ELEC-A7200

Signals and Systems

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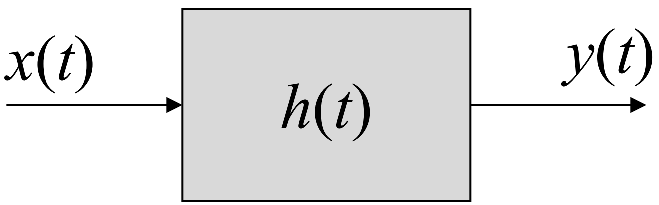
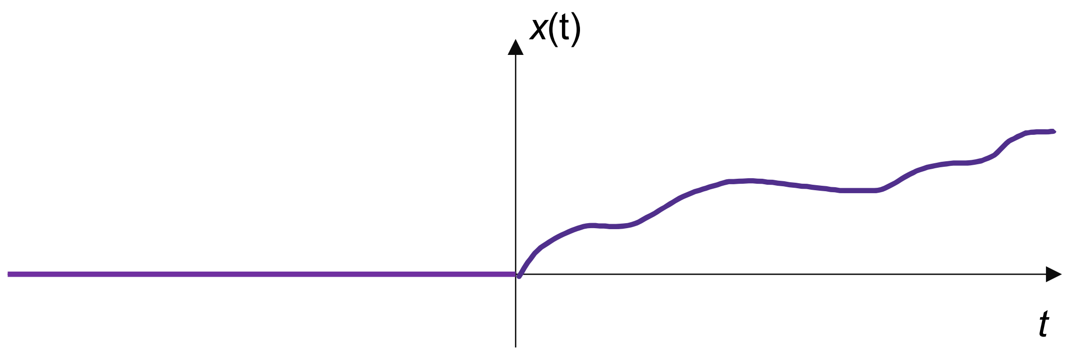


Lecture 8
Linear Time Invariant Systems – Part I

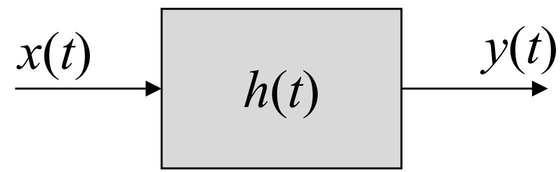
Causal signals

A continuous time signal $x(t)$ is called causal signal if the signal $x(t) = 0$ for $t < 0$. Therefore, a causal signal does not exist for negative time.



Continuous time Linear Time Invariant (LTI) Systems

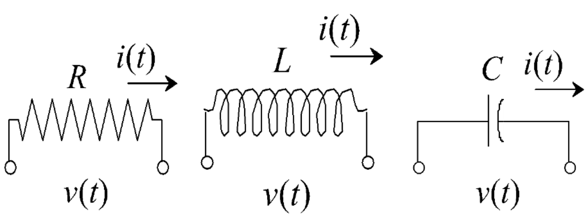
Are described in time domain by a linear differential equation with constant parameters



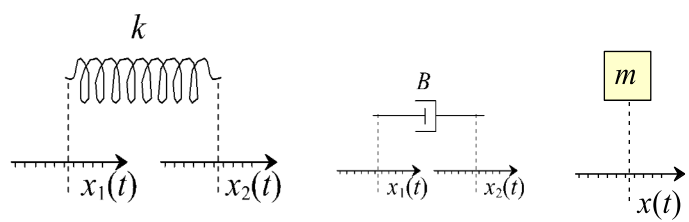
$$\frac{d^n}{dt^n}y(t) = -a_1\frac{d^{n-1}}{dt^{n-1}}y(t) - \cdots - a_n y(t) + b_0\frac{d^m}{dt^m}x(t) + b_1\frac{d^{m-1}}{dt^{m-1}}x(t) + \cdots + b_m x(t)$$

n order of the system

- Improper system $m>n$
- Proper system $m\leq n$
- Strictly proper system $m<n$



$$v(t) = Ri(t) \qquad v(t) = L\frac{di(t)}{dt} \qquad i(t) = C\frac{dv(t)}{dt}$$



$$F_k(t) = k(x_1(t) - x_2(t)) = k\Delta x(t) \qquad F_b(t) = B\frac{d\Delta x(t)}{dt} \qquad F_m(t) = m\frac{d^2x(t)}{dt^2}$$

Proper system ($m \leq n$)

- **Response of the system does not depend on future values or time derivatives of the input signal.**
- **Output signal y(t) depends directly on input signal x(t).**
- **Example: PI controller**

$$u(t) = Pe(t) + I\int_0^t e(\tau)d\tau$$

Improper system (m>n)

