Mathematics for Economists

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Comparative Statics

Envelope theorem

J. Viner

Motivating example: consumer model

Consumer model

$$\max_{c,l} \; U(c,l) \text{ s.t. } pc \leq w(h-l)+I$$

$U : \mathbb{R}^2_+ \mapsto \mathbb{R}$ utility function

$c \geq 0$ consumption

$l \in [0,h]$ leisure, $h - l =$ hours worked

p price of the consumption bundle

w wage, $w \times (h-l)$=wage income

I non-wage income

- Which of the variables are endogenous/exogenous?
- How does the welfare change when exogenous variables vary?
- How does the optimal utility change when exogenous variables vary?

Motivating example: income tax

Consumer model

$$\max_{c,l} \; U(c,l) \text{ s.t. } pc \leq (1-t)w(h-l)+I$$

- Note: w is replaced with $(1-t)w$, t is the tax rate
- What is the impact of changing taxes? [see, e.g., Mirrlees 1971, Saez and Piketty 2012, Henden 2020]

What happens when t is marginally increased or decreased?

- behavioral response; changes of c and l
- How is the optimal utility changed?

Comparative statics of optimal values

Optimization problem $\max_{x \in \mathbb{R}^n} f(x, a)$, where $a \in \mathbb{R}^m$ is the vector of exogenous variables

- the solution depends on a, assuming uniqueness: $x(a)$

What happens to the optimal f when a is changed?

- comparative statics of the (optimal) value function $v(a) = \max_x f(x, a) = f(x(a), a)$
- What are $\frac{\partial v(a)}{\partial a_i}$, $i = 1, \dots, m$?
- note $\frac{\Delta v}{\Delta a_i} \approx \frac{\partial v(a)}{\partial a_i}$ for small Δa_i's, or $\Delta v \approx \left(\frac{\partial v(a)}{\partial a_i}\right) \Delta a_i$
- note $\frac{\Delta v}{\Delta a_i} \approx \frac{\partial v(a)}{\partial a_i}$ for small Δa_i's, or $\boxed{\Delta v \approx \left(\frac{\partial v(a)}{\partial a_i}\right) \Delta a_i}$

The problem is unconstrained, but what about the consumer problem that is constrained?

- $\max\limits_{c,l} \; U(c, l)$ s.t. $pc \leq w(h - l) + I$
- $\max\limits_{l} \; U\left([w(h - l) + I]/p, l\right)$ after elimination of c

Example 1: direct computation of v

- $f(x, a) = -x^2 + 2ax + 4a^2$
- First order condition

$$\frac{\partial f(x,a)}{\partial x} = 0 \ \text{ holds at } \ x = x(a)$$

$$-2x + 2a = 0, \quad \text{which gives} \quad x(a) = a$$

- Plug the solution into f to form v, and differentiate w.r.t a:

$$v(a) = f(x(a), a) = -a^2 + 2a^2 + 4a^2 = 5a^2$$

$$\Longrightarrow$$

$$\frac{dv(a)}{da} = 10a$$

- What if $x(a)$ cannot be found analytically or finding it is hard?

Rowing ...

Assume $x, a \in \mathbb{R}$

$$\boxed{\frac{dv(a)}{da} = \frac{df(x(a),a)}{da} = ?}$$

Chain rule: $\frac{df(x(a),a)}{da} = x'(a)\frac{\partial f(x(a),a)}{\partial x} + [d(a)/da]\frac{\partial f(x(a),a)}{\partial a}$

