Mathematics for Economists

Mitri Kitti

Aalto University

Optimization Problems with Inequality Constraints

Optimization Problem with Inequality Constraints

Proposition (Necessary and sufficient conditions for concave problems)

Let $f, g_1, \ldots, g_k$ be C^1 functions defined over $\mathbb{R}^n$, and let $b_1, \ldots, b_k$ be real numbers. Consider the problem of maximizing f on the constraint set defined by the inequalities

$$g_1(\boldsymbol{x}) \leq b_1, g_2(\boldsymbol{x}) \leq b_2, \ldots, g_k(\boldsymbol{x}) \leq b_k.$$

Suppose that:

(1) *f is concave*

(2) **either** *each g_i is linear* **or** *each g_i is convex and there exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $g_i(\boldsymbol{x}) < b_i$ for $i = 1, \ldots, k$.*

Form the Lagrangian $L(\boldsymbol{x}, \mu_1, \ldots, \mu_k) = f(\boldsymbol{x}) - \sum_{i=1}^k \mu_i \left[g_i(\boldsymbol{x}) - b_i\right].$

(Continued on next page)

Concave Problems

Proposition (Necessary and sufficient conditions for concave problems)

(Cont'd)

Then $\mathbf{x}^* \in \mathbb{R}^n$ *solves the constrained maximization problem under consideration* **if and only if** *there exist multipliers* $\mu_1^*, \ldots, \mu_k^*$ *such that*

1. $\frac{\partial L}{\partial x_1}(\mathbf{x}^*, \boldsymbol{\mu}^*) = 0, \ldots, \frac{\partial L}{\partial x_n}(\mathbf{x}^*, \boldsymbol{\mu}^*) = 0$
2. $\mu_1^* [g_1(\mathbf{x}^*) - b_1] = 0, \ldots, \mu_k^* [g_k(\mathbf{x}^*) - b_k] = 0$
3. $\mu_1^* \geq 0, \ldots, \mu_k^* \geq 0$
4. $g_1(\mathbf{x}^*) \leq b_1, \ldots, g_k(\mathbf{x}^*) \leq b_k$.

Note: The NDCQ is replaced by:

(2) **either** each g_i is linear **or** each g_i is convex and there exists $\mathbf{x} \in \mathbb{R}^n$ such that $g_i(\mathbf{x}) < b_i$ for $i = 1, \ldots, k$.

Concave Problems

- **Example.** Consider the constrained maximization problem:

$$\begin{aligned}\max_{x,y,z} \quad & f(x,y,z) = x + y - 2z \\ \text{s.t.} \quad & g_1(x,y,z) = x^2 + y^2 - z \leq 0 \\ & g_2(x,y,z) = -x \leq 0 \\ & g_3(x,y,z) = -y \leq 0 \\ & g_4(x,y,z) = -z \leq 0\end{aligned}$$

- The objective function f is concave
- Each g_i is convex and there exists a point, e.g. $\boldsymbol{x} = (1,1,3)$, such that $g_i(\boldsymbol{x}) < 0$ for $i = 1, \ldots, 4$
- Thus a solution to this problem is fully identified by first order conditions

Concave Problems

- **Example (cont'd).** The Lagrangian is

$$L = x + y - 2z - \lambda_1(x^2 + y^2 - z) + \lambda_2 x + \lambda_3 y + \lambda_4 z$$

- The first order conditions are

$$2x\lambda_1 = 1 + \lambda_2 \quad (1)$$

$$2y\lambda_1 = 1 + \lambda_3 \quad (2)$$

$$\lambda_1 + \lambda_4 = 2 \quad (3)$$

$$\lambda_1(x^2 + y^2 - z) = 0 \quad (4)$$

$$\lambda_2 x = 0 \quad (5)$$

$$\lambda_3 y = 0 \quad (6)$$

$$\lambda_4 z = 0 \quad (7)$$

$$\lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_3 \geq 0,\ \lambda_4 \geq 0 \quad (8)$$

$$x^2 + y^2 - z \leq 0,\ x \geq 0,\ y \geq 0,\ z \geq 0 \quad (9)$$

Concave Problems

- **Example (cont'd).** If $\lambda_1 = 0$ or $x = 0$, then $\lambda_2 = -1$ by (1), so contradicting (8). Thus we must have $\lambda_1 > 0$ and $x > 0$
- By the same token, we can use (2) to conclude that $y > 0$
- $x > 0$ and $y > 0$ imply $\lambda_2 = \lambda_3 = 0$ via (5) and (6)
- Since $\lambda_1 > 0$, we get $x = y = \frac{1}{2\lambda_1}$ from (1) and (2). Consequently, $z = \frac{1}{2\lambda_1^2} > 0$, which in turn implies $\lambda_4 = 0$ via (7)
- Finally, we get $\lambda_1 = 2$ from (3)
- Thus the unique solution is

