition implies $p_1 x_1^* + p_2 x_2^* = w$. That is, the budget constraint is binding

- In general, when the utility function represents monotone preferences, one can treat the budget constraint as an equality constraint

Constrained Optimization

- The general formulation of a constrained maximization problem with n variables and k *inequality constraints* is to
  - **maximize** the objective function $f(x_1, \ldots, x_n)$ with respect to $(x_1, \ldots, x_n)$
  - subject to the constraints:

$$
\begin{aligned}
g_1(x_1, \ldots, x_n) &\leq b_1 \\
g_2(x_1, \ldots, x_n) &\leq b_2 \\
\ldots \quad \ldots \quad & \ldots \quad \ldots \\
g_k(x_1, \ldots, x_n) &\leq b_k
\end{aligned}
$$

- The constraint set is

$$C = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \leq b_1, g_2(\mathbf{x}) \leq b_2, \ldots, g_k(\mathbf{x}) \leq b_k\},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$

Constrained Optimization

- The non-degenerate constraint qualification (NDCQ) at a given point $\mathbf{x}$ is formulated as follows:
  - Without loss of generality, suppose that the first k_0 inequality constraints ($k_0 \leq k$) are binding at $\mathbf{x}$, and the last $k - k_0$ are inactive at $\mathbf{x}$
  - The Jacobian of the binding constraints is

$$D\mathbf{g}(\mathbf{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial g_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial g_{k_0}}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

  - We say that the NDCQ is satisfied at $\mathbf{x}$ if the rank of $D\mathbf{g}(\mathbf{x})$ is as large as it can be

Constrained Optimization

Proposition (First order necessary conditions)

Let $f, g_1, \ldots, g_k$ be C^1 functions defined over $\mathbb{R}^n$. Suppose that:

1. $\boldsymbol{x}^*$ *is a* **local maximizer** *of f on the constraint set defined by the inequalities*

$$g_1(\boldsymbol{x}) \leq b_1, g_2(\boldsymbol{x}) \leq b_2, \ldots, g_k(\boldsymbol{x}) \leq b_k$$

2. *the NDCQ is satisfied at $\boldsymbol{x}^*$.*

Form the Lagrangian $L(\boldsymbol{x}, \mu_1, \ldots, \mu_k) = f(\boldsymbol{x}) - \sum_{i=1}^k \mu_i \left[g_i(\boldsymbol{x}) - b_i\right]$.
Then, there exists multipliers $\mu_1^*, \ldots, \mu_k^*$ such that

1. $\frac{\partial L}{\partial x_1}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = 0, \ldots, \frac{\partial L}{\partial x_n}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = 0$
2. $\mu_1^* \left[g_1(\boldsymbol{x}^*) - b_1\right] = 0, \ldots, \mu_k^* \left[g_k(\boldsymbol{x}^*) - b_k\right] = 0$
3. $\mu_1^* \geq 0, \ldots, \mu_k^* \geq 0$
4. $g_1(\boldsymbol{x}^*) \leq b_1, \ldots, g_k(\boldsymbol{x}^*) \leq b_k$.

Constrained Optimization

Proposition (First order necessary conditions)

Let $f, g_1, \ldots, g_k$ be C^1 functions defined over $\mathbb{R}^n$. Suppose that:

1. $\boldsymbol{x}^*$ *is a* **local minimizer** *of f on the constraint set defined by the inequalities*

$$g_1(\boldsymbol{x}) \geq b_1, g_2(\boldsymbol{x}) \geq b_2, \ldots, g_k(\boldsymbol{x}) \geq b_k$$

2. *the NDCQ is satisfied at $\boldsymbol{x}^*$.*

Form the Lagrangian $L(\boldsymbol{x}, \mu_1, \ldots, \mu_k) = f(\boldsymbol{x}) - \sum_{i=1}^{k} \mu_i \left[g_i(\boldsymbol{x}) - b_i\right].$
Then, there exists multipliers $\mu_1^*, \ldots, \mu_k^*$ such that