Chain rule: $\frac{df(x(a),a)}{da} = x'(a)\frac{\partial f(x(a),a)}{\partial x} + 1\frac{\partial f(x(a),a)}{\partial a}$

$$\text{Chain rule: } \underbrace{\frac{df(x(a),a)}{da}}_{\text{Total effect}} = \underbrace{x'(a)\frac{\partial f(x(a),a)}{\partial x}}_{\text{indirect effect}} + \underbrace{\frac{\partial f(x(a),a)}{\partial a}}_{\text{direct effect}}$$

$$\text{Chain rule: } \underbrace{\frac{df(x(a),a)}{da}}_{\text{Total effect}} = \underbrace{x'(a) \times 0}_{\text{indirect effect}} + \underbrace{\frac{\partial f(x(a),a)}{\partial a}}_{\text{direct effect}}$$

How to handle the indirect effect, which contains $x(a)$?

by the first order condition $\frac{\partial f(x(a),a)}{\partial x} = 0$

$\Rightarrow$ Indirect effect vanishes!

Envelope theorem

From $$\underbrace{\frac{df(x(a),a)}{da}}_{\text{Total effect}} = \underbrace{x'(a)\frac{\partial f(x(a),a)}{\partial x}}_{\text{indirect effect}} + \underbrace{\frac{\partial f(x(a),a)}{\partial a}}_{\text{direct effect}}$$

to

$$\boxed{\frac{df(x(a),a)}{da} = \frac{\partial f(x(a),a)}{\partial a}}$$

$\boxed{\text{total effect} = \text{direct effect}}$

A version of the envelope theorem

Back to Example 1

- $f(x, a) = -x^2 + 2ax + 4a^2$
- By invoking the envelope theorem

$$\frac{dv(a)}{da} = 2x(a) + 8a = 10a$$

- By invoking the envelope theorem

$$\frac{dv(a)}{da} = 2x(a) + 8a \quad (= 10a)$$

- No need to find $v(a)$ analytically!
- If the signs of $x(a)$ and a were known, we would also know the sign of $dv(a)/da$
  using the envelope theorem it is possible to obtain "qualitative" results of this type, without actually ever finding $x(a)$ explicitly

The envelope theorem

Assume that the optimum of f is unique in the neighborhood of a^*, f is differentiable at $(x(a^*), a^*)$, and $x(a)$ is differentiable at a^*. Then

$$\frac{\partial v(a^*)}{\partial a_i} = \frac{\partial f(x(a^*), a^*)}{\partial a_i}$$

for all $i = 1, \ldots, m$, where v is the value function

- Indirect effects do not matter
- Changes of the behavior can be ignored

Geometric interpretation

- The graph of the value function v is the envelope of the family of graphs of $f(\cdot, a)$
- The slope of v is the slope of $f(\cdot, a)$ to which it is a tangent
- Example $f(x, a) = -x^2 + 2ax + 4a^2$: video

Application: Wage increase of Wal-Mart

In 2015 Wal-Mart increased its minimum wages from \$9/hr to \$10/hr

- outcome: lower turnover of employees, more work applications
- note: there was exogenous pressure coming from competitors

Efficiency wages

- worker effort dependent on wages $e(w)$ (increasing)
- profit function $\pi(L, w) = R(L \times e(w)) - wL$

1. What is the effect of a marginal increase in the wage?
   - assume the optimality of \$9/hr and a small change, what happens to profits?
2. What would happen in the competitive case if w increases?
   - $e(w) = 1$ and w is exogeneous
   - Note: $\Delta v \approx \left(\frac{\partial v(a)}{\partial a_i}\right) \times (\Delta a_i)$

Paul Krugman: Wal-Mart's Visible Hand

Interpretation of Lagrange multipliers

Proposition (Envelope Theorem for Constrained Problems)

Let $f, h_1, \ldots, h_m$ be C^1 functions on $\mathbb{R}^n$. Let $\boldsymbol{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ be parameters, and consider the problem of **maximizing or minimizing** *$f(\boldsymbol{x})$ w.r.t. $\boldsymbol{x}$ subject to the constraints:*

$$h_1(\boldsymbol{x}) = a_1, \ldots, h_m(\boldsymbol{x}) = a_m.$$

Let $(x_1^*(\boldsymbol{a}), \ldots, x_n^*(\boldsymbol{a}))$ be the solution to this problem, with corresponding Lagrange multipliers $\lambda_1^*(\boldsymbol{a}), \ldots, \lambda_m^*(\boldsymbol{a})$.