- **Response of the system does depend on the time derivatives of the input signal.**
- **Example: Texbook PID controller**

$$u(t) = Pe(t) + I\int_0^t e(\tau)d\tau + D\frac{d}{dt}e(t)$$

- **Improper systems cannot be physically realized**

Strictly proper system ($m<n$)

- **Response of the system does not depend on future values or time derivatives of the input signal.**
- **Output signal y(t) does not depend directly on input signal x(t).**
- **Example: RC filter**

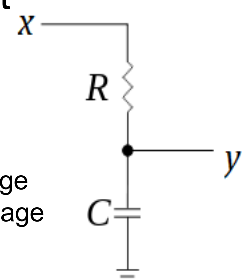
$$\frac{dy(t)}{dt} = -x(t) \Rightarrow y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

assuming causal input signal

Example 1st order systems

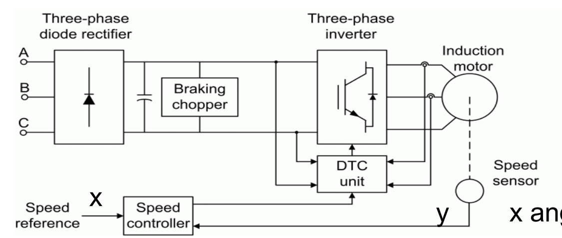
$$\frac{dy(t)}{dt} = -ay(t) + bx(t)$$

RC circuit



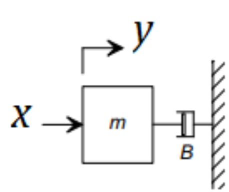
x input voltage
y output voltage

Direct torque controlled drive



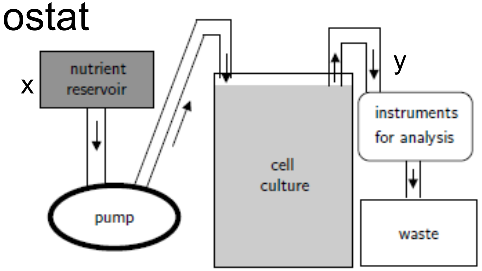
x angular speed references
y angular speed of the motor

Shock absorber



x force
y velocity

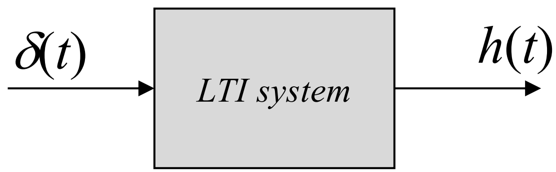
Chemostat



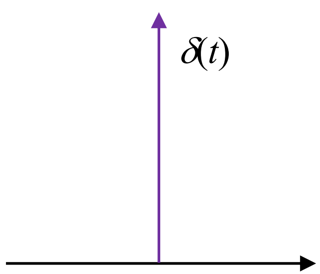
x dietary concentration
y biomass

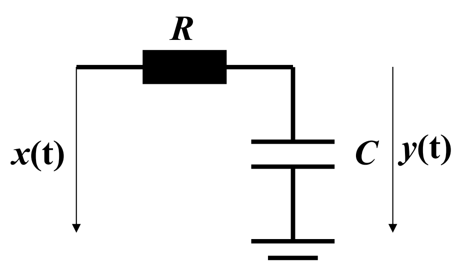
Impulse response

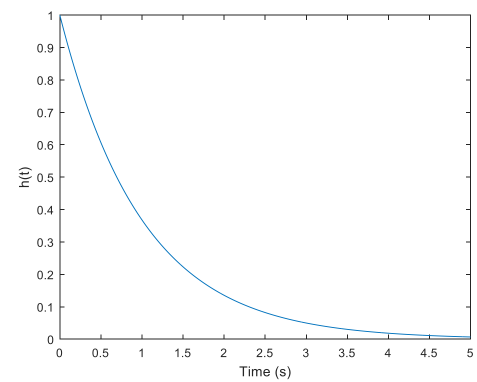
Impulse response h(t)



Example: RC filter







Impulse response corresponds physically discharging of the fully charged capacitor

Response of LTI system to general input

- In time domain the response of an LTI system to a general input $x(t)$ is given by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$$

where $h(t)$ denotes impulse response of the system.

