MEC-E1050

FINITE ELEMENT METHOD IN SOLIDS 2022

WEEK 44: DISPLACEMENT ANALYSIS

2 DISPLACEMENT ANALYSIS

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LEARNING OUTCOMES

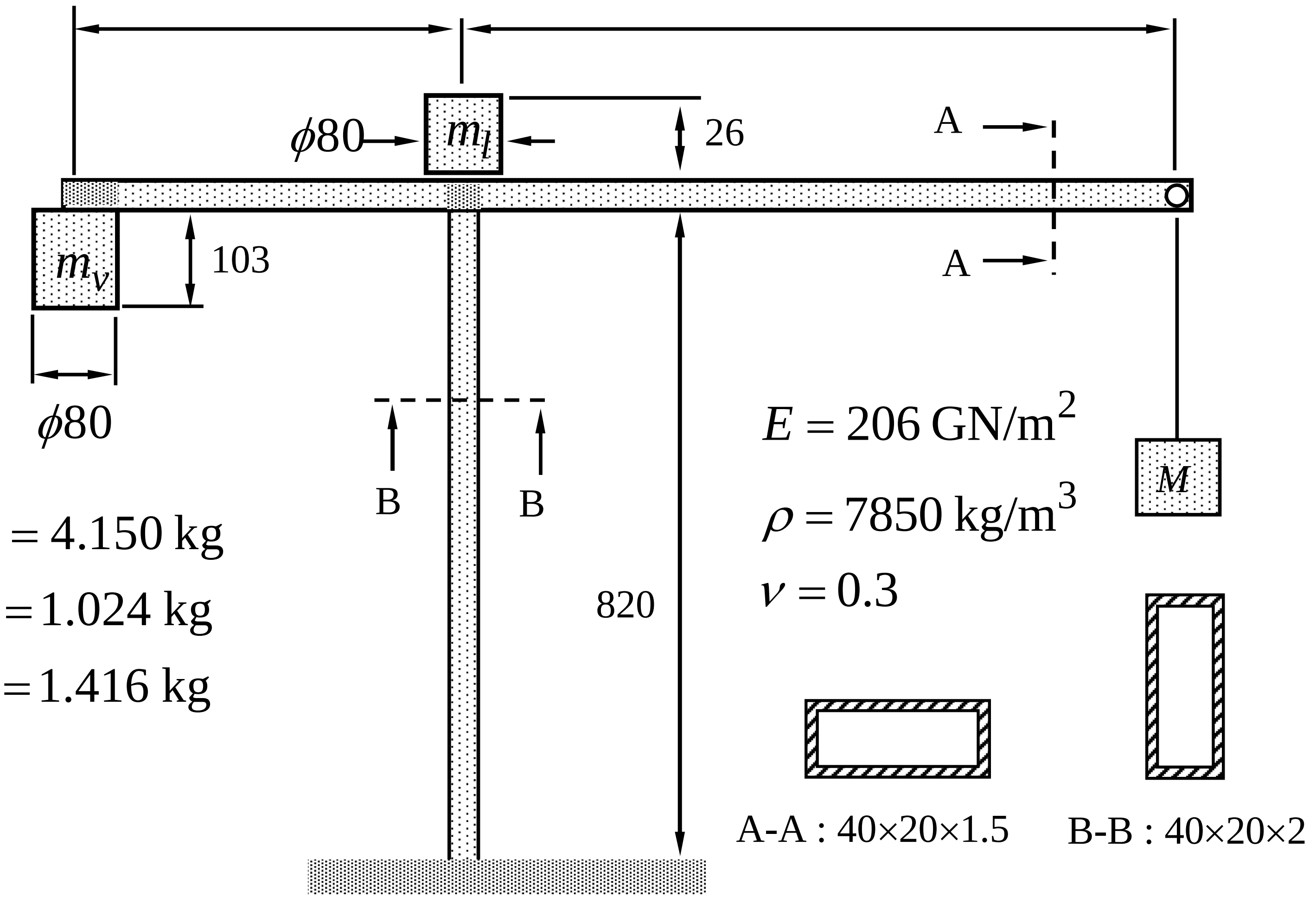
Students are able to solve the weekly lecture problems, home problems, and exercise problems on the topic:

- ☐ Engineering paradigm, elements and nodes, structural and material coordinate systems, displacements and rotations

- ☐ Equilibrium equations of nodes and element contributions (force-displacement relationships)

- ☐ Derivation of tension bar, torsion bar, and bending beam element contributions from the exact solutions to the corresponding boundary value problems.

