Model solutions 6

1. (a) The profit function of Acme is:

$$\begin{aligned}\Pi_A(Q_A, Q_B) =& P^D(Q_A, Q_B)Q_A - TC_A(Q_A)\\ =& (100 - 2(Q_A + Q_B))Q_A - 20Q_A - 200\\ =& -2Q_A^2 - 2Q_AQ_B + 80Q_A - 200\end{aligned}$$

The profit function of Bonk is otherwise identical, but with a lower marginal cost:

$$\begin{aligned}\Pi_B(Q_A, Q_B) =& P^D(Q_A, Q_B)Q_B - TC_B(Q_B)\\ =& (100 - 2(Q_A + Q_B))Q_B - 10Q_B - 200\\ =& -2Q_B^2 - 2Q_AQ_B + 90Q_B - 200\end{aligned}$$

The same actions are available to both players, $Q_i \in \{0, 1, 2\}$. However, the game is not symmetric because Bonk faces lower variable cost. Plugging in the different combinations of actions in in the above profit functions we get the payoff matrix:

		Bonk		
	€k	0	1	2
Acme	0	0,0	0,392	0,728
	1	312,0	184,264	56,472
	2	568,0	312,136	56,216

(b) Bonk has a dominant strategy of sending out two vessels. Knowing this, Acme is indifferent between sending out one or two vessels. So we know that, in Nash equilibrium, Acme sends out either one or two vessels and Bonk sends out two vessels.[1]

(c) Bonk has a dominant strategy so any espionage capabilities and their disclosure can make no difference to it.

[1] Acme would also be indifferent with any mixing probabilities between sending out one or two vessels. Here mixing would not add any Nash equilibrium outcomes to the game.

2. (a) Taking into account the costs, the payoff function for a scooter firm participating in a market of n firms is (in €m):

$$\Pi(n) = 24/n - 15$$

The number of active firms n is either 0, 1 or 2. By plugging in the above function $n = 1$ and $n = 2$ and by knowing that action "Out" yields a zero payoff, we get the following payoff matrix (where n corresponds to the number of players choosing "In"):

	€m	B: Out	B: In
A	Out	0,0	0,9
A	In	9,0	-3,-3

(b) There are two pure strategy Nash equilibria: {In,Out} and {Out,In}. Both firms would prefer to be the one that stays in while the other one moves out of the market. The game is symmetric so there is must also be a symmetric Nash equilibrium, and the only remaining possibility is that it is in mixed strategies. Denote the probability of choosing "Out" with p.

	€m	*p* Out	*1-p* In
p	Out	0,0	0,9
1-p	In	9,0	-3,-3

For example, to make player B mix, player A has to choose p so that B is indifferent between playing "Out" and "In". Let's solve for the p that satisfies this condition:

$$0 \times p + 0 \times (1 - p) = 9 \times p - 3 \times (1 - p)$$
$$0 = 9p - 3 + 3p$$
$$p = 1/4$$

Due to symmetry, in the mixed strategy Nash equilibrium, both players stay "Out" with probability 1/4 and "In" with probability 3/4.

(c) Let's express the three-player game as two payoff matrices, where the left matrix shows payoffs for A, B and C when C plays "Out", and the right matrix shows payoffs for A, B and C when C plays "In". We also need to compute one additional payoff value for the case where all three players play "In": $\Pi(3) = -7$. The payoff matrices become:

C plays "Out"

	€m	B: Out	B: In
A	Out	0,0,0	0,9,0
A	In	9,0,0	-3,-3,0

C plays "In"

	€m	B: Out	B: In
A	Out	0,0,9	0,-3,-3
A	In	-3,0,-3	-7,-7,-7

(d) In a symmetric equilibrium all players use the same strategy. In pure strategies, the symmetric cases would be {Out,Out,Out} and {In,In,In}. However, these are not Nash equilibria, since players have profitable deviations from them.