Stability

Bounded input – Bounded output BIBO stability

- A system is said to be stable if its response of the system *y*(*t*) is bounded, $|y(t)| < \infty$,when ever the amplitude of the input x(t) is bounded $|x(t)| < \infty$.
- For an LTI system, this is equivalent to requiring that the impulse response fulfills

$$\int_{-\infty}^{\infty} |h(\lambda)|\, d\lambda < \infty$$

Laplace transform vs Fourier Transform

One-sided Laplace transform

$$\hat{X}(s) = \int_{0}^{\infty} x(t)e^{-st}dt \stackrel{\text{def}}{=} \mathrm{L}[x(t)]$$

$s = \gamma + i2\pi f$

Inverse transform

$$x(t) = \int_{\gamma+-i\infty}^{\gamma+i\infty} \hat{X}(s)e^{-st}ds \stackrel{\text{def}}{=} \mathrm{L}^{-1}[x(t)] \quad t \geq 0$$

a.k.a. Fourier-Mellin integral

for causal signals

Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt \stackrel{\text{def}}{=} \mathrm{F}[x(t)]$$

$= \hat{X}(i2\pi f)$ If signal is causal $x(t)=0$ $t<0$ and $\int_{-\infty}^{\infty} |x(t)|dt < \infty$

Inverse transform

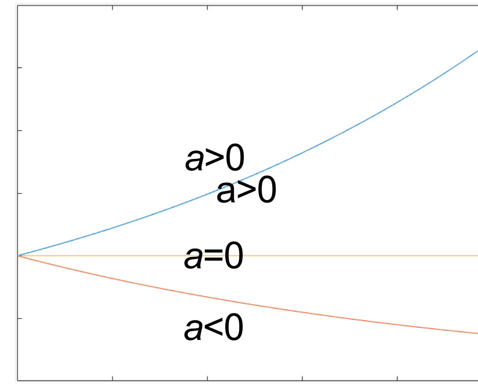
$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df \stackrel{\text{def}}{=} \mathrm{F}^{-1}[x(t)]$$

Set $\gamma=0$
and perform change of variables $s = i2\pi f$

Laplace transform vs Fourier Transform

Consider a signal $x(t) = e^{at}u(t)$

$$\int_{-\infty}^{\infty}|x(t)|dt = \int_{0}^{\infty} e^{at}dt = \lim_{t\to\infty}\frac{1}{a}e^{at} - \frac{1}{a}e^{a0} = \begin{cases} -\frac{1}{a} & a < 0 \\ \infty & a \geq 0 \end{cases}$$



$X(f) = \frac{1}{i2\pi f - a}$, $a<0$

$\hat{X}(s) = \frac{1}{s-a}$, $-\infty < a < \infty$

Fourier transform does not exist is $a \geq 0$

Laplace transform exists for all a

Laplace transform

LTI system (differential equation)

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1}x(t)}{dt^{m-1}} + \cdots + b_0 x(t)$$

Laplace transform when Initial values are set to 0:

$$s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_0 Y(s) = b_m s^m X(s) + b_{m-1}s^{m-1}X(s) + \cdots + s^m X(s)$$

Transfer function

$$H(s) \overset{\text{def}}{=} \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_m s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^n + \cdots + a_0}$$

$$= L[h(t)]$$

Causal signal f(t)
f(t)=0, t<0

Laplace transform F(s)
F(s)=L[f(t)]

Time domain	s domain
$af(t)+bg(t)$	$aF(s)+bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f'(t)$	$sF(s)-f(0^-)$
$f''(t)$	$s^2F(s)-sf(0^-)-f'(0^-)$
$f^{(n)}(t)$	$s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0^-)$
$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)\,d\sigma$
$\int_0^t f(\tau)\,d\tau=(u*f)(t)$	$\frac{1}{s}F(s)$
$e^{at}f(t)$	$F(s-a)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
$f(t)g(t)$	$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT}F(\sigma)G(s-\sigma)\,d\sigma$
$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)\cdot G(s)$

Transfer function

Transfer function of a strictly proper (m<n) **LTI system**

$$H(s) = \frac{M(s)}{N(s)} = \frac{b_m s^m + b_m s^{n-1} + \cdots + b_0}{s^n + a_{n-1} s^n + \cdots + a_0}$$

$$= K \frac{(s - z_1)^{M_1} (s - z_2)^{M_2} \cdots \left(s - z_{n_z}\right)^{M_{n_z}}}{(s - p_1)^{N_1} (s - p_2)^{N_2} \left(s - p_{n_p}\right)^{N_{n_p}}}$$

$$\sum_{i=1}^{n_z} M_i = m$$

$$\sum_{i=1}^{n_p} N_i = n$$

where the ‘*zeros*’ z_i are the zeros of the polynomial $M(s)$: $M(z_i) = 0$

and *‘poles’* p_i are the zeros of the polynomial $N(s)$: $N(p_i)$=0

Transfer function

Using partial-fraction expansion, the transfer function can be written as

$$H(s) = K \sum_{i=1}^{n_p} \sum_{k=1}^{N_i} \frac{C_{ik}}{(s - p_i)^k}$$

where

$$C_{ik} = \left[\frac{1}{(N_i - k)!} \cdot \frac{d^{N_i - k}}{ds^{N_i - k}} \left((s - p_i)^{N_i} \frac{M(s)}{N(s)} \right) \right]_{s=p_i}$$