2.1 STRUCTURE ANALYSIS

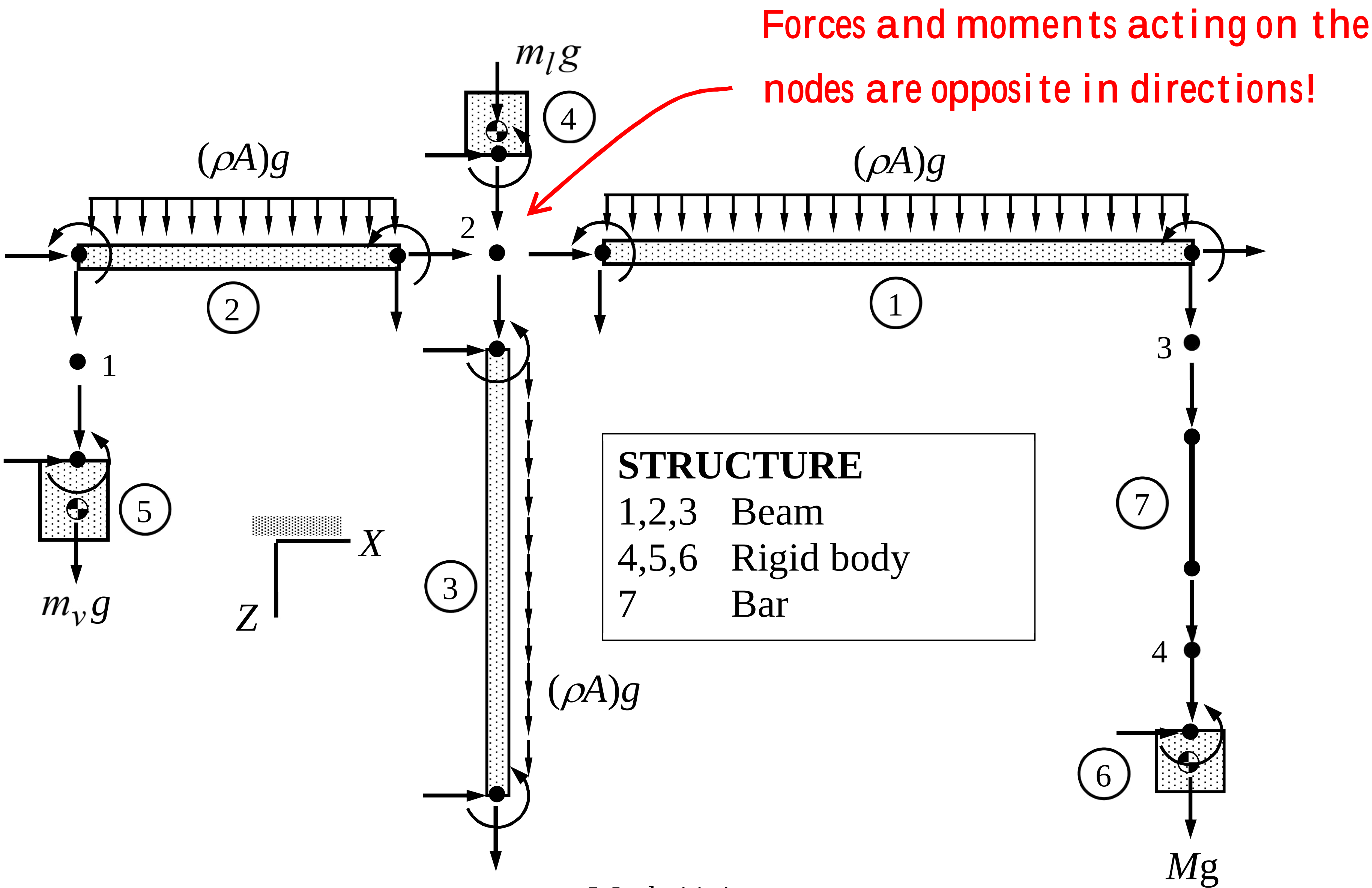


$m_v = 4.150\ \text{kg}$

$m_l = 1.024\ \text{kg}$

$M = 1.416\ \text{kg}$

ELEMENTS AND NODES



NEWTON's LAWS OF MOTION

I In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

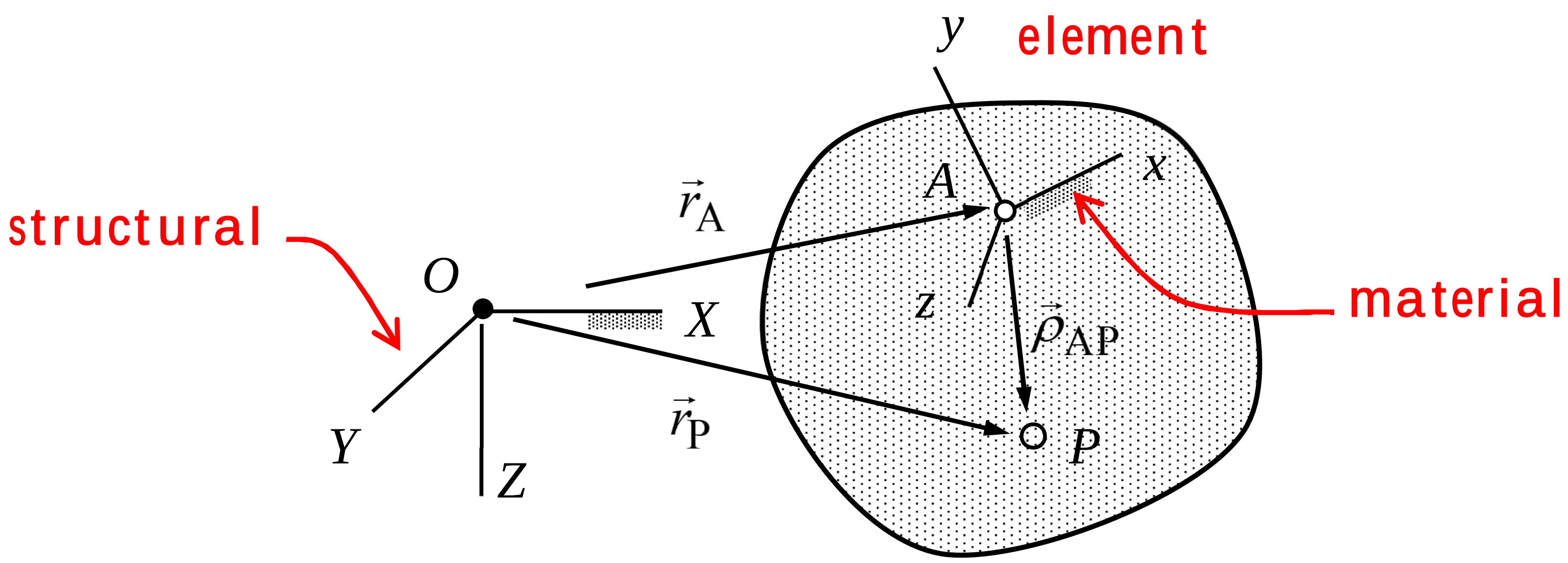
II The vector sum of the forces on an object is equal to the mass of that object multiplied by the acceleration of the object (assuming that the mass is constant).

III When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

Newton's laws in the original form apply to particles only. The formulation for rigid bodies and deformable bodies require slight modifications.

COORDINATE SYSTEMS

The particles of elements are identified by coordinates (x, y, z) of the *material coordinate system* which moves and deforms with the body (in principle). The unique *structural coordinate system* (X,Y,Z) is needed, e.g., in description of geometry.



The basis vectors of the material and structural systems are denoted by $\vec{i}, \vec{j}, \vec{k}$ and $\vec{I}, \vec{J}, \vec{K}$, respectively!

SIGN CONVENTIONS AND NOTATIONS

Displacements, rotations, forces and moments are vector quantities (magnitude and direction) so the components are taken to be positive in the directions of the chosen coordinate axes. The convention may differ from that used in mechanics of materials courses (be careful).

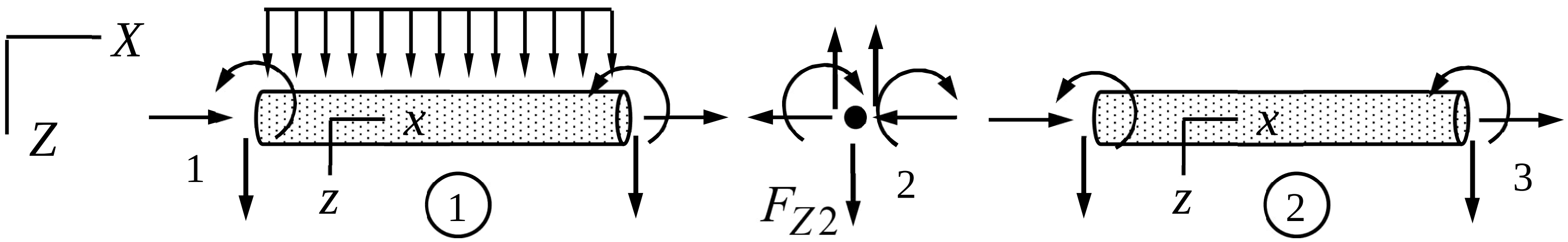
	Displacement	**Force**	**Rotation**	**Moment**
Material	u_x, u_y, u_z	F_x, F_y, F_z	$\theta_x, \theta_y, \theta_z$	M_x, M_y, M_z
Structural	u_X, u_Y, u_Z	F_X, F_Y, F_Z	$\theta_X, \theta_Y, \theta_Z$	M_X, M_Y, M_Z

Representation in one system can be transformed into another assuming that the relative orientations of the axes are known (example).

FREE BODY DIAGRAMS

Index e refers to an *element* (in a figure ○) and i to a *node* (in a figure •):

$$F_{X1}^{1}, F_{Z1}^{1}, M_{Y1}^{1} \qquad F_{X2}^{1}, F_{Z2}^{1}, M_{Y2}^{1} \qquad F_{X2}^{2}, F_{Z2}^{2}, M_{Y2}^{2} \qquad F_{X3}^{2}, F_{Z3}^{2}, M_{Y3}^{2}$$



External known forces and moments are acting on the nodes. Internal forces between the elements satisfy the law of action and reaction (Newton III) and act through the nodes. Components acting on the elements are considered as positive (with respect to the material coordinate system of the element).

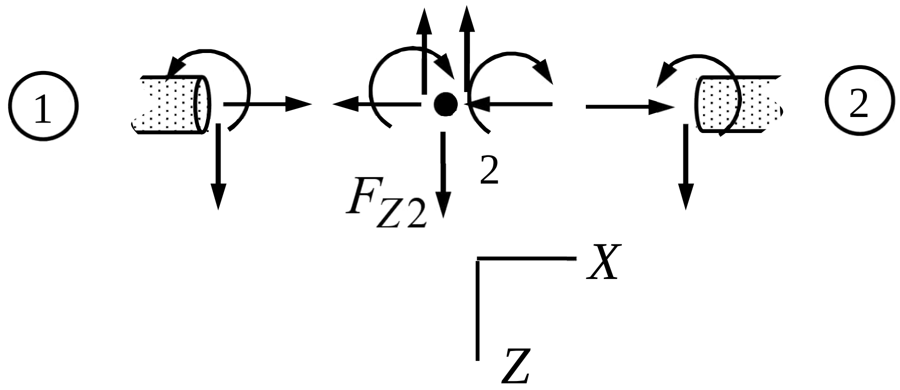
EQUILIBRIUM EQUATIONS

Principles of momentum and moment of momentum are applied to the nodes of structure: As nodes are massless particles (or control points), the sums of the forces and moments acting on the nodes $i \in I$ must vanish

$$-\sum_{e\in E} F_{Xi}^{e} + F_{Xi} = 0, \quad -\sum_{e\in E} M_{Xi}^{e} + M_{Xi} = 0$$

$$-\sum_{e\in E} F_{Yi}^{e} + F_{Yi} = 0, \quad -\sum_{e\in E} M_{Yi}^{e} + M_{Yi} = 0$$

$$-\sum_{e\in E} F_{Zi}^{e} + F_{Zi} = 0, \quad -\sum_{e\in E} M_{Zi}^{e} + M_{Zi} = 0$$



The sums extend over the elements connected to node i. Equilibrium equations for the constrained displacements and rotations may contain constraint forces and moments treated as unknown external forces in MEC-E1050.