Let's then look at mixed strategies. The game is still symmetric, because we can swap the labels between players without having to change the numbers in the payoff matrices. Therefore we can still just look for a symmetric mixed strategy Nash equilibrium, where each player plays "Out" with probability p. In equilibrium, p needs to be such that a firm is indifferent between playing "Out" and playing "In". We know that the payoff of playing "Out" is zero. Thus, we need to find the p that yields zero expected profits for playing "In".

Consider the game from the point of view of a firm choosing to stay "In" the market. The below table describes the probabilities for the number of other firms staying "Out" and the resulting payoffs for this firm.

No. of other firms staying "Out"	Probability of event	Payoff for this firm staying "In"
2	p^2	$\Pi(1) = 9$
1	$2p(1-p)$	$\Pi(2) = -3$
0	$(1-p)^2$	$\Pi(3) = -7$

Combining the probabilities and payoffs from the above table, we get the equation:

$$\underbrace{9p^2 - 3 \times 2p(1-p) - 7(1-p)^2}_{\text{Expected payoff from playing "In"}} = \underbrace{0}_{\text{Payoff from playing "Out"}}$$
$$9p^2 - 6p(1-p) - 7(1 - 2p + p^2) = 0$$
$$8p^2 + 8p - 7 = 0$$

Solving the equation on the third line (using the formula for solving quadratic equations) yields two roots, of which one is not plausible since it is not between zero and one. Thus, in the symmetric Nash equilibrium, p is $(3\sqrt{2}-2)/4 \approx 0.56$. The probability of playing "In" for an individual firm is then approximately $1 - 0.56 = 0.44$. The probability of all three staying in is $(1-p)^3 \approx 0.08$. Expected profits for all firms are zero.

Once again, free entry and free exit drive expected profits to zero. Probabilistic entry and exit remove even those expected profits that might survive in some markets due to the integer constraint.

3. In the example, two grocery store chains, FIRM 1 and FIRM 2 are deciding whether to locate a store in Town A or Town B. FIRM 1 gets to make its location choice first. If FIRM 2 decides to locate in the same town as FIRM 1, FIRM 1 has an additional decision to make: to start or not to start a price war.

   The payoffs (see Figure 1) are yearly profits in millions of euros. In equilibrium, FIRM 1 locates in Town A and FIRM 2 in Town B. FIRM 1 makes a higher profit, since the demographics of the town are more suitable to it.