Suppose further that all the x_i^*'s and λ_i^*'s are differentiable functions of $\boldsymbol{a}$ and that the NDCQ holds. Then, for each $j = 1, \ldots, m$,

$$\frac{d}{da_j} f(x_1^*(\boldsymbol{a}), \ldots, x_n^*(\boldsymbol{a})) = \lambda_j^*(\boldsymbol{a}).$$

Interpretation of Lagrange multipliers

The previous proposition can be easily generalized to the case with inequality constraints.

Proposition (Envelope Theorem for Constrained Problems)

Let $f, g_1, \dots, g_m$ be C^1 functions on $\mathbb{R}^n$. Let $\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{R}^m$ be parameters, and consider the problem of **maximizing** *$f(\boldsymbol{x})$ w.r.t. $\boldsymbol{x}$ subject to the constraints:*

$$g_1(\boldsymbol{x}) \leq a_1, \dots, g_m(\boldsymbol{x}) \leq a_m.$$

Let $(x_1^*(\boldsymbol{a}), \dots, x_n^*(\boldsymbol{a}))$ be the solution to this problem, with corresponding Lagrange multipliers $\mu_1^*(\boldsymbol{a}), \dots, \mu_m^*(\boldsymbol{a})$.
Suppose further that all the x_i^*'s and μ_i^*'s are differentiable functions of $\boldsymbol{a}$ and that the NDCQ holds. Then, for each $j = 1, \dots, m$,

$$\frac{d}{da_j} f(x_1^*(\boldsymbol{a}), \dots, x_n^*(\boldsymbol{a})) = \mu_j^*(\boldsymbol{a}).$$

Interpretation of Lagrange multipliers

Proposition

Let $f, g_1, \ldots, g_m$ be C^1 functions on $\mathbb{R}^n$. Let $\boldsymbol{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ be parameters, and consider the problem of **minimizing** *$f(\boldsymbol{x})$ w.r.t. $\boldsymbol{x}$ subject to the constraints:*

$$g_1(\boldsymbol{x}) \geq a_1, \ldots, g_m(\boldsymbol{x}) \geq a_m.$$

Let $(x_1^*(\boldsymbol{a}), \ldots, x_n^*(\boldsymbol{a}))$ be the solution to this problem, with corresponding Lagrange multipliers $\mu_1^*(\boldsymbol{a}), \ldots, \mu_m^*(\boldsymbol{a})$.
Suppose further that all the x_i^*'s and μ_i^*'s are differentiable functions of $\boldsymbol{a}$ and that the NDCQ holds. Then, for each $j = 1, \ldots, m$,

$$\frac{d}{da_j} f(x_1^*(\boldsymbol{a}), \ldots, x_n^*(\boldsymbol{a})) = \mu_j^*(\boldsymbol{a}).$$

Example

- Consider the problem

$$\begin{aligned} \max_{x,y,z} \quad & f(x,y,z) = xyz \\ \text{s.t.} \quad & x + y + z \leq 1 \\ & x \geq 0 \\ & y \geq 0 \\ & z \geq 0 \end{aligned}$$

- The Lagrangian is

$$L = xyz - \mu_1(x + y + z - 1) + \mu_2 x + \mu_3 y + \mu_4 z$$

- You can verify that the solution is $x^* = y^* = z^* = \frac{1}{3}$, with $\mu_1^* = \frac{1}{9}$ and $\mu_2^* = \mu_3^* = \mu_4^* = 0$