2.2 DISPLACEMENT ANALYSIS

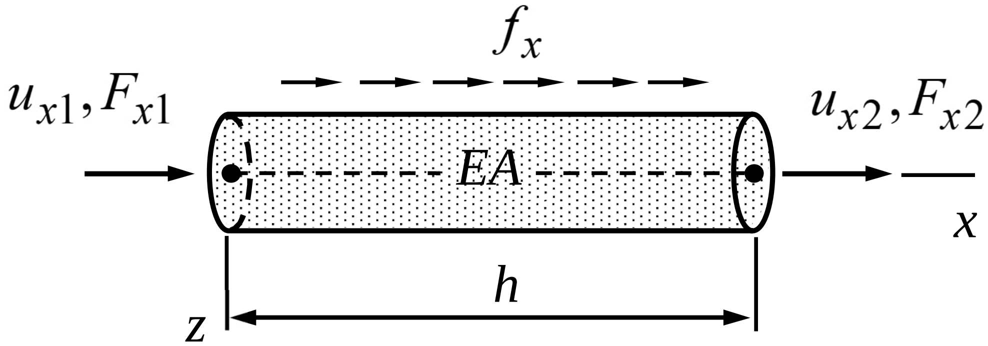
- Idealize a complex structure as a set of elements, whose behavior can be approximated by using the usual engineering models (bar, beam, plate, rigid body etc.).

- Write down the equilibrium equations of the nodes, the force-displacement relationships of the elements (element contributions), and constraints concerning the nodal displacements.

- Solve the nodal displacements and rotations and the internal forces and moments acting on the elements from the equation system.

- Determine the stress in elements one-by-one (optional step).

BAR ELEMENT

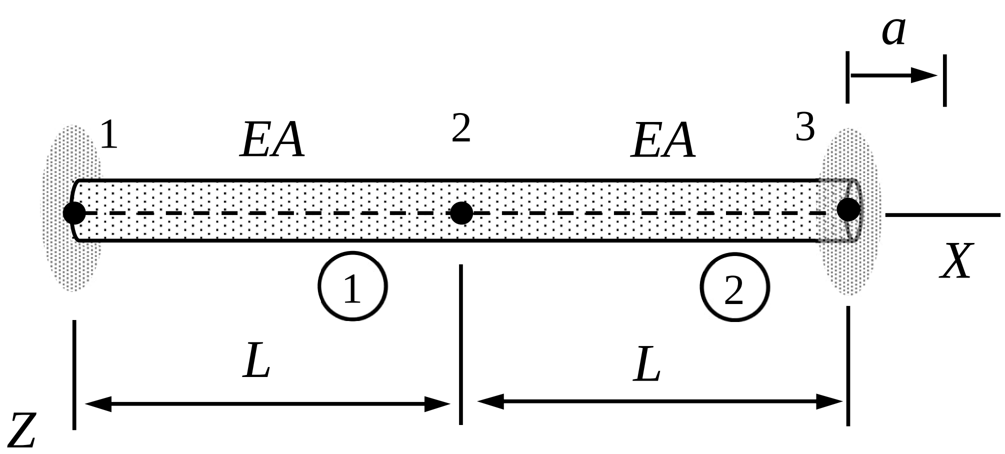


$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

force-displacement relationship of a bar element!

The force-displacement relationships of elements are always expressed in material coordinate systems. In calculations, the displacement, rotation, force, and moment components of material coordinate systems need to be expressed in terms of those of the structural system.

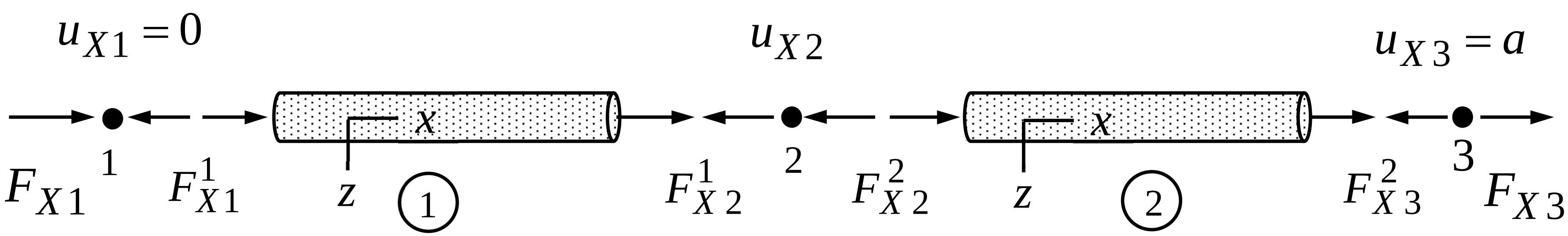
EXAMPLE 2.1 Determine the nodal displacements and the forces acting on the elements 1 and 2 of the figure. The displacement of node 3 is known to be a, EA is constant and the structure consists of two bars.



Answer $u_{X2} = \frac{a}{2}$, $F_{X1}^{1} = -\frac{EA}{2L}a$, $F_{X2}^{1} = \frac{EA}{2L}a$, $F_{X2}^{2} = -\frac{EA}{2L}a$, $F_{X3}^{2} = \frac{EA}{2L}a$,

$F_{X1} = -\frac{EA}{2L}a$, $F_{X3} = \frac{EA}{2L}a$.

- Free body diagram shows all the forces acting on the two bar elements and three nodes. External constraint forces F_{X1} and F_{X3} acting on points 1 and 3 due to the walls are unknown quantities of the problem, whereas displacements of points 1 and 3 are known ($u_{X1} = 0$, $u_{X3} = a$).



- As the axes of the structural and material coordinate systems coincide in this case so for element 1 $F^1_{x1} = F^1_{X1}$, $F^1_{x2} = F^1_{X2}$, $u^1_{x1} = u_{X1} = 0$, $u^1_{x2} = u_{X2}$ and for element 2 $F^2_{x2} = F^2_{X2}$, $F^2_{x3} = F^2_{X3}$, $u^2_{x2} = u_{X2}$, and $u^2_{x3} = u_{X3} = a$. In terms of the force and displacement components in the structural system, bar element contributions and the equilibrium equations of nodes are

$$\begin{Bmatrix} F_{X1}^{1} \\ F_{X2}^{1} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix}, \quad \begin{Bmatrix} F_{X2}^{2} \\ F_{X3}^{2} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ a \end{Bmatrix},$$

$$F_{X1} - F_{X1}^{1} = 0, \quad -F_{X2}^{1} - F_{X2}^{2} = 0, \quad -F_{X3}^{2} + F_{X3} = 0$$

- The seven unknowns $F_{X1}, F_{X1}^{1}, F_{X2}^{1}, F_{X2}^{2}, F_{X3}^{2}, F_{X3}, u_{X2}$ can be solved from the system of seven equations above. The unknown displacement follows from the equilibrium equation of node 2 after elimination of the internal forces:

$$F_{X2}^{1} + F_{X2}^{2} = \frac{EA}{L}u_{X2} + \frac{EA}{L}u_{X2} - \frac{EA}{L}a = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{a}{2}. \quad \leftarrow$$

- After that, internal forces follow from the element contributions (the components in the material coordinate system are more useful, e.g., in stress calculations)

$$\begin{Bmatrix} F_{x1}^{1} \\ F_{x2}^{1} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ a/2 \end{Bmatrix} = \frac{EA}{L}\frac{a}{2}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix},$$

$$\begin{Bmatrix} F_{x2}^{2} \\ F_{x3}^{2} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} a/2 \\ a \end{Bmatrix} = \frac{EA}{L}\frac{a}{2}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}.$$

- Finally, the constraint forces (boundary reactions) follow from the remaining two equilibrium equations of the boundary nodes 1 and 2.

$$F_{X1} = F_{X1}^{1} = -\frac{EA}{2L}a, \quad F_{X3} = F_{X3}^{2} = \frac{EA}{2L}a.$$

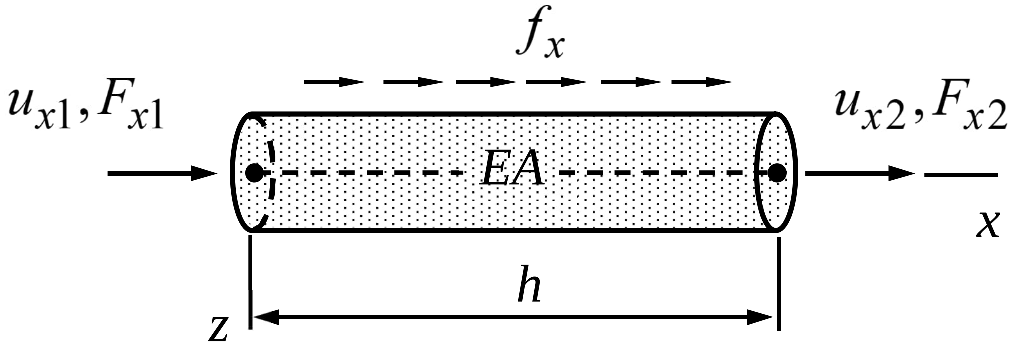
2.3 ELEMENT CONTRIBUTION

- Find the generic solution to the differential equation of the model. Assume that the external distributed forces are simple (for example constant or linear).