1. $\frac{\partial L}{\partial x_1}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = 0, \ldots, \frac{\partial L}{\partial x_n}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = 0$
2. $\mu_1^* \left[g_1(\boldsymbol{x}^*) - b_1\right] = 0, \ldots, \mu_k^* \left[g_k(\boldsymbol{x}^*) - b_k\right] = 0$
3. $\mu_1^* \geq 0, \ldots, \mu_k^* \geq 0$
4. $g_1(\boldsymbol{x}^*) \geq b_1, \ldots, g_k(\boldsymbol{x}^*) \geq b_k.$

Constrained Optimization

- **Example.** Consider the constrained maximization problem

$$\begin{aligned} \max_{x,y} \quad & x^2 + y^2 \\ \text{s.t.} \quad & 2x + y \leq 2 \\ & x \geq 0, y \geq 0 \end{aligned}$$

- Here we can invoke Weierstrass's theorem and claim that a solution exists (why?)

- The Lagrangian is

$$L = x^2 + y^2 - \mu(2x + y - 2) + \lambda_1 x + \lambda_2 y$$

Constrained Optimization

- **Example (cont'd).** The first order conditions are

$$\begin{aligned} 2x - 2\mu + \lambda_1 &= 0 \\ 2y - \mu + \lambda_2 &= 0 \\ \mu(2x + y - 2) &= 0 \\ \lambda_1 x &= 0 \\ \lambda_2 y &= 0 \\ \mu \geq 0, \lambda_1 \geq 0, \lambda_2 &\geq 0 \\ 2x + y \leq 2,\ x \geq 0,\ y &\geq 0 \end{aligned}$$

Constrained Optimization

- **Example (cont'd).** To find all the candidates for a solution, we can consider four distinct cases.

  1. $\mathbf{x} = \mathbf{y} = \mu = \lambda_1 = \lambda_2 = \mathbf{0}$ satisfies all the first order conditions, hence $(0, 0)$ is a candidate for a solution.

  2. Suppose there is a solution with $\mathbf{x} = \mathbf{0}$ and $\mathbf{y} > \mathbf{0}$. If so, then $\lambda_1 = 2\mu$ and $y = 2$. If $y = 2$, then $\lambda_2 = 0$, so $\mu = 4$ and $\lambda_1 = 8$, which is consistent with all the first order conditions. Thus $(x, y) = (0, 2)$ is a candidate for a solution.

  3. Suppose there is a solution with $\mathbf{y} = \mathbf{0}$ and $\mathbf{x} > \mathbf{0}$. If so, then $\lambda_2 = \mu$ and $x = 1$. If $x = 1$, then $\lambda_1 = 0$, so $\mu = 1$ and $\lambda_2 = 1$, which is consistent with all the first order conditions. Thus $(x, y) = (1, 0)$ is a candidate for a solution.

  4. Suppose there is a solution with $\mathbf{y} > \mathbf{0}$ and $\mathbf{x} > \mathbf{0}$. If so, then $\lambda_1 = \lambda_2 = 0$. Consequently, $x = \mu = \frac{4}{5}$ and $y = \frac{2}{5}$, which is consistent with all the first order conditions. Thus $(x, y) = (\frac{4}{5}, \frac{2}{5})$ is a candidate for a solution.

Constrained Optimization

- **Example (cont'd).** If we evaluate the objective function at the four candidates for a solution, we find that the global constrained maximizer is $(x^*, y^*) = (0, 2)$
- **Exercise.** Check whether the NDCQ is satisfied
- **Exercise.** Solve the problem of *minimizing* f over the same set of constraints of this example

Constrained Optimization

- Suppose we want to solve the following problem with k inequality constraints and n non-negativity constraints:
  - **maximize** $f(x_1, \dots, x_n)$ with respect to $(x_1, \dots, x_n)$
  - subject to the constraints:

$$\begin{aligned} g_1(x_1, \dots, x_n) &\leq b_1 \\ g_2(x_1, \dots, x_n) &\leq b_2 \\ \dots \quad \dots \quad & \dots \quad \dots \\ g_k(x_1, \dots, x_n) &\leq b_k \\ x_1 \geq 0, x_2 \geq 0, \dots, x_n &\geq 0 \end{aligned}$$