  $$x = y = \frac{1}{4}, \ z = \frac{1}{8}$$

  with multipliers

  $$\lambda_1 = 2, \ \lambda_2 = \lambda_3 = \lambda_4 = 0.$$

Concave Problems

- **Exercise.** Consider the constrained maximization problem:

$$\begin{aligned}
\max_{x,y,z} \quad & f(x,y,z) = 3\ln(z+1) - z - 2x - y \\
\text{s.t.} \quad & g_1(x,y,z) = z^2 - x - y \leq 0 \\
& g_2(x,y,z) = -x \leq 0 \\
& g_3(x,y,z) = -y \leq 0 \\
& g_4(x,y,z) = -z \leq 0
\end{aligned}$$

- Can you apply the Proposition at pp. 2-3? Why or why not?

- Show that the unique solution to this problem is

$$(x,y,z) = \left(0, \frac{1}{4}, \frac{1}{2}\right)$$

Concave Problems

- **Exercise.** Consider the constrained maximization problem:

$$\begin{aligned} \max_{x,y} \quad & f(x,y) = x + ay \\ \text{s.t.} \quad & g_1(x,y,z) = x^2 + y^2 \leq 1 \\ & g_2(x,y,z) = -x - y \leq 0, \end{aligned}$$

  where $a \in \mathbb{R}$ is a parameter
- Can you apply the Proposition at pp. 2-3? Why or why not?
- Show that:
  - when $a \geq -1$, the unique solution is

$$(x,y) = \left(\frac{1}{\sqrt{1+a^2}}, \frac{a}{\sqrt{1+a^2}} \right);$$

  - when $a < -1$, the unique solution is

$$(x,y) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$

Mixed Constraints

- Suppose we have to solve the following constrained maximization problem:

$$\begin{aligned} \max_{x,y} \quad & 3xy - x^3 \\ \text{s.t.} \quad & 2x - y = -5 \\ & -5x - 2y \leq -37 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

- This is a problem with **mixed** constraints: one *equality* and three *inequality* constraints

Mixed Constraints

- We can rewrite the problem as one with inequality constraints only and then solve it. That is,

$$
\begin{aligned}
\max_{x,y} \quad & 3xy - x^3 \\
\text{s.t.} \quad & \mathbf{2x - y \leq -5} \\
& \mathbf{-2x + y \leq 5} \\
& -5x - 2y \leq -37 \\
& x \geq 0 \\
& y \geq 0
\end{aligned}
$$

- *Alternatively*, we can combine results from previous lectures and formulate a general proposition that will enable us to solve a problem like this without doing any rewriting/transformation

Mixed Constraints

- The general formulation of a constrained maximization problem with n variables and *mixed constraints* (k inequality and m equality constraints) is to
  - **maximize** the objective function $f(x_1, \ldots, x_n)$ with respect to $(x_1, \ldots, x_n)$
  - subject to the constraints:

$$
\begin{aligned}
g_1(x_1, \ldots, x_n) &\leq b_1 \\
g_2(x_1, \ldots, x_n) &\leq b_2 \\
\ldots \quad \ldots \quad &\ldots \quad \ldots \\
g_k(x_1, \ldots, x_n) &\leq b_k \\
h_1(x_1, \ldots, x_n) &= c_1 \\
h_2(x_1, \ldots, x_n) &= c_2 \\
\ldots \quad \ldots \quad &\ldots \quad \ldots \\
h_m(x_1, \ldots, x_n) &= c_m
\end{aligned}
$$

Mixed Constraints

- The non-degenerate constraint qualification (NDCQ) at a given point $\mathbf{x} = (x_1, \ldots, x_n)$ is formulated as follows:
  - Without loss of generality, suppose that the first k_0 inequality constraints ($k_0 \leq k$) are binding at $\mathbf{x}$, and the last $k - k_0$ are inactive at $\mathbf{x}$
  - The Jacobian of the *equality* constraints and the *binding inequality* constraints is

$$D\mathbf{g}(\mathbf{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial g_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial g_{k_0}}{\partial x_n}(\mathbf{x}) \\ \frac{\partial h_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