- Express the integration constants of the generic solution in terms of the nodal displacement and rotations. The number of integration constants and the number of nodal displacement and rotations should naturally match.

- Substitute the displacement back into the force-displacement relationship of the model and rearrange to get a matrix representation of the form $\mathbf{R} = \mathbf{K}\mathbf{a} - \mathbf{F}$ (to be called as the element contribution) in which $\mathbf{R}$ contains the nodal forces and moments and $\mathbf{a}$ the nodal displacements and rotations.

BAR ELEMENT



$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

the force-displacement relationship of bar element!

Element contributions are always expressed in material coordinate systems. However, in calculations, the displacement, rotation, force, and moment components of material coordinate systems need to be expressed in terms of those of the structural system.

- Boundary value problem for a bar element of length h

$$EA\frac{d^2u}{dx^2}+f_x=0 \quad x\in]0,h[\qquad \text{(equilibrium equation)}$$

$$u(0)=u_{x1} \text{ and } u(h)=u_{x2} \qquad \text{(given nodal displacements)}$$

$$EA\frac{du}{dx}(0)=-F_{x1} \text{ and } EA\frac{du}{dx}(h)=F_{x2} \quad \text{(force-displacement relationship)}$$

- The generic solution to the equilibrium equation (f_x and EA are constants) is given by

$$u=a+bx-\frac{f_x}{2EA}x^2=\{1 \quad x\}\begin{Bmatrix}a\\b\end{Bmatrix}-\frac{f_x}{2EA}x^2.$$

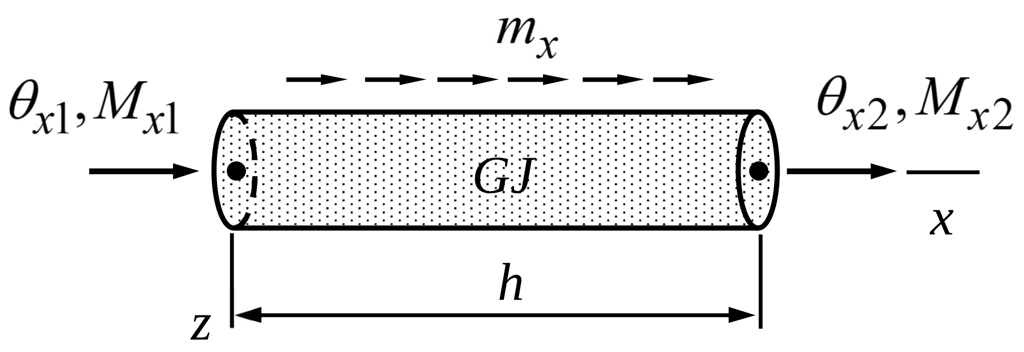
- Integration constants a and b need to be expressed in terms of the nodal displacements u_{x1} and u_{x2}

$$\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = \begin{Bmatrix} u(0) \\ u(h) \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & h \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} - \frac{f_x}{2EA} \begin{Bmatrix} 0 \\ h^2 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & h \end{bmatrix}^{-1} (\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} + \frac{f_x}{2EA} \begin{Bmatrix} 0 \\ h^2 \end{Bmatrix})$$

- The relationship $\mathbf{R} = \mathbf{Ka} - \mathbf{F}$ between the nodal forces and displacement (element contribution) follows from the force-displacement relationship of the model

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = EA \begin{Bmatrix} -\frac{du}{dx}(0) \\ \frac{du}{dx}(h) \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \quad \Leftarrow$$

TORSION ELEMENT

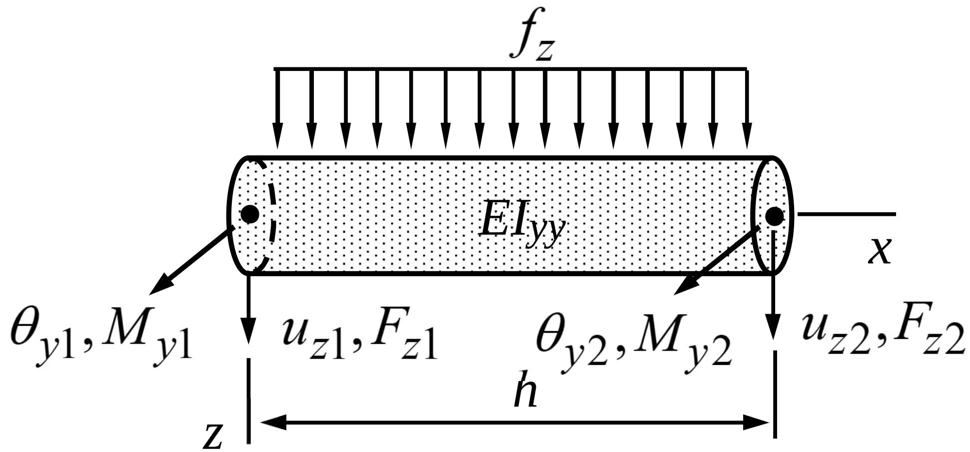


$$\begin{Bmatrix} M_{x1} \\ M_{x2} \end{Bmatrix} = \frac{GJ}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

the force-displacement relationship of torsion bar element!

The force-displacement relationships of elements are always expressed in material coordinate systems. However, in calculations, the displacement, rotation, force, and moment components of material coordinate systems need to be expressed in terms of those of the structural system.

BENDING BEAM ELEMENT



$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ \hdashline F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \left[\begin{array}{cc:cc} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ \hdashline -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{array}\right] \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ \hdashline u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ \hdashline 6 \\ h \end{Bmatrix}$$

the force-displacement relationship of bending beam element!

Notice that the displacements, rotations, forces, and moments are components of vectors of the material coordinate system!

- Boundary value problem for a bending beam (Bernoulli) element of length h

$$\frac{d^2}{dx^2}\left(EI_{yy}\frac{d^2w}{dx^2}\right) - f_z = 0 \quad x \in]0,h[\qquad \text{(equilibrium equation)}$$

$$\frac{dw}{dx}(0) = -\theta_{y1},\ \frac{dw}{dx}(h) = -\theta_{y2},\ w(0) = u_{z1}, \text{ and } w(h) = u_{z2} \quad \text{(boundary conditions)}$$

$$\frac{d^2w}{dx^2}(0) = \frac{M_{y1}}{EI_{yy}},\ \frac{d^2w}{dx^2}(h) = -\frac{M_{y2}}{EI_{yy}},\ \frac{d^3w}{dx^3}(0) = \frac{F_{z1}}{EI_{yy}}, \text{ and } \frac{d^3w}{dx^3}(h) = -\frac{F_{z2}}{EI_{yy}}.$$

- First, the generic solution to the equilibrium equation is given by (f_z and EI_{yy} constants)

$$w(x) = a + bx + cx^2 + dx^3 + \frac{f_z}{24EI_{yy}}x^4 \quad (a,b,c,d \text{ are integration constants})$$

- Second, the integration constants a,b,c,d are expressed in terms of displacements u_{z1}, u_{z2} and rotations θ_{y1}, θ_{y2} by using conditions

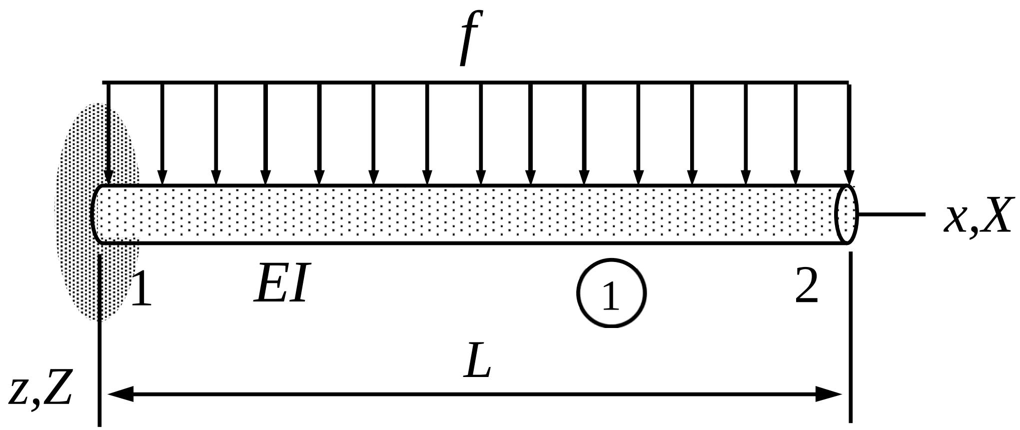
$$w(0)=u_{z1},\ w(h)=u_{z2},\ \frac{dw}{dx}(0)=-\theta_{y1},\ \text{and}\ \frac{dw}{dx}(h)=-\theta_{y2}$$

  Notice that derivatives and rotations are positive in opposite directions.