- Clearly, one can solve this problem by using the first order conditions we already saw
- Alternatively, an equivalent way to solve the problem is to use the **Kuhn-Tucker formulation**

Constrained Optimization

- The Kuhn-Tucker Lagrangian of this problem is

$$\tilde{L}(\boldsymbol{x}, \mu_1, \ldots, \mu_k) = f(\boldsymbol{x}) - \sum_{i=1}^{k} \mu_i \left[g_i(\boldsymbol{x}) - b_i \right],$$

  in which non-negativity constraints are not included.

Constrained Optimization

- The first order conditions in terms of the Kuhn-Tucker Lagrangian are

$$
\begin{aligned}
\frac{\partial \tilde{L}}{\partial x_1} \leq 0, \ldots, \frac{\partial \tilde{L}}{\partial x_n} \leq 0, &\qquad \frac{\partial \tilde{L}}{\partial \mu_1} \geq 0, \ldots, \frac{\partial \tilde{L}}{\partial \mu_k} \geq 0 \\
x_1 \frac{\partial \tilde{L}}{\partial x_1} = 0, \ldots, x_n \frac{\partial \tilde{L}}{\partial x_n} = 0, &\qquad \mu_1 \frac{\partial \tilde{L}}{\partial \mu_1} = 0, \ldots, \mu_k \frac{\partial \tilde{L}}{\partial \mu_k} = 0 \\
x_1 \geq 0, \ldots, x_n \geq 0, &\qquad \mu_1 \geq 0, \ldots, \mu_k \geq 0.
\end{aligned}
$$

- If a solution $\boldsymbol{x}^*$ exists, and if the NDCQ is satisfied, the above system is satisfied at $\boldsymbol{x}^*$

Constrained Optimization

- **Example.** Consider the utility maximization problem

$$\begin{aligned}\max_{x_1,x_2} \quad & u(x_1,x_2) \\ \text{s.t.} \quad & p_1x_1 + p_2x_2 \leq w \\ & x_1 \geq 0, x_2 \geq 0\end{aligned}$$

- The Kuhn-Tucker Lagrangian is

$$\tilde{L} = u(x_1,x_2) - \mu\,[p_1x_1 + p_2x_2 - w]$$

Constrained Optimization

- **Example (cont'd).** The first order conditions are:

$$
\begin{aligned}
\frac{\partial u}{\partial x_1} - \mu p_1 \leq 0, &\qquad \frac{\partial u}{\partial x_2} - \mu p_2 \leq 0, \\
x_1 \left(\frac{\partial u}{\partial x_1} - \mu p_1 \right) = 0, &\qquad x_2 \left(\frac{\partial u}{\partial x_2} - \mu p_2 \right) = 0, \\
\frac{\partial \tilde{L}}{\partial \mu} = (w - p_1 x_1 - p_2 x_2) \geq 0, &\qquad \mu \frac{\partial \tilde{L}}{\partial \mu} = \mu (w - p_1 x_1 - p_2 x_2) = 0 \\
x_1 \geq 0, x_2 \geq 0, &\qquad \mu \geq 0.
\end{aligned}
$$

- **Exercise.** Solve the constrained optimization problem at p. 20 by using the Kuhn-Tucker formulation

Constrained Optimization

- **Example.** Consider the constrained maximization problem:

$$\begin{aligned} \max_{x,y} \quad & f(x,y) = x^2 + 2y \\ \text{s.t.} \quad & g_1(x,y) = x^2 + y^2 \le 5 \\ & g_2(x,y) = -y \le 0 \end{aligned}$$

- Before solving this problem, we want to check the NDCQ. That is, we want to check if there are points in the constraint set in which the NDCQ fails. If such points exist, we include them in our set of candidates for a solution together with the points that satisfy the first order necessary conditions

Constrained Optimization

- **Example (cont'd).** We have to check four cases:

  (1) *Both constraints are inactive.* In this case the NDCQ holds trivially

  (2) *Constraint 1 is active and constraint 2 is inactive.* This means that $x^2 + y^2 = 5$ and $y > 0$. The Jacobian is
  $$Dg(x, y) = \begin{pmatrix} 2x & 2y \end{pmatrix}$$
  The rank is zero if and only if $x = y = 0$, which contradicts $y > 0$. Hence the NDCQ is satisfied in this case

Constrained Optimization

- **Example (cont'd).**

  (3) *Constraint 2 is active and constraint 1 is inactive.* This means that $x^2 + y^2 < 5$ and $y = 0$. The Jacobian is
  $$Dg(x, y) = \begin{pmatrix} 0 & -1 \end{pmatrix}$$
  The rank is always equal to 1. Hence the NDCQ is satisfied in this case

  (4) *Both constraints are active.* This means that $x^2 + y^2 = 5$ and $y = 0$. The Jacobian is
  $$Dg(x, y) = \begin{pmatrix} 2x & 2y \\ 0 & -1 \end{pmatrix}$$
  Since $y = 0$, $x = \pm\sqrt{5}$. At both $(\sqrt{5}, 0)$ and $(-\sqrt{5}, 0)$, then rank is 2. Hence the NDCQ is satisfied in this case

- We can conclude that the NDCQ is satisfied at every point in the constraint set

Constrained Optimization

- **Example (cont'd).** To solve the problem, we first observe that a solution exists by Weierstrass's Theorem
- The Lagrangian is

$$L = x^2 + 2y - \lambda_1 \left(x^2 + y^2 - 5\right) + \lambda_2 y$$

- The first order conditions are

$$\begin{align} 2x - 2x\lambda_1 &= 0 \tag{2} \\ 2 - 2y\lambda_1 + \lambda_2 &= 0 \tag{3} \\ \lambda_1(x^2 + y^2 - 5) &= 0 \tag{4} \\ \lambda_2 y &= 0 \tag{5} \\ \lambda_1 \geq 0,\ \lambda_2 &\geq 0 \tag{6} \\ x^2 + y^2 - 5 \leq 5,\ y &\geq 0 \tag{7} \end{align}$$

- **Example (cont'd).** Like we did when checking the NDCQ, we consider four cases.

  (1) *Both constraints are inactive.* This implies $\lambda_1 = \lambda_2 = 0$, which contradicts (3).

  (2) *Constraint 1 is active and constraint 2 is inactive.* This means that $x^2 + y^2 = 5$ and $y > 0$, which implies $\lambda_2 = 0$. From (2), $x(1 - \lambda_1) = 0$. Thus $x = 0$ or $\lambda_1 = 1$, or both. If $x = 0$, then $y = \pm\sqrt{5}$. Since $y > 0$, only $y = \sqrt{5}$ is possible. Then $\lambda_1 = \frac{1}{\sqrt{5}}$. Hence $(x, y) = (0, \sqrt{5})$ is a solution candidate.

      If $\lambda_1 = 1$, then we get $y = 1$ from (3). Furthermore, the first constraint implies $x = \pm 2$. Thus the points $(2, 1)$ and $(-2, 1)$ are two more solution candidates.

Constrained Optimization

- **Example (cont'd).**

  (3) *Constraint 2 is active and constraint 1 is inactive.* This implies $\lambda_1 = 0$. By (3), $\lambda_2 = -2$, which contradicts (6).

  (4) *Both constraints are active.* If $y = 0$, then (3) implies $\lambda_2 = -2$, which contradicts (6).

- In sum, we have three solution candidates:

$$(x, y) = (0, \sqrt{5}), \quad (x, y) = (2, 1), \quad (x, y) = (-2, 1)$$

- Since we have

$$f(2,1) = f(-2,1) = 6 > 2\sqrt{5} = f(0, \sqrt{5}),$$

  we can conclude that both $(2, 1)$ and $(-2, 1)$ solve this constrained maximization problem