  - We say that the NDCQ is satisfied at $\mathbf{x}$ if the rank of $D\mathbf{g}(\mathbf{x})$ is as large as it can be

Mixed Constraints

Proposition (First order necessary conditions)

Let $f, g_1, \ldots, g_k, h_1, \ldots, h_m$ be C^1 functions defined on $\mathbb{R}^n$. Suppose that:

1. $\boldsymbol{x}^*$ *is a* **local maximizer** *of f on the constraint set defined by*

$$g_1(\boldsymbol{x}) \leq b_1, \ldots, g_k(\boldsymbol{x}) \leq b_k, h_1(\boldsymbol{x}) = c_1, \ldots, h_m(\boldsymbol{x}) = c_m$$

2. *the NDCQ is satisfied at $\boldsymbol{x}^*$.*

Form the Lagrangian $L(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) - \sum_{i=1}^{k} \mu_i [g_i(\boldsymbol{x}) - b_i] - \sum_{i=1}^{m} \lambda_i [h_i(\boldsymbol{x}) - c_i]$.
Then, there exist multipliers $\mu_1^*, \ldots, \mu_k^*, \lambda_1^*, \ldots, \lambda_m^*$ such that:

1. $\frac{\partial L}{\partial x_1}(\boldsymbol{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = 0, \ldots, \frac{\partial L}{\partial x_n}(\boldsymbol{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = 0$
2. $\mu_1^* [g_1(\boldsymbol{x}^*) - b_1] = 0, \ldots, \mu_k^* [g_k(\boldsymbol{x}^*) - b_k] = 0$
3. $h_1(\boldsymbol{x}^*) = c_1, \ldots, h_m(\boldsymbol{x}^*) = c_m$
4. $\mu_1^* \geq 0, \ldots, \mu_k^* \geq 0$
5. $g_1(\boldsymbol{x}^*) \leq b_1, \ldots, g_k(\boldsymbol{x}^*) \leq b_k$.

Mixed Constraints

- Back to the maximization problem:

$$\begin{aligned} \max_{x,y} \quad & 3xy - x^3 \\ \text{s.t.} \quad & 2x - y = -5 \\ & -5x - 2y \leq -37 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

- The Lagrangian is

$$L = 3xy - x^3 - \lambda(2x - y + 5) - \mu_1(-5x - 2y + 37) + \mu_2 x + \mu_3 y$$

Mixed Constraints

- The first order conditions are:

$$\frac{\partial L}{\partial x} = 0 \iff 3y - 3x^2 - 2\lambda + 5\mu_1 + \mu_2 = 0$$
$$\frac{\partial L}{\partial y} = 0 \iff 3x + \lambda + 2\mu_1 + \mu_3 = 0$$
$$\mu_1 \left(-5x - 2y + 37\right) = 0$$
$$\mu_2 x = 0$$
$$\mu_3 y = 0$$
$$\mu_1, \mu_2, \mu_3 \geq 0$$
$$2x - y + 5 = 0$$
$$-5x - 2y + 37 \leq 0, \quad x \geq 0, \quad y \geq 0$$

- **Exercise:** Show that the only point that satisfies the first order conditions is such that $x = 5$, $y = 15$, $\lambda = -15$, $\mu_1 = \mu_2 = \mu_3 = 0$
- **Exercise:** Show that the NDCQ is always satisfied

Verifying the Optimality

- Assume $\mathbf{x}^*$ is a candidate for an optimal point (satisfies FOCs), is it optimal (locally or globally)?

1. Is the problem concave (or convex)?
   - in maximization f should be concave and the feasible set convex
   - note 1: inequality constraints are $g_i(\mathbf{x}) \leq 0$, $i = 1, \ldots, m$ and g_i are convex functions, and inequality constraints are linear, the feasible set is convex
   - $\mathbf{x}^*$ is a global maximizer
   - note 2: sometimes equality constraints can be turned into inequalities without affecting the optimality, which may help
2. Can the problem be transformed into a concave problem?
   - for example Cobb-Douglas functions are log-concave
   - note: with log-transformation variables need to be > 0

Verifying the Optimality

3. Is the feasible set compact and objective function continuous? Are all the critical points known?
   - If yes, and NDCQ does not fail in the feasible set, evaluate the objective function at critical points and find the global maximizer
4. Try the second order conditions
   - If the Hessian of the Lagrangian is neg. def. you have a local maximizer
   - If you cant directly say anything about the definiteness of the Hessian of L, try the Bordered Hessian