- Third, by using the force/moment-displacement/rotation relationship of the model

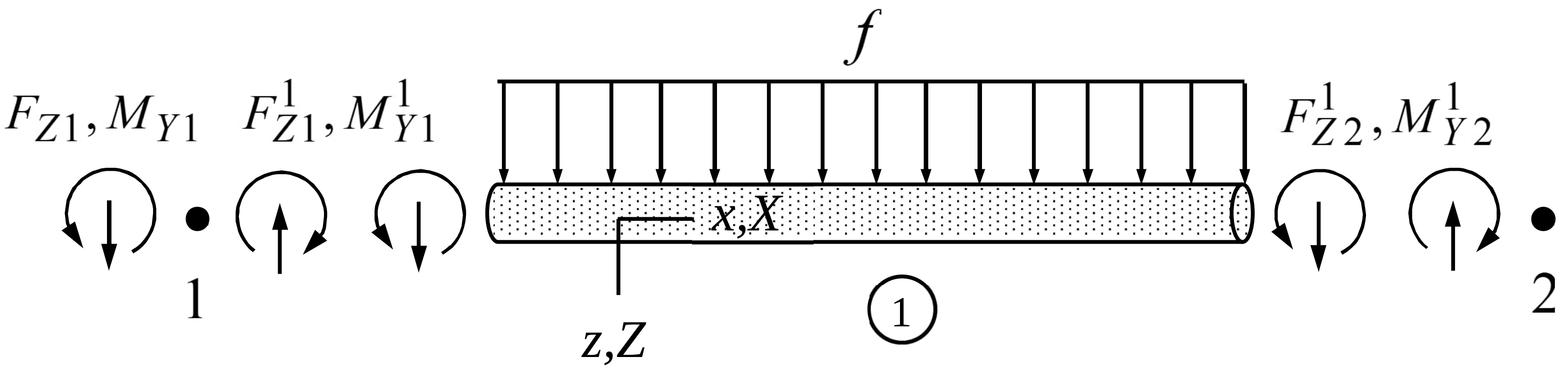
$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ \hline F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3}\left[\begin{array}{cc|cc} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ \hline -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{array}\right]\begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ \hline u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12}\begin{Bmatrix} 6 \\ -h \\ \hline 6 \\ h \end{Bmatrix}. \leftarrow$$

EXAMPLE 2.2. Consider a bending beam loaded by its own weight and clamped on its left end (figure). Determine the displacement and rotation at the right-end by using one beam element. Bending rigidity of the beam EI is constant.



Answer $u_{Z2} = \frac{1}{8}\frac{fL^4}{EI}$ and $\theta_{Y2} = -\frac{1}{6}\frac{fL^3}{EI}$

- Free body diagram shows all the forces acting on the beam element and the two nodes. External constraint force and moment F_{Z1} and M_{Y1} acting on node 1 are unknown quantities of the problem, whereas displacement and rotation at the wall are known ($u_{Z1}=0,\ \theta_{Y1}=0$). At node 2, external forces are known (zeros), whereas the displacement are rotation u_{Z2} and θ_{Y2} are unknown.



- Element contribution, when the known displacement and rotation at the left end $u_{Z1}=\theta_{Y1}=0$ are substituted there, and the equilibrium equations of nodes 1 and 2:

$$\begin{Bmatrix} F_{Z1}^{1} \\ M_{Y1}^{1} \\ F_{Z2}^{1} \\ M_{Y2}^{1} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}.$$

Node 1: $F_{Z1}-F_{Z1}^{1}=0$ and $M_{Y1}-M_{Y1}^{1}=0$

Node 2: $-F_{Z2}^{1}=0$ and $-M_{Y2}^{1}=0$

- By eliminating the internal forces from the equilibrium equations which do NOT contain constraint forces (node 2) with the expressions of the element contribution

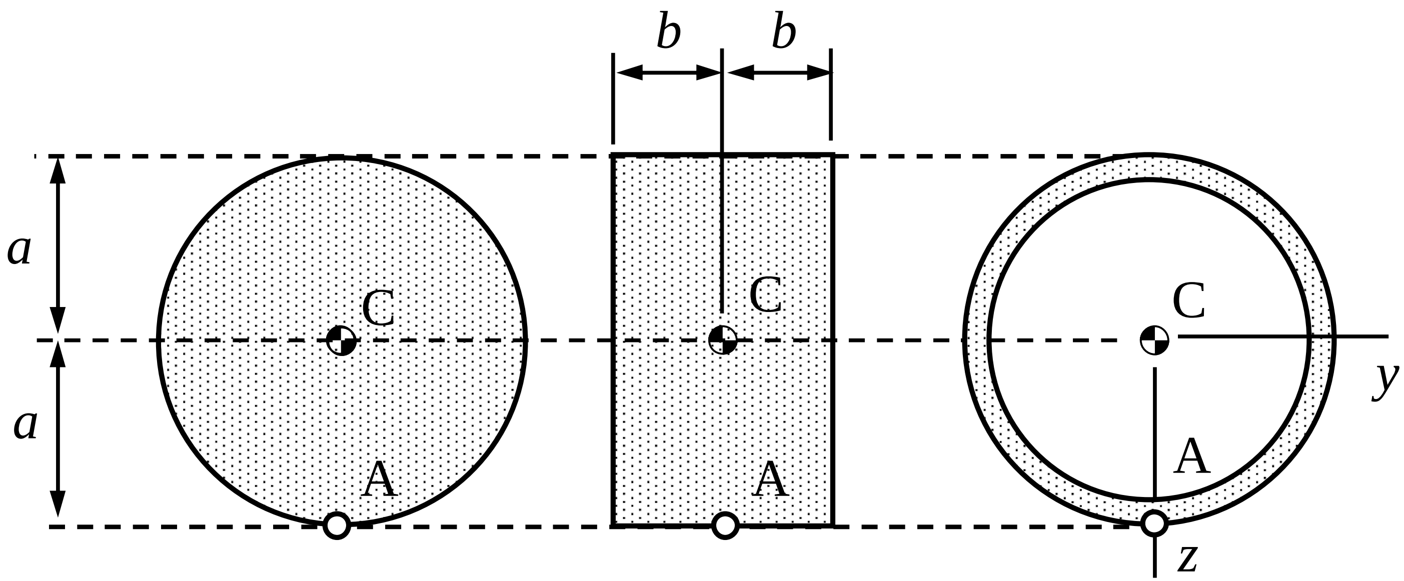
$$-\begin{Bmatrix} F_{Z2}^{1} \\ M_{Y2}^{1} \end{Bmatrix} = -\left(\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix}\right) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

- The equilibrium equations in this form give the solution to the displacement and rotation at the right-end

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \frac{fL^4}{12EI}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}^{-1}\begin{Bmatrix} 6 \\ L \end{Bmatrix} \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \frac{fL^4}{12EI}\begin{Bmatrix} 3/2 \\ -2/L \end{Bmatrix}. \quad \leftarrow$$

SECOND MOMENTS OF CROSS-SECTION



$$I_{zz} = \frac{\pi}{4} a^4 \qquad I_{zz} = \frac{4}{3} ab^3 \qquad I_{zz} = \frac{\pi}{4}[a^4 - (a-t)^4]$$

MATERIAL PARAMETERS

Material	ρ **[**kg / m^3**]**	E [GN / m^2]	ν **[** 1 **]**
Steel	7800	210	0.3
Aluminum	2700	70	0.33
Copper	8900	120	0.34
Glass	2500	60	0.23
Granite	2700	65	0.23
Birch	600	16	-
Rubber	900	10^{-2}	0.5
Concrete	2300	25	0.1