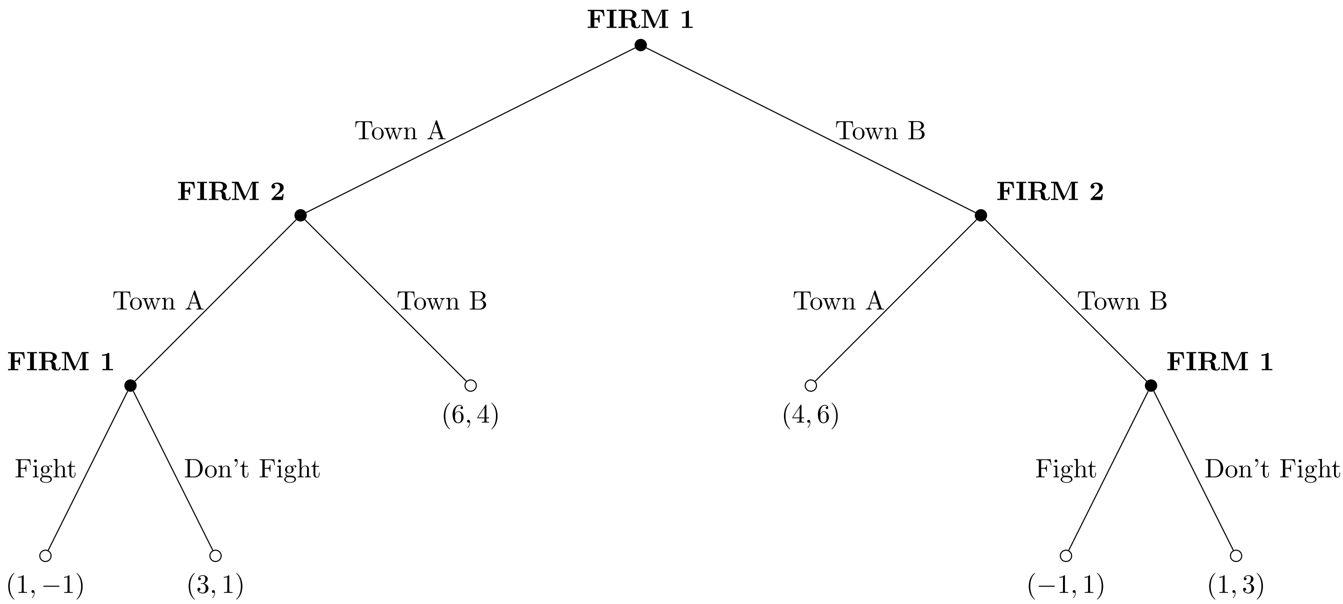


Figure 1: Game tree example: Grocery chains

4. (a) Let's formulate the payoff function for OneGulp and derive its best-response function (since the firms are identical, the payoff and best-response functions for TwoSips are identical). Denoting OneGulp's price by p_1 its payoff function is:

$$\begin{aligned}\Pi_1(p_1, p_2) &= (p_1 - \mathrm{MC})Q_1(p_1, p_2) \\ &= (p_1 - 5)(20 - 2p_1 + p_2) \\ &= -2p_1^2 + 30p_1 + p_1p_2 - 5p_2 - 100\end{aligned}$$

To find the best-response function, let's differentiate the payoff function wrt. p_1 and find the root:

$$\begin{aligned}\frac{\Pi_1(p_1, p_2)}{\partial p_1} &= 20 - 4p_1 + p_2 + 10 = 0 \\ \implies \mathrm{BR}_1(p_2) &= \frac{30 + p_2}{4}\end{aligned}$$

Since the problem is symmetric, $\mathrm{BR}(p) = (30 + p)/4$ and we know that in equilibrium, $p_1^* = p_2^* = p^*$. Then the equilibrium price is solved from:

$$p = \mathrm{BR}(p) \implies p = \frac{30 + p}{4} \implies p^* = 10.$$

As both firms charge the equilibrium price 10 €/l, they both sell $Q_i(p^*, p^*) = 20 - 2 \times 10 + 10 = 10$ thousand litres. Equilibrium profit per firm is then $\Pi_i(p, p) = 10 \times 10 - 5 \times 10 = 50$ thousand euros.

(b) Let's solve for the profits of the first-mover by substituting the second-mover's best-response function into the first-mover's payoff function:

$$\begin{aligned}\Pi_1(p_1) &= (p_1 - 5)(20 - 2p_1 + \mathrm{BR}(p_1)) \\ &= (p_1 - 5)\Big(20 - 2p_1 + \frac{30 + p_1}{4}\Big) \\ &= -\frac{7}{4}p_1^2 + \frac{145}{4}p_1 - \frac{550}{4}\end{aligned}$$

where the subscripts $i = 1, 2$ now denote the order of moves (which TwoSips gets to choose, but we don't know its choice yet!).

Let's differentiate wrt. p_1 to get the optimal price for the first-mover:

$$\begin{aligned}\frac{\Pi_1(p_1)}{\partial p_1} = -\frac{14}{4}p_1 + \frac{145}{4} &= 0 \implies \\ p_1^* &= \frac{580}{56} \approx 10.36 \text{ €/l}\end{aligned}$$

The second-mover will use its best response and charge

$$p_2^* = \mathrm{BR}_2(p_1^*) = \frac{30 + 580/56}{4} = \frac{565}{4} \approx 10.09 \text{ €/l.}$$