Example

- Suppose that the first constrained is changed to $x + y + z \leq 0.9$. What is the corresponding change in the value function $f(x^*, y^*, z^*)$?
- Write the constraint in parametric form $x + y + z \leq a$. We know the solution when $a = 1$, and now we want to estimate the change in the value function when $da = -0.1$
- By the envelope theorem,

$$df(x^*(1), y^*(1), z^*(1)) = \mu_1^* da = \frac{1}{9}\left(-\frac{1}{10}\right) = -\frac{1}{90}$$

- So by decreasing a from 1 to 0.9, the value function decreases approximately by 0.0111
- Notice that the envelope theorem enables us to estimate the change without solving the problem with $a = 0.9$. If we solved the new problem, we would find the *exact* change in the value function

Shadow Prices

- We can use the envelope theorem to give an economic interpretation to Lagrange multipliers
- Consider a firm producing n different final goods. Those final goods are using as inputs m different resources whose total supplies are $a_1, \dots, a_m$
- Given the quantities $x_1, \dots, x_n$ of the final goods, let $\pi(\boldsymbol{x})$ denote the firm's profit when goods $\boldsymbol{x} = (x_1, \dots, x_n)$ are produced, and let $g_i(\boldsymbol{x})$ be the corresponding number of units of resource number i required, with $i = 1, \dots, m$

Shadow Prices

- The firm's profit maximization problem is

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & \pi(\boldsymbol{x}) \\ \text{s.t.} \quad & g_1(\boldsymbol{x}) \leq a_1 \\ & \dots \\ & g_m(\boldsymbol{x}) \leq a_m \end{aligned}$$

- By the envelope theorem,

$$\frac{d\pi}{da_i}(x_1^*(\boldsymbol{a}), \dots, x_n^*(\boldsymbol{a})) = \mu_i^*(\boldsymbol{a})$$

- In words, the multiplier $\mu_i^*(\boldsymbol{a})$ tells how valuable another unit of input i would be to the firm's profit

- $\mu_i^*(\boldsymbol{a})$ is often called the *shadow price* or *internal value* of input i

Shadow Prices

- **Exercise.** If x thousand Euro is spent on labor and y thousand Euro is spent on equipment, a certain factory produces $Q(x, y) = 50x^{\frac{1}{2}}y^2$ units of output.

  (a) How should $80,000$ Euro be allocated between labor and equipment to yield the largest possible output?

  (b) Use the envelope theorem to estimate the change in maximum output if this allocation decreased by 1000 Euro

  (c) Compute the exact change in (b)

Net National Product

- Consider a two period consumption/investment model of a social planner

$$\max_{c_0, c_1, i_0} u(c_0) + \delta u(c_1)$$
$$c_0 + i_0 \leq k_0$$
$$c_1 \leq f(i_0)$$

- First order conditions

$$u'(c_0^*) = \lambda_0$$
$$\delta u'(c_1^*) = \lambda_1$$
$$-\lambda_0 + \lambda_1 f'(i_0^*) = 0$$

- Observation 1: if the objective function was linearized at c_0^*, c_1^*, we would get an objective function $\lambda_0(c_0 - c_0^*) + \lambda(c_1 - c_1^*)$
- Observation 2: The first order conditions hold for the linearized objective function hold at c_0^*, c_1^*

Net National Product

- Observation 3: assuming $c_1^* = f(i_0^*)$ we have
  $\lambda_0 c_0^* + \lambda_1 c_1^* = \lambda_0 c_0^* + \lambda_0 f(i_0^*)/f'(i_0^*) \approx \lambda_0 c_0 + \lambda_0 f'(i_0^*) i_0^*/f'(i_0^*) = \lambda_0(c_0^* + i_0^*)$
- $c_0^* + i_0^*$ is the net national product
- Net national product is an approximation of the optimal welfare!
  - national accounting system provides a way to measure welfare
  - BUT: this is only in an idealized world
- What if the national accounting system is missing something (goods with no markets/prices)?
  - find shadow prices! (e.g. green national accounting) (more)