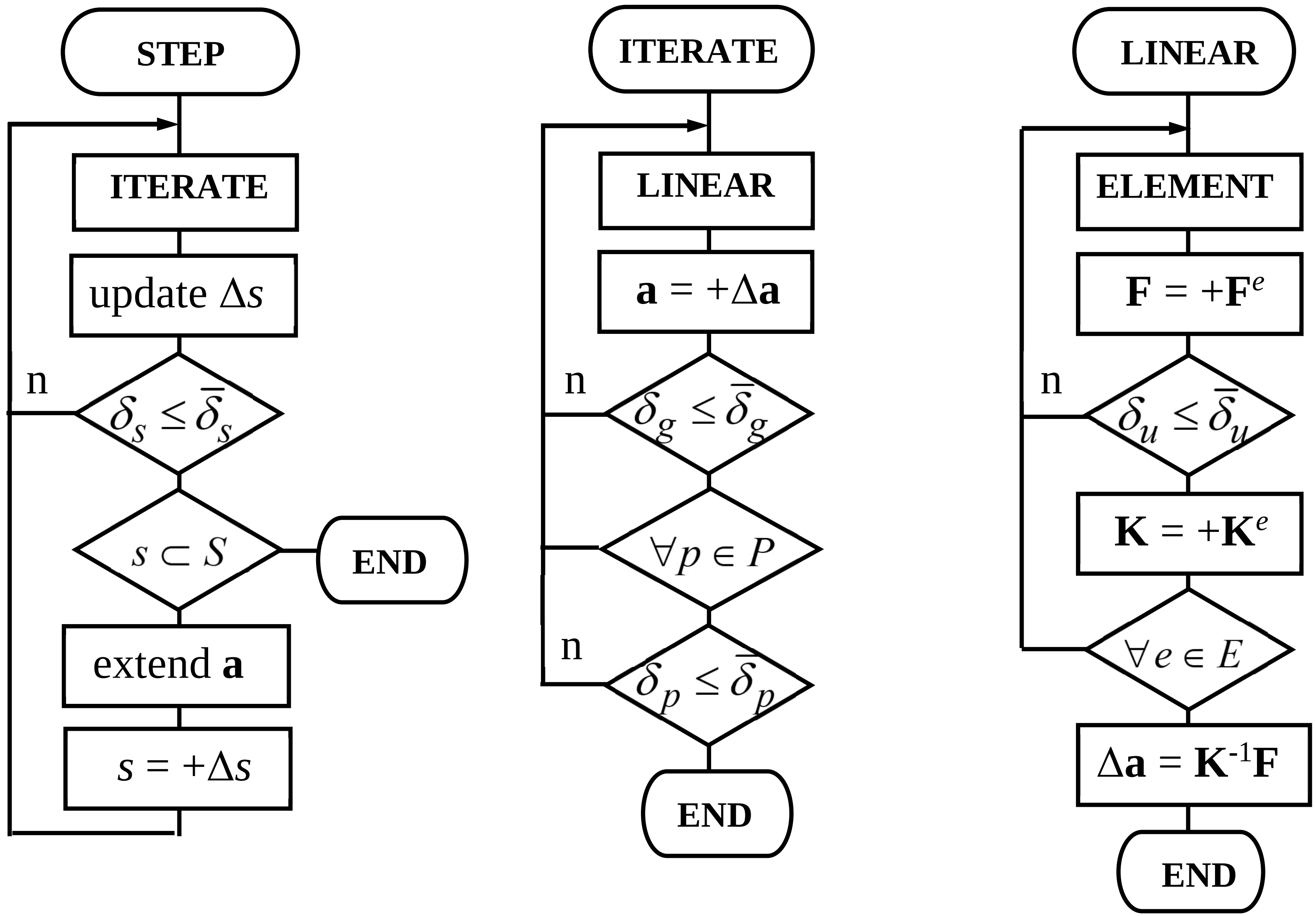
FORCE ELEMENT

In MEC-E1050, point forces and moments are taken into account by using a one-node force-moment element. The element contribution is given by

$$\begin{Bmatrix} F_{Xi} \\ F_{Yi} \\ F_{Zi} \end{Bmatrix} = -\begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} \text{ and } \begin{Bmatrix} M_{Xi} \\ M_{Yi} \\ M_{Zi} \end{Bmatrix} = -\begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}.$$

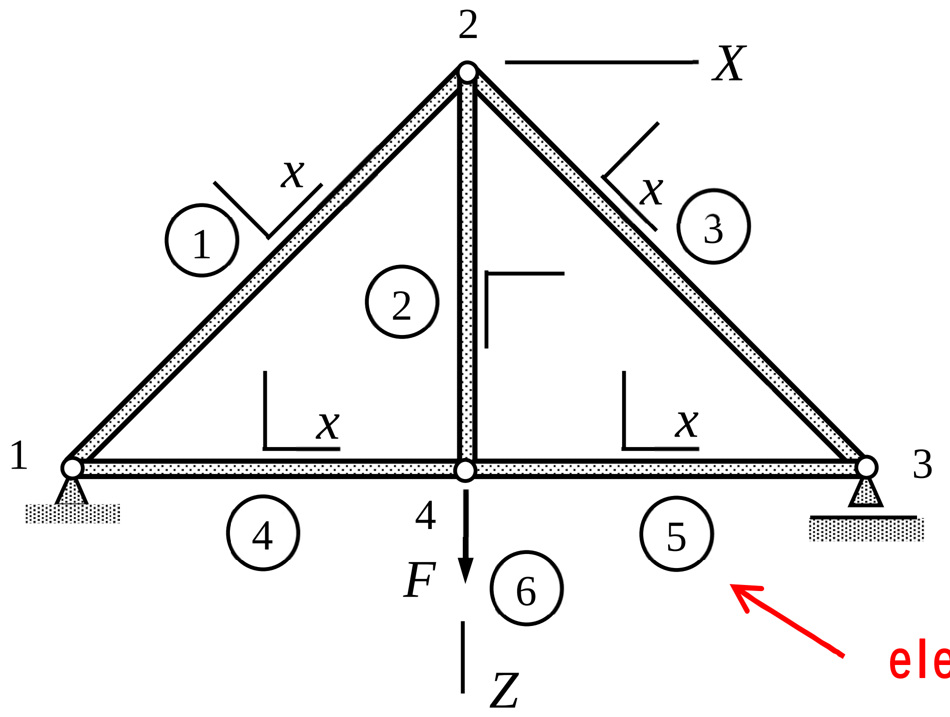
The quantities on the right-hand side are known whereas those on the left-hand side represent internal forces acting on the nodes. These “element contributions” are written directly in the structural system as that is needed finally.

2.4 ALGORITHM AND DATA STRUCTURE OF FEA



ELEMENT TABLE

Element table contains the quantities associated with elements. The table indicates also the topology of the structure (how elements are connected).



	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	{{E}, {A}}	Line[{2, 4}]
3	BAR	{{E}, {A}}	Line[{2, 3}]
4	BAR	{{E}, {A}}	Line[{1, 4}]
5	BAR	{{E}, {A}}	Line[{4, 3}]
6	FORCE	{0, 0, F}	Point[{4}]

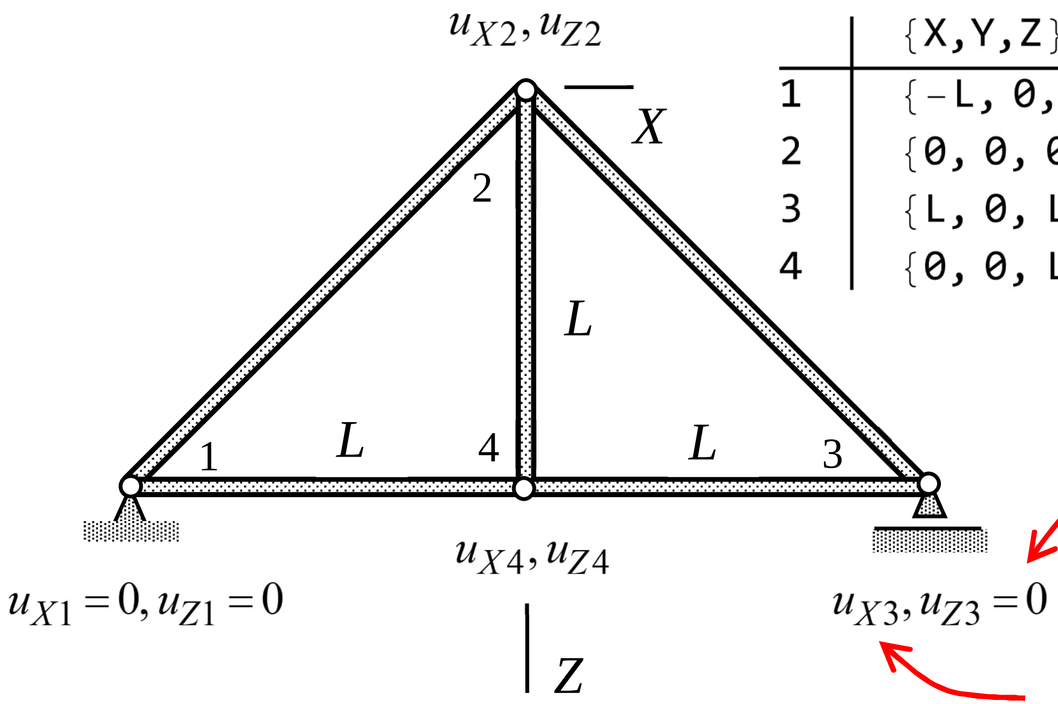
node number

element number

The order of the nodal numbers in the table depends on the orientation of the elementwise material coordinate systems.