The resulting quantities are

$$\begin{aligned}&\text{First-mover: } Q_1^*(p_1^*, p_2^*) = 20 - 2 \times \frac{580}{56} + \frac{565}{56} = \frac{75}{8} = 9.375 \\ &\text{Second-mover: } Q_2^*(p_1^*, p_2^*) = 20 - 2 \times \frac{565}{56} + \frac{580}{56} = \frac{285}{28} \approx 10.18\end{aligned}$$

and profits

$$\begin{aligned}&\text{First-mover: } \Pi_1^*(p_1^*, p_2^*) = \Big(\frac{580}{56} - 5\Big) \times \frac{75}{8} \approx 50.22 \\ &\text{Second-mover: } \Pi_2^*(p_1^*, p_2^*) = \Big(\frac{565}{56} - 5\Big) \times \frac{285}{28} \approx 51.80\end{aligned}$$

Profits are higher for the second-mover, so TwoSips should commit to a price only after OneGulp has already done so.

(c) The marginal cost of production has increased for TwoSips so we need to solve for its new best-response function by the same steps as in problem 4a:

$$\Pi_2(p_1, p_2) = (p_2 - 6.5)(20 - 2p_2 + p_1)$$

The best-response function:

$$\frac{\Pi_2(p_1, p_2)}{\partial p_2} = 20 - 4p_2 + p_1 + 13 = 0$$
$$\implies \text{BR}_2(p_1) = \frac{33 + p_1}{4}$$

OneGulp knows about the increase in TwoSips' marginal cost, but the game is no longer symmetric. The Nash equilibrium condition that both players are using their best responses simultaneously can no longer be reduced to just one equation. Instead we have a system of two equations and two unknowns: $p_1 = \text{BR}_1(p_2)$ and $p_2 = \text{BR}_2(p_1)$. Plugging in the solved best responses from before,

$$p_1 = \frac{30 + p_2}{4}$$
$$p_2 = \frac{33 + p_1}{4}$$

The solution is $p_1^* = 10.20, p_2^* = 10.80$. OneGulp's profits are

$$\Pi_1(p_1^*, p_2^*) = (10.20 - 5) \times (20 - 2 \times 10.20 + 10.80) = 54.08 \text{ €k}.$$

TwoSips' problems cause OneGulp also to increase its price (although by less than TwoSips' price hike). As a result OneGulp's profits go up by about 4.1 €k.

Classroom games Oct 28th

6.5a) (6p) Stopping game with sequential choice

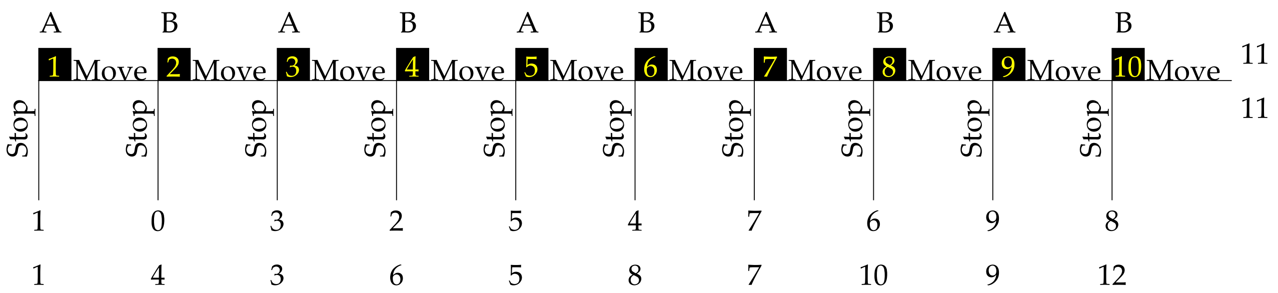


Figure 1: Stop or Move game. Payoffs to A on top.

To receive any points in the stopping game the player had to choose a valid strategy within the rules encoded in the game tree. The first stop node has to be an odd number for A and an even number for B; always choosing Move, i.e., Never Stop, is also a valid strategy.
The first round results of the classroom game were discarded as a "warm-up" game. Only the *Stopping game B* and *Stopping game A - Round 2* were recorded and graded.