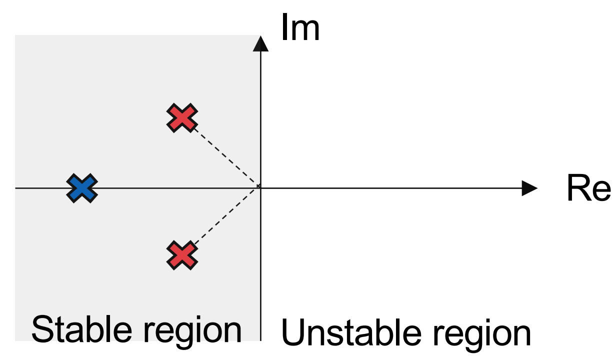
Stability

Impulse response of a LTI system

$$h(t) = L^{-1}[H(s)] = K \sum_{i=1}^{n_p} \sum_{k=1}^{N_i} C_{ik}\, t^{k-1} e^{p_i t}$$

is *BIBO stable* if the poles have negative real parts Re{p_i}<0

Poles plotted in complex plane



For real LTI system, complex poles always appear in complex conjugate pairs.

Transfer function of a second order system

Transfer function

$$H(s) = \frac{\omega_0^2}{s^2 + 2\,\zeta\omega_0 s + \omega_0^2} = \frac{M(s)}{N(s)}$$

Characteristic function $N(s)$**=** $s^2 + 2\,\zeta\omega_0 s + \omega_0^2$

Poles = solution to $N(s)$**=** $s^2 + 2\,\zeta\omega_0 s + \omega_0^2 = 0$:

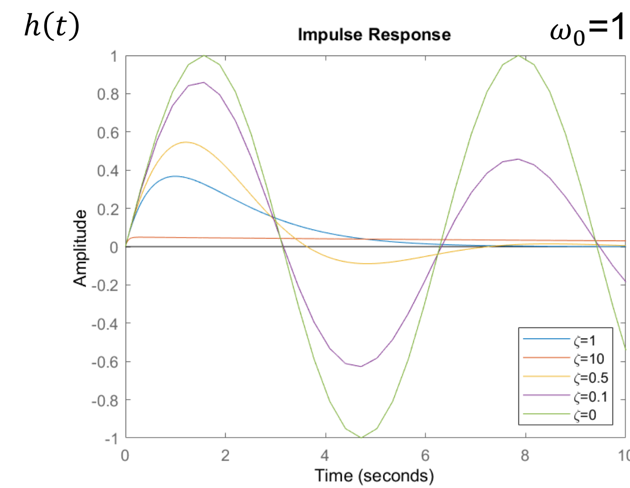
$$s = -\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_0$$

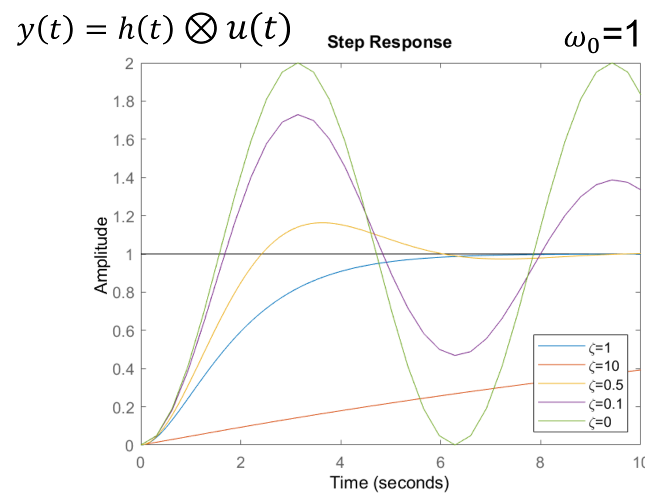
$\zeta\omega_0 > 0$ Stable

$\zeta = 0,\ \omega_0 > 0$ Marginally stable (Oscillator)

$|\zeta| < 1$ Underdamped (complex valued poles)

$|\zeta| > 1$ Overdamped (real valued poles)