NODE TABLE

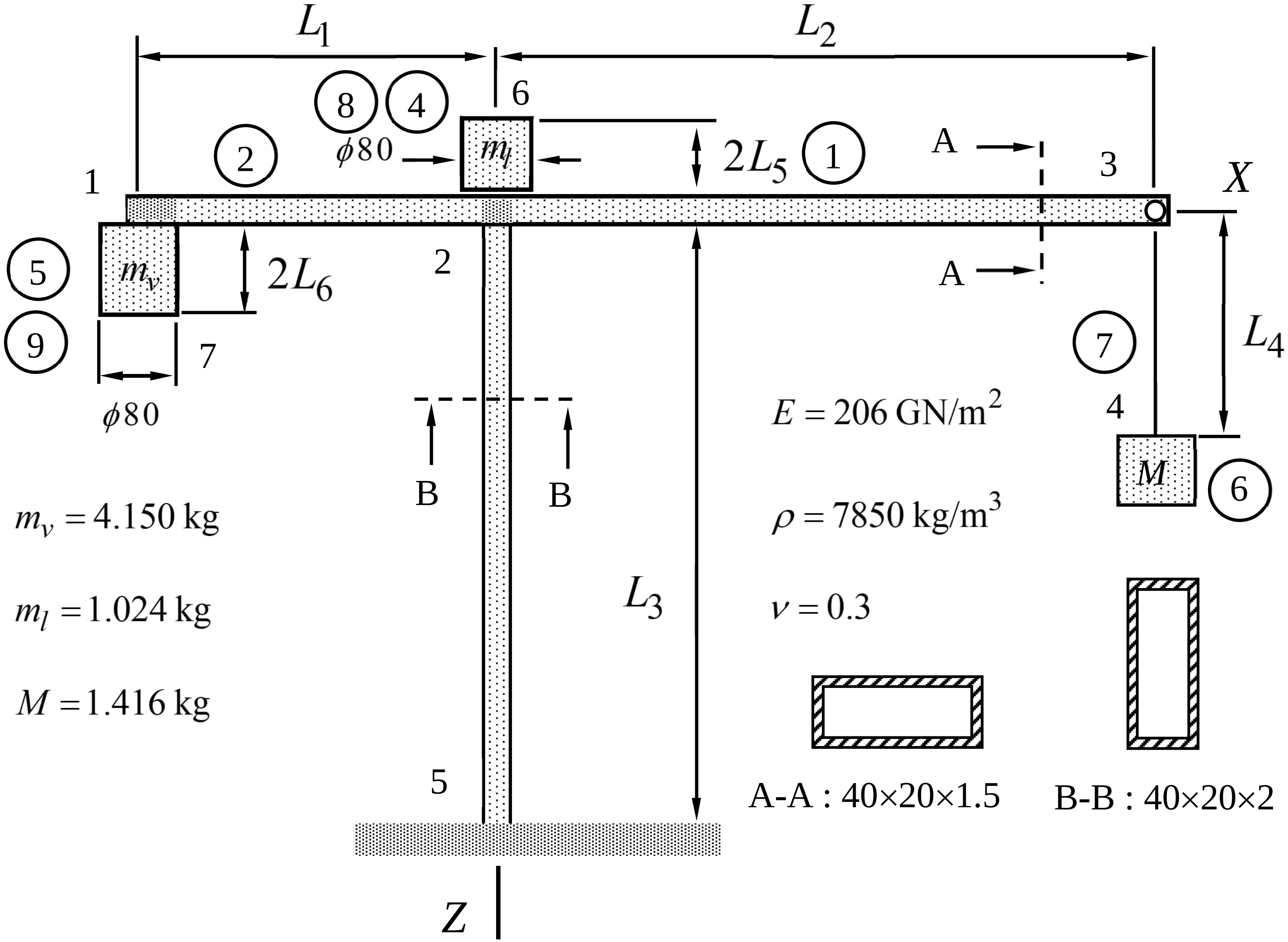
Node table contains the quantities associated with nodes. Nodal coordinates define the actual geometry. Nodal displacements and rotations represent the unknowns of the problem.



	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{-L, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{0, 0, 0}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{L, 0, L}	{uX[3], 0, 0}	{0, 0, 0}
4	{0, 0, L}	{uX[4], 0, uZ[4]}	{0, 0, 0}

If the value of a nodal displacement or rotation is known, the value is used instead of a symbol in the table!

MINIATURE MODEL OF A CRANE



$m_v = 4.150$ kg

$m_l = 1.024$ kg

$M = 1.416$ kg

$E = 206$ GN/m^2

$\rho = 7850$ kg/m^3

$\nu = 0.3$

A-A : 40×20×1.5

B-B : 40×20×2

- Problem description consists of the element and node tables:

	model	properties	geometry
1	BEAM	{{E}, {A, I, I}, {0, 0, A g ρ}}	Line[{2, 3}]
2	BEAM	{{E}, {A, I, I}, {0, 0, A g ρ}}	Line[{1, 2}]
3	BEAM	{{E}, {A, I, I}, {0, 0, A g ρ}}	Line[{2, 5}]
4	FORCE	{0, 0, g ml}	Point[{6}]
5	FORCE	{0, 0, g mv}	Point[{7}]
6	FORCE	{0, 0, g M}	Point[{4}]
7	BAR	{{E}, {a}}	Line[{3, 4}]
8	RIGID	{}	Line[{6, 2}]
9	RIGID	{}	Line[{7, 1}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{-L1, 0, 0}	{uX[1], 0, uZ[1]}	{0, θY[1], 0}
2	{0, 0, 0}	{uX[2], 0, uZ[2]}	{0, θY[2], 0}
3	{L2, 0, 0}	{uX[3], 0, uZ[3]}	{0, θY[3], 0}
4	{L2, 0, L4}	{uX[3], 0, uZ[4]}	{0, 0, 0}
5	{0, 0, L3}	{0, 0, 0}	{0, 0, 0}
6	{0, 0, -L5}	{uX[6], 0, uZ[6]}	{0, θY[6], 0}
7	{-L1, 0, L6}	{uX[7], 0, uZ[7]}	{0, θY[7], 0}

ASSEMBLY OF SYSTEM EQUATIONS

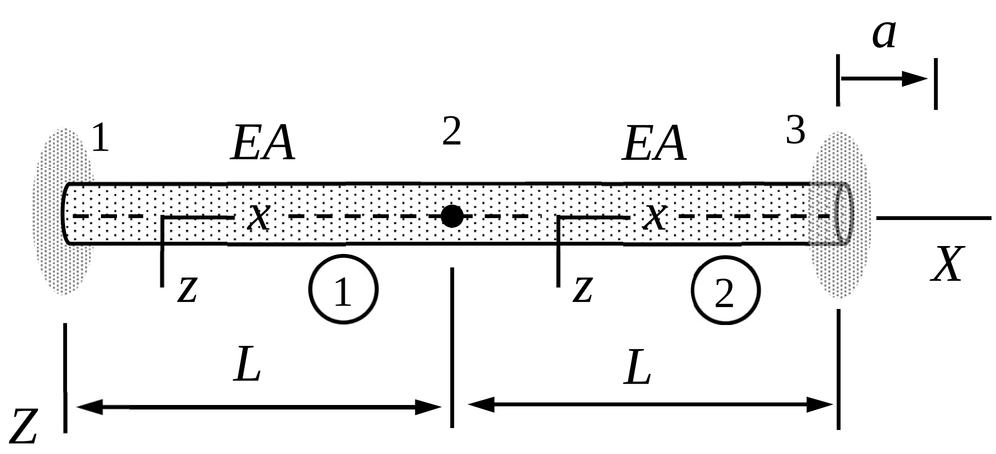
- ☐ Number the elements and nodes of the structure and express the problem data in the form of element and node tables (unless the structure is very simple),

- ☐ Write the element contributions $\mathbf{R}^e = \mathbf{K}\mathbf{a} - \mathbf{F}$ in terms of the displacement and rotation components of the structural coordinate system.

- ☐ Assemble the system equations $\mathbf{R} = \sum_{e \in E} \mathbf{R}^e = 0$ by summing the internal forces acting on the nodes in directions where displacements and rotations are not constrained.

- ☐ Solve the unknown displacements and rotations from the system equations $\mathbf{K}\mathbf{a} - \mathbf{F} = 0$ ($\mathbf{a} = \mathbf{K}^{-1}\mathbf{F}$).