Harvesting a Resource Stock

- Two period with consumptions c_0 and c_2
- Objective function (NPV) $\sum_{t=0}^{1}[B(c_t) - C(c_t)]/(1+r)^t$
- Resoure constraint $c_0 + c_1 = S$
- FOCs:

$$\begin{aligned} B'(c_0) - C'(c_0) - \lambda = 0 \\ [B'(c_1) - C'(c_1)]/(1+r) - \lambda = 0 \\ c_0 + c_1 = S \end{aligned}$$

- Observation 1: marginal wtp $\neq$ marginal cost!
- Observation 2: present value of $MWTP - MC$ is constant in each period
  - note B' can be interpreted as the market price (why?)
  - the difference $B'(c_t) - C'(c_t)$ is the scarcity rent (which equals the shadow price)

A General Envelope Theorem

The following Proposition combines the envelope theorem and the interpretation of Lagrange multipliers

Proposition (Envelope Theorem for Constrained Problems)

Let $f, h_1, \ldots, h_m$ be C^1 functions on $\mathbb{R}^n$. Let $a \in \mathbb{R}$ be a parameter, and consider the problem of **maximizing** *$f(\boldsymbol{x}; a)$ w.r.t. $\boldsymbol{x}$ subject to the constraints:*

$$h_1(\boldsymbol{x}; a) = 0, \ldots, h_m(\boldsymbol{x}; a) = 0.$$

Let $(x_1^*(a), \ldots, x_n^*(a))$ be the solution to this problem, with corresponding Lagrange multipliers $\mu_1^*(a), \ldots, \mu_m^*(a)$.
Suppose further that all the x_i^*'s and μ_i^*'s are differentiable functions of a and that the NDCQ holds. Then,

$$\frac{d}{da} f(x_1^*(a), \ldots, x_n^*(a); a) = \frac{\partial \mathcal{L}}{\partial a}\left(x_1^*(a), \ldots, x_n^*(a), \mu_1^*(a), \ldots, \mu_m^*(a); a\right),$$

where $\mathcal{L}$ is the Lagrangian function for this problem.

Example

- Consider the utility maximization problem

$$\begin{aligned} \max_{x_1, x_2} \quad & u(x_1, x_2) \\ \text{s.t.} \quad & p_1 x_1 + p_2 x_2 \leq w, \end{aligned}$$

  where $p_1 > 0$ and $p_2 > 0$ are prices, and $w > 0$ is income or wealth

- Notice that x_1 and x_2 are unknown variables, whereas p_1, p_2 and w are parameters

- Suppose this problem has a unique solution, at which the budget constraint is binding:

$$x_1^*(p_1, p_2, w), \quad x_2^*(p_1, p_2, w)$$

Example

- Define the value function v of this problem:

$$v(p_1, p_2, w) := u\left(x_1^*(p_1, p_2, w), x_2^*(p_1, p_2, w)\right)$$

- The function v is called **indirect utility function**

- By using the envelope theorem, we can estimate how v changes when we change one of the problem's parameters

- Recall that the Lagrangian is

$$\mathcal{L}(x_1, x_2, \mu; p_1, p_2, w) = u(x_1, x_2) - \mu(p_1 x_1 + p_2 x_2 - w)$$

Example

- Thus we have:

$$\frac{dv}{dp_1}(p_1, p_2, w) = \frac{\partial \mathcal{L}}{\partial p_1}(x_1^*, x_2^*, \mu^*; p_1, p_2, w) = -\mu^* x_1^*$$

$$\frac{dv}{dp_2}(p_1, p_2, w) = \frac{\partial \mathcal{L}}{\partial p_2}(x_1^*, x_2^*, \mu^*; p_1, p_2, w) = -\mu^* x_2^*$$

$$\frac{dv}{dw}(p_1, p_2, w) = \frac{\partial \mathcal{L}}{\partial w}(x_1^*, x_2^*, \mu^*; p_1, p_2, w) = \mu^*$$