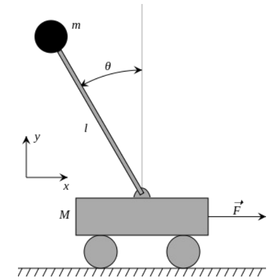


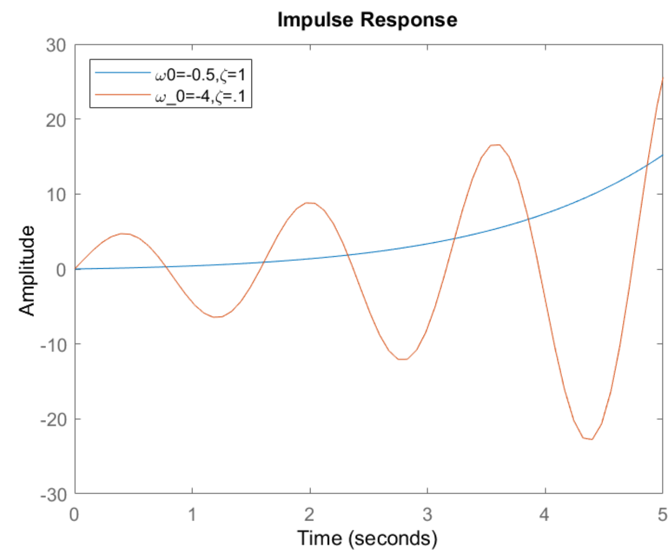


Unstable systems

System is unstable if the real part of its poles are positive.

Impulse response of an unstable system grows without bound.





Partial fractions example (1/2)

Consider a third order system with a transfer function

$$H(s) = \frac{1}{s^3 + 5s^3 + 7s + 3}$$

Poles: $s^3 + 5s^3 + 7s + 3 = 0$ => s={-3,-1,-1}

$$H(s) = \frac{1}{(s+3)(s+1)^2} = \frac{C_{11}}{(s+3)} + \frac{C_{21}}{(s+1)} + \frac{C_{22}}{(s+1)^2}$$

$$C_{11} = \lim_{s\to-3} \frac{1}{(1-1)!}(s+3)H(s) = \lim_{s\to-3} \frac{1}{0!}\frac{1}{(s+1)^2} = \mathbf{1}\,\frac{1}{(-3+1)^2} = \frac{1}{4}$$

$$C_{21} = \lim_{s\to-\mathbf{1}} \frac{1}{(2-1)!}\frac{d^{2-1}}{ds^{2-1}}(s+1)^2 H(s) = \lim_{s\to-\mathbf{1}} \frac{1}{(2-1)!}\frac{d}{ds}\frac{1}{(s+3)} = \lim_{s\to-\mathbf{1}} \frac{1}{1!}\frac{-1}{(s+3)^2} = -\frac{1}{4}$$

$$C_{22} = \lim_{s\to-\mathbf{1}} \frac{1}{(2-\mathbf{2})!}\frac{d^{2-2}}{ds^{2-2}}(s+1)^2 H(s) = \lim_{s\to-\mathbf{1}} \frac{1}{0!}\frac{1}{(s+3)} = \frac{1}{-1+3} = \frac{1}{2}$$

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$$C_{ik} = \left[\frac{1}{(N_i-k)!} \cdot \frac{d^{N_i-k}}{ds^{N_i-k}}\left((s-p_i)^{N_i}\frac{M(s)}{N(s)}\right)\right]_{s=p_i}$$

Partial fractions example (2/2)

Transfer function

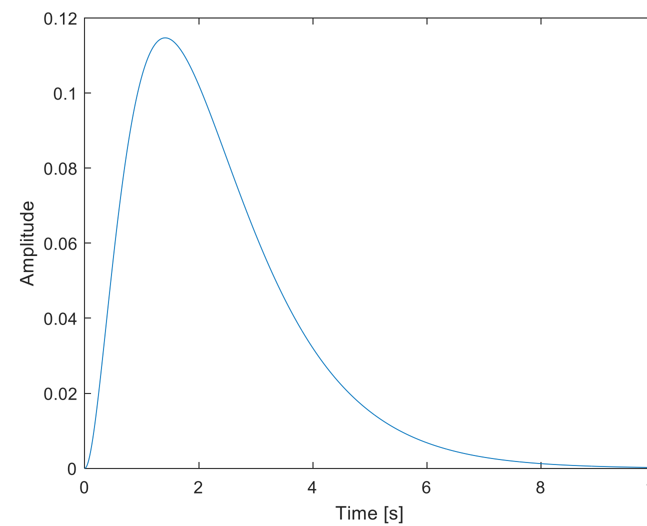
$$H(s) = \frac{1}{(s+3)(s+1)^2}$$

$$= \frac{1}{4}\frac{1}{(s+3)} - \frac{1}{4}\frac{1}{(s+1)} + \frac{1}{2}\frac{1}{(s+1)^2}$$

Inverse Laplace transform using formulas (B) and (H) gives the impulse response of the system

$$h(t) = \left(\frac{1}{4}e^{-3t} - \frac{1}{4}e^{-t} + \frac{1}{2}te^{-t}\right)u(t)$$