EXAMPLE 2.3. Solve the nodal displacement u_{X2} of the bar shown, if the displacement of node 3 is a and rigidity EA is constant.



Answer $u_{X2} = \frac{a}{2}$

- Problem description consists of the element and node tables. In this case, the number of unknown displacements is one as the displacement of the right end is known.

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	{{E}, {A}}	Line[{2, 3}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}
3	{2 L, 0, 0}	{a, 0, 0}	{0, 0, 0}

- Element contributions need to be written in terms of displacement and force components of the structural coordinate system. As the material and structural systems coincide and $f_x = 0$

$$\text{Bar 1}: \begin{Bmatrix} F_{X1}^{1} \\ F_{X2}^{1} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

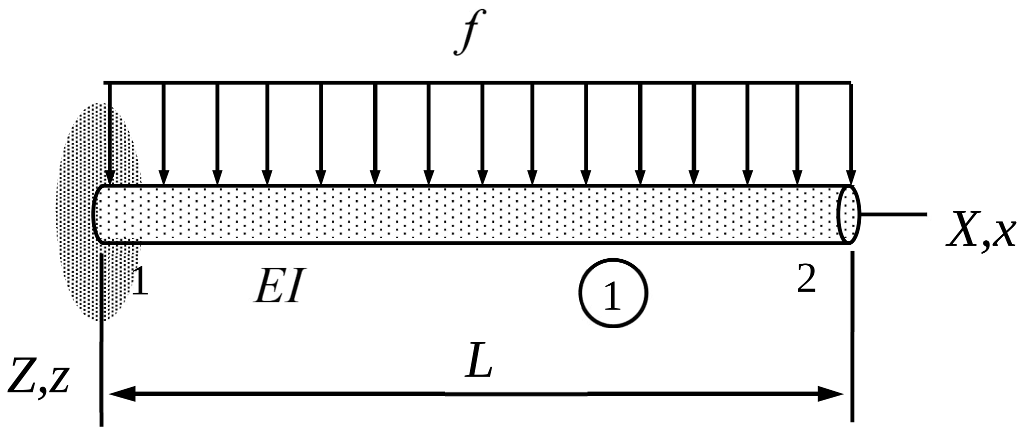
$$\text{Bar 2}: \begin{Bmatrix} F_{X2}^{2} \\ F_{X3}^{2} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ a \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

- Sum of the internal forces, external and constraint forces acting on the nodes should vanish for the equilibrium. According to the algorithm, it is enough to consider the non-constrained directions for displacements and rotations. Hence

$$F_{X2}^{1} + F_{X2}^{2} = \frac{EA}{L}u_{X2} + \frac{EA}{L}(u_{X2} - a) = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{a}{2}.$$

The problem can also be solved by the Mathematica code of the course. The code takes the problem description tables as input and returns the solution.

EXAMPLE 2.4. The beam of the figure is loaded by its own weight f (per unit length). Determine the end displacement and rotation by using a two-node beam element. Bending rigidity of the beam EI is constant.



Answer $u_{Z2} = \frac{1}{8}\frac{fL^4}{EI}$ and $\theta_{Y2} = -\frac{1}{6}\frac{fL^3}{EI}$

- Problem description consists of the element and node tables. In this case, the number of unknown displacements and rotations is 2.

	type	properties	geometry
1	BEAM	{{E, G}, {A, I, I}, {0, 0, f}}	Line[{1, 2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, θY[2], 0}

- Element contributions need to be written in the structural coordinate system

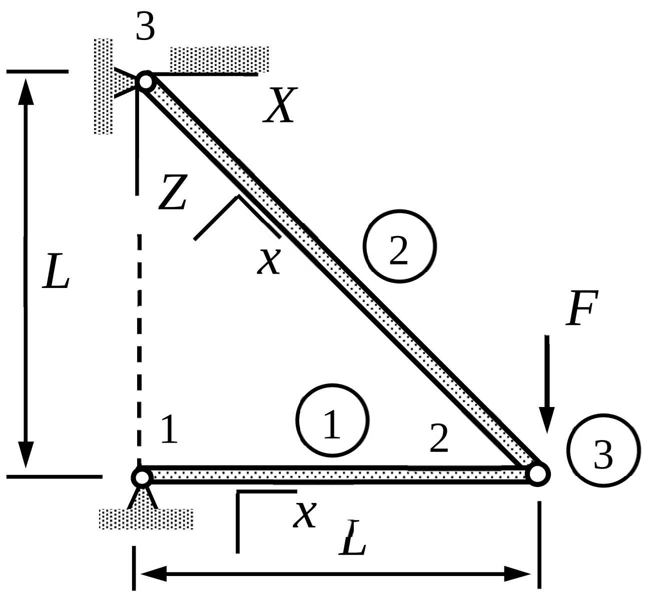
Beam 1: $$\begin{Bmatrix} F_{Z1}^{1} \\ M_{Y1}^{1} \\ F_{Z2}^{1} \\ M_{Y2}^{1} \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{f_z L}{12} \begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix} \quad (h = L)$$

- Sums of the internal forces, external and constraint forces acting on the nodes should vanish for the equilibrium. According to the algorithm, it is enough to consider the non-constrained directions for displacements and rotations (now displacement and rotation at the right end)

$$\begin{Bmatrix} F_{Z2}^{1} \\ M_{Y2}^{1} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \frac{L^3}{EI}\frac{f_z L}{12}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}^{-1}\begin{Bmatrix} 6 \\ L \end{Bmatrix} = \frac{fL^4}{12EI}\begin{Bmatrix} 3/2 \\ -2/L \end{Bmatrix}. \quad \Leftarrow$$

EXAMPLE 2.5. A bar structure is loaded by a point force having magnitude F as shown in the figure. Determine the nodal displacements of the bars. Cross-sectional area of bar 1 is A and that for bar 2 $\sqrt{8}A$. Young's modulus is E and weight is omitted.



Answer $u_{X2} = -\dfrac{FL}{EA}$ and $u_{Z2} = 2\dfrac{FL}{EA}$

- Problem description consists of the element and node tables

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	$\{\{E\}, \{2\sqrt{2}A\}\}$	Line[{3, 2}]
3	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, L}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}

- The nodal displacements of the material and structural coordinate systems are related by (orientation angle β of the material coordinate is rotation in the positive direction along the Y-axis)

$$\begin{Bmatrix} u_x \\ u_z \end{Bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} u_X \\ u_Z \end{Bmatrix} \text{ and } \begin{Bmatrix} F_X \\ F_Z \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} F_x \\ F_z \end{Bmatrix}.$$

- For bar 1, the relationships between the displacement and force components of the material and structural system and the bar element contribution are as $\beta = 0$ (notice that a bar element takes forces only in its direction and therefore $F_{z2} = 0$)

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{x2} \end{Bmatrix}, \text{ where } u_{x2} = u_{X2} \text{ and } \begin{Bmatrix} F^1_{X2} \\ F^1_{Z2} \end{Bmatrix} = \begin{Bmatrix} F_{x2} \\ 0 \end{Bmatrix} \Rightarrow$$

$$\begin{Bmatrix} F^1_{X2} \\ F^1_{Z2} \end{Bmatrix} = \frac{EA}{L}\begin{Bmatrix} u_{X2} \\ 0 \end{Bmatrix}.$$

- For bar 2, the relationships between the displacement and force components of the material and structural system and the element contribution are ($\beta = -45°$)

$$\begin{Bmatrix} F_{x3} \\ F_{x2} \end{Bmatrix} = \frac{E\sqrt{8}A}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{x2} \end{Bmatrix} \text{ where } u_{x2} = \frac{u_{X2} + u_{Z2}}{\sqrt{2}}, \begin{Bmatrix} F^2_{X2} \\ F^2_{Z2} \end{Bmatrix} = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} F_{x2} \Rightarrow$$

$$\begin{Bmatrix} F_{X2}^2 \\ F_{Z2}^2 \end{Bmatrix} = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\frac{2EA}{L}\frac{1}{\sqrt{2}}\begin{Bmatrix} 1 & 1 \end{Bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix}.$$

- Element contribution of the point force is

$$\begin{Bmatrix} F_X^3 \\ F_Z^3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -F \end{Bmatrix}.$$

- Equilibrium requires that the sum of the forces acting on the non-constrained node 2 vanish:

$$\begin{Bmatrix} F_X^1 \\ F_Z^1 \end{Bmatrix} + \begin{Bmatrix} F_X^2 \\ F_Z^2 \end{Bmatrix} + \begin{Bmatrix} F_X^3 \\ F_Z^3 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{FL}{EA}\begin{Bmatrix} -1 \\ 2 \end{Bmatrix}.$$ ←