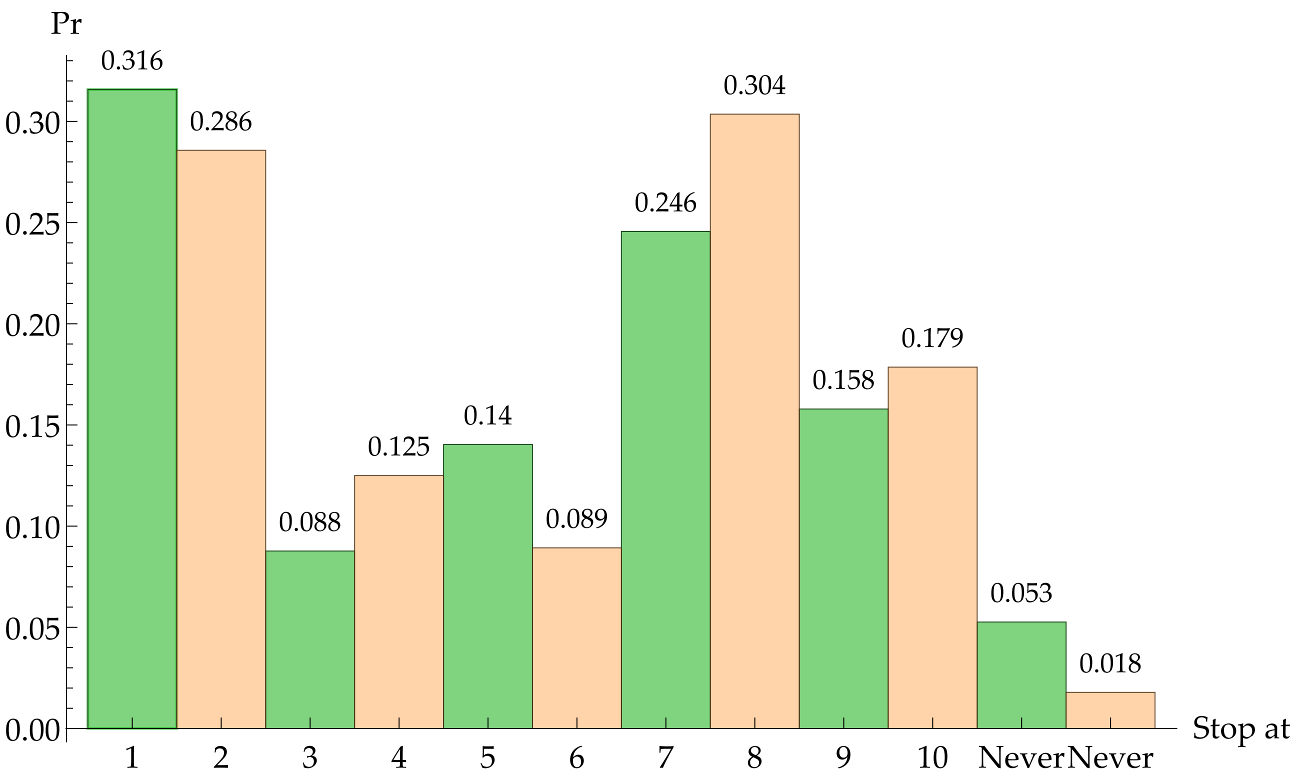


Figure 2: Distribution of choices in the "stop or move" game. Player A in odd and B in even turns.

Player A has no dominated strategies; even Never Stop can be rationalized if A expects B to play Never Stop. The most common choice (32%) was to stop at the first opportunity (node 1), with a further 40% stopping on one of the two last opportunities (nodes 7 and 9). While stopping at the first opportunity is the equilibrium choice under backward induction (subgame perfect Nash equilibrium), it was actually the worst choice for expected value against this classroom population of players in the role of player B. Thus everyone in the role of player A performed at least as well as the SPNE strategy and obtained full 3 points for Round A.