$f(t)$	$F(s) = \mathcal{L}[f(t)]$		Formula
$f(t) = 1$	$F(s) = \frac{1}{s}$	$s > 0$	A
$f(t) = e^{at}$	$F(s) = \frac{1}{(s-a)}$	$s > a$	B
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$	$s > 0$	C
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2+a^2}$	$s > 0$	D
$f(t) = \cos(at)$	$F(s) = \frac{s}{s^2+a^2}$	$s > 0$	E
$f(t) = \sinh(at)$	$F(s) = \frac{a}{s^2-a^2}$	$s > \|a\|$	F
$f(t) = \cosh(at)$	$F(s) = \frac{s}{s^2-a^2}$	$s > \|a\|$	G
$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s-a)^{(n+1)}}$	$s > a$	H
$f(t) = e^{at}\sin(bt)$	$F(s) = \frac{b}{(s-a)^2+b^2}$	$s > a$	I
$f(t) = e^{at}\cos(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2+b^2}$	$s > a$	J
$f(t) = e^{at}\sinh(bt)$	$F(s) = \frac{b}{(s-a)^2-b^2}$	$s - a > \|b\|$	K
$f(t) = e^{at}\cosh(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2-b^2}$	$s - a > \|b\|$	L



Discrete time systems

Discrete time systems are described by difference equations

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

Continuous time system

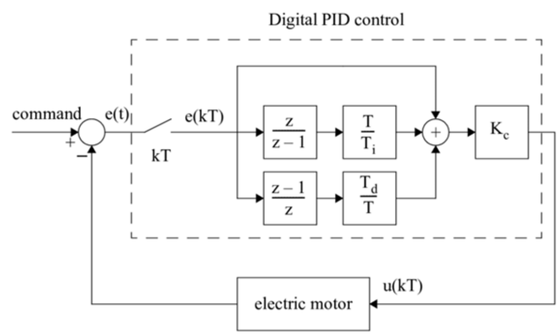
$$\frac{dy(t)}{dt} = -ay(t) + b \;\; x(t)$$

Discretized (sampled) system

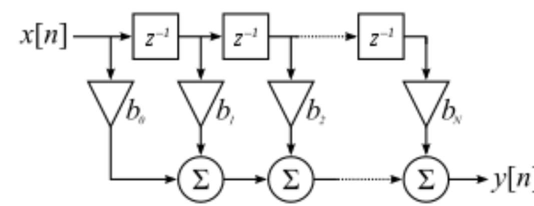
$$y[n] = y(nT_s),\ x[n] = x(nT_s)$$

$$y[n+1] = e^{aT_s} \mathrm{y[n]} + \frac{e^{aT_s}-1}{a} b \;\; \mathrm{x[n]}$$

Digital control systems



Digital filters



Z-transform

Discrete time systems are described by difference equations

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

Z-transform

$$Y(z) + a_1 z^{-1} Y(z) + \cdots + a_N z^{-N} Y(z)$$
$$= b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

Transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= Z[h[n]]$$

Z-transform

$$Z[x[n]] \overset{\text{def}}{=} X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

		Sequence	z - transform
	definition	$x_n = x[n]$	$X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$
1	addition	$x_n + y_n$	$X(z) + Y(z)$
2	constant multiple	$c\,x_n$	$c\,X(z)$
3	linearity	$c\,x_n + d\,y_n$	$c\,X(z) + d\,Y(z)$
4	delayed unit step	$u[n-m]$	$\frac{z^{1-m}}{z-1}$
5	time delay 1 tap	$x_{n-1}\,u[n-1]$	$\frac{1}{z}\,X(z)$
6	time delayed shift	$x_{n-m}\,u[n-m]$	$z^{-m}\,X(z)$
7	forward 1 tap	x_{n+1}	$z\,(X(z) - x_0)$
8	forward 2 taps	x_{n+2}	$z^2\,(X(z) - x_0 - x_1 z^{-1})$
9	time forward	x_{n+m}	$z^m \left(X(z) - \sum_{i=0}^{m-1} x_i\, z^{-i}\right)$
10	complex translation	$e^{a n} x_n$	$X(z\,e^{-a})$
11	frequency scale	$b^n x_n$	$X\left(\frac{z}{b}\right)$
12	differentiation	$n\,x_n$	$-z\,X'(z)$
13	integration	$\frac{1}{n}\,x_n$	$-\int \frac{X(z)}{z}\,dz$
14	integration shift	$\frac{1}{n+m}\,x_n$	$-z^{-m} \int \frac{X(z)}{z^{m+1}}\,dz$
15	discrete time convolution	$x_n * y_n = \sum_{i=0}^{n} x_i\, y_{n-i}$	$X(z)\,Y(z)$
16	convolution with $y_n == 1$	$\sum_{i=0}^{n} x_i$	$\frac{z}{z-1}\,X(z)$
17	initial time	x_0	$\lim_{z\to\infty} X(z)$
18	final value	$\lim_{n\to\infty} x_n$	$\lim_{z\to 1}\,(z-1)\,X(z)$