In the role of player B, stopping at the 2nd to last opportunity (node 8) was slightly more popular than stopping at the first opportunity (node 2). Stopping at node 2 resulted in the lowest EV for player B. However, for player B the strategy of Never Stop is dominated and resulted in only 2 points (full 3 points for any other choice) in Round B. Choosing Never Stop did in fact quite well against this population of A-players, but it is logically guaranteed that Stop at 10 must do better. In other words, no matter what are B's beliefs about the probability distribution of choices by A, choosing Never Stop can never be the payoff-maximizing choice.

The highest expected value for player B was achieved by playing Stop at 8, and for A by Stop at 9. These expected values come largely from the chance of meeting someone who is stopping very late on the other side which more than offsets the low payoffs from meeting early stoppers.

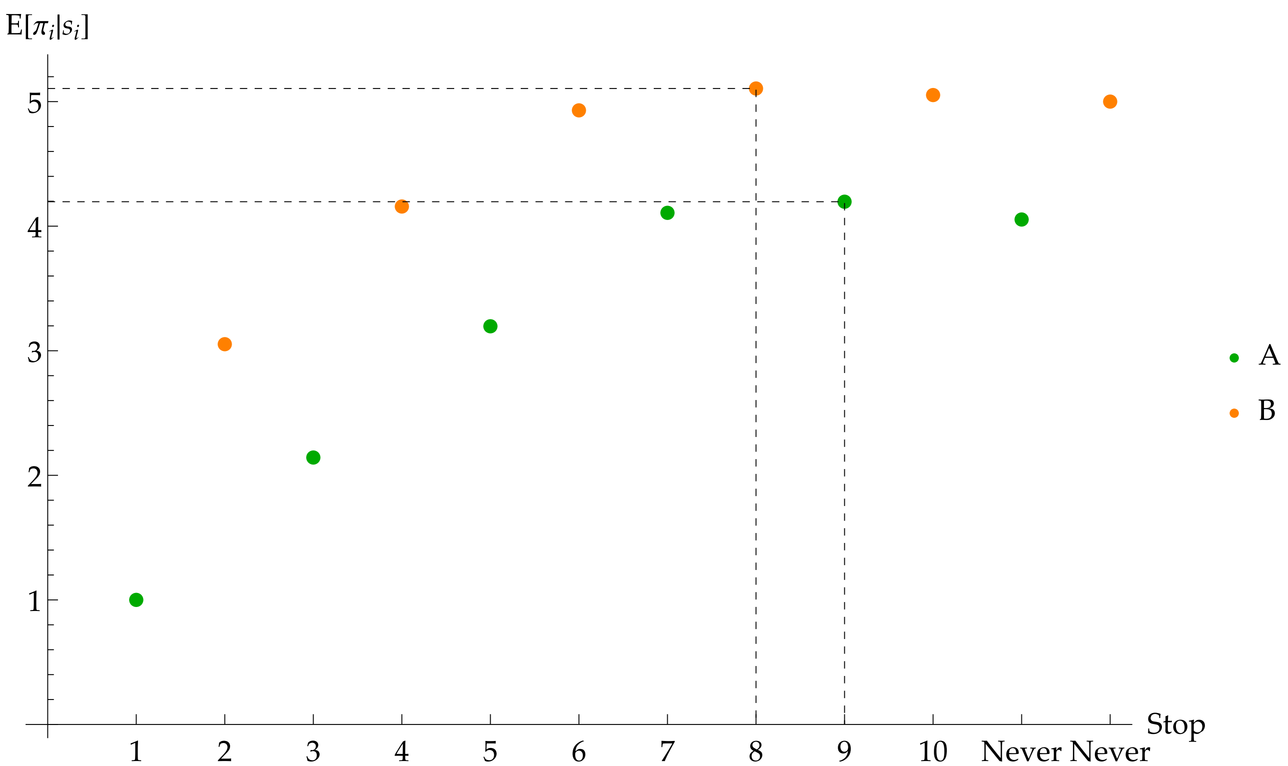