Discrete time transfer function

Transfer function

$$H(z) = \frac{M(z)}{N(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{z^N b_0 + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

Case M=N

$$H(z) = \frac{M(z)}{N(z)} = b_0 \frac{z^N + \frac{b_1}{b_0} z^{N-1} + \cdots + \frac{b_M}{b_0}}{z^N + a_1 z^{N-1} + \cdots + a_N} = b_0 + b_0 \frac{\left(\frac{b_1}{b_0} - a_1\right) z^{N-1} + \cdots + \left(\frac{b_M}{b_0} - a_1\right)}{z^N + a_1 z^{N-1} + \cdots + a_N} = b_0 + b_0 \frac{\frac{(z-z_1)^{M_1}(z-z_2)^{M_2}\cdots(z-z_{n_z})^{M_{n_z}}}{(z-p_1)^{N_1}(z-p_2)^{N_2}(z-p_{n_p})^{N_{n_p}}}}{\frac{M(z)}{N(z)}}$$

$$M_1 + M_2 + \ldots + M_{n_z} = N-1$$

$$N_1 + N_2 + \ldots + N_{n_p} = N$$

$$H(z) = b_0 + b_0 \sum_{i=1}^{n_p} \sum_{k=1}^{N_i} \frac{C_{ik}}{(z - p_i)^k}$$

$$C_{ik} = \left[\frac{1}{(N_i - k)!} \cdot \frac{d^{N_i - k}}{ds^{N_i - k}} \left((z - p_i)^{N_i} \frac{M(z)}{N(z)} \right) \right]_{z = p_i}$$

Discrete time transfer function

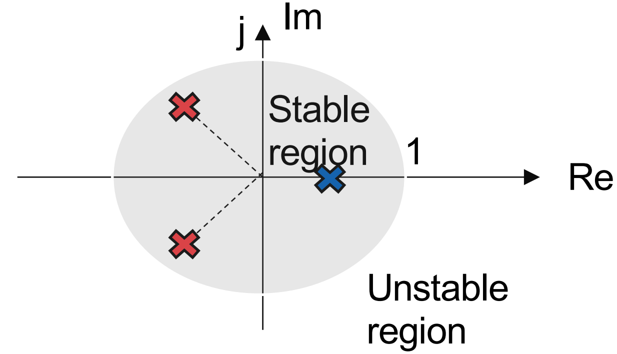
Impulse response (inverse z-transform)

$$h[n] = B\delta(n) + K\sum_{i=1}^{n_p}\sum_{k=1}^{N_i} C_{ik}\frac{(-1)^k\delta(n) + \binom{n-1}{k-1}{p_i}^n}{{p_i}^k}$$

$\delta(n) = 1\ if\ n = 0$; otherwise $\delta(n) = 0$

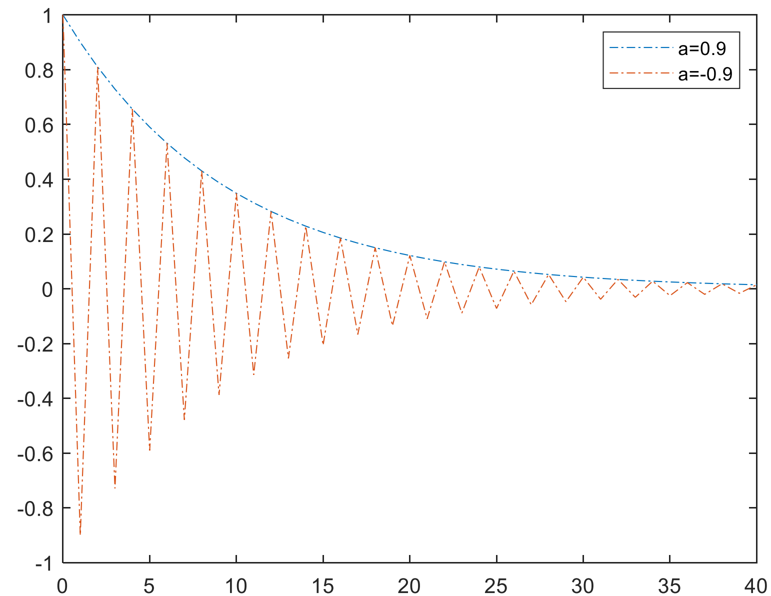
Impulse response stays bounded if the poles are inside the unit disc in complex plane: $|p_i| < 1$

Poles plotted in complex plane



Example

- First order discrete time system
  $$y[n] = ay[n-1] + bx[n]$$
- Z-transform
  $$Y(z) = az^{-1}Y(z) + bX(z)$$
- Transfer function
  $$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1-az^{-1}} = \frac{bz}{z-a}$$
- Pole: $z - a = 0 \Rightarrow z = a$
- System is stable if $|a| < 1$



First order discrete time system can exhibit oscillations.

Discretization of a continuous time system

- **Zero order hold:**

  Keep signal output constant during sample time ΔT:

  $y_{ZOH}(t) = x[k](u(t-k\Delta T) - u(t-(k+1)\Delta T))$

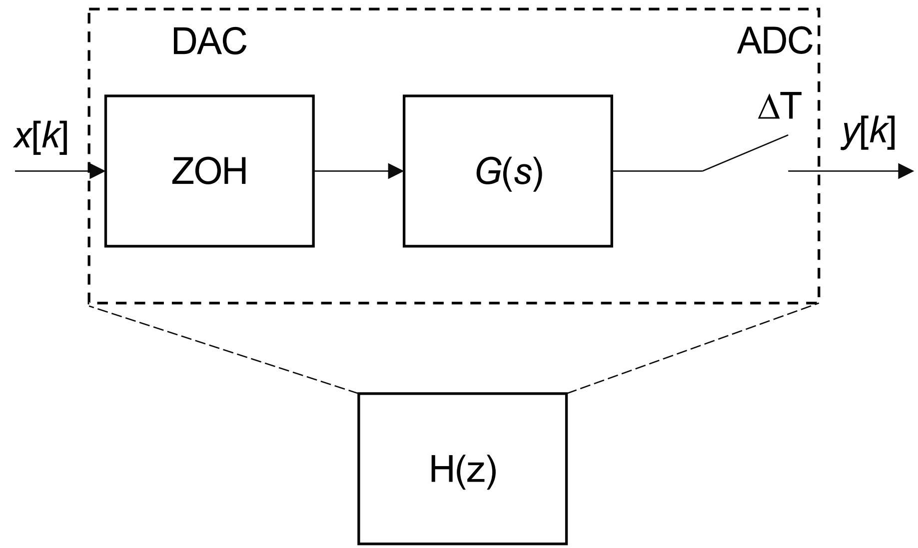
  for $k\Delta T \leq t < (k+1)\Delta T$

  Laplace transform:

  $Y_{ZOH}(s) = \dfrac{1-e^{-s\Delta T}}{s}$

- **Sampled system**

  $y[k] = L^{-1}\left\{\dfrac{1-e^{-s\Delta T}}{s}G(s)\right\}\Bigg|_{t=k\Delta T}$

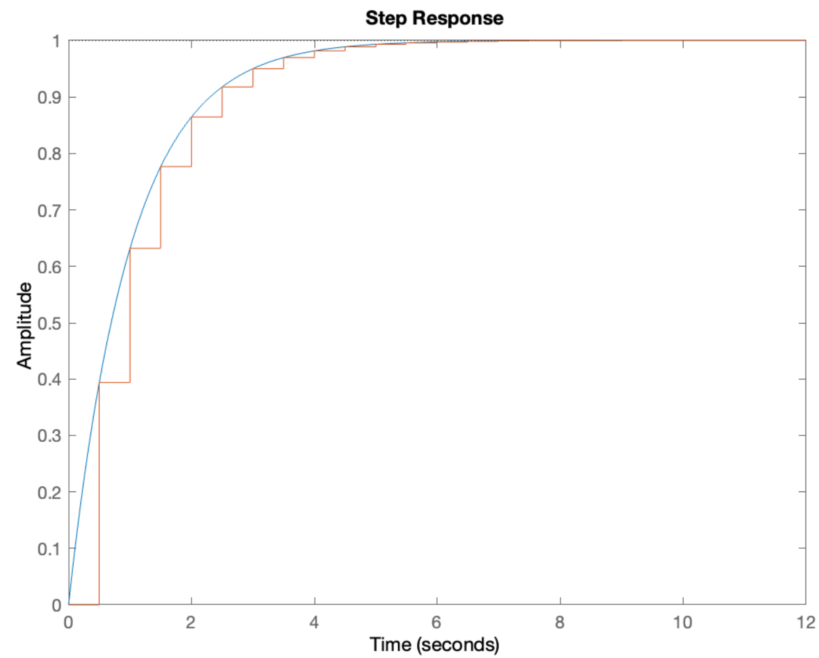


Discretation of continuous time system

Example

$$G(s) = \frac{1}{s+1}$$

```
G=tf(1,[1 1]); %G(s)=1/(s+1) First order sy
Ts=0.5; %Sampling time interval
Gd=c2d(G,Ts,'zoh'); %ZOH method
step(G,Gd) %Step response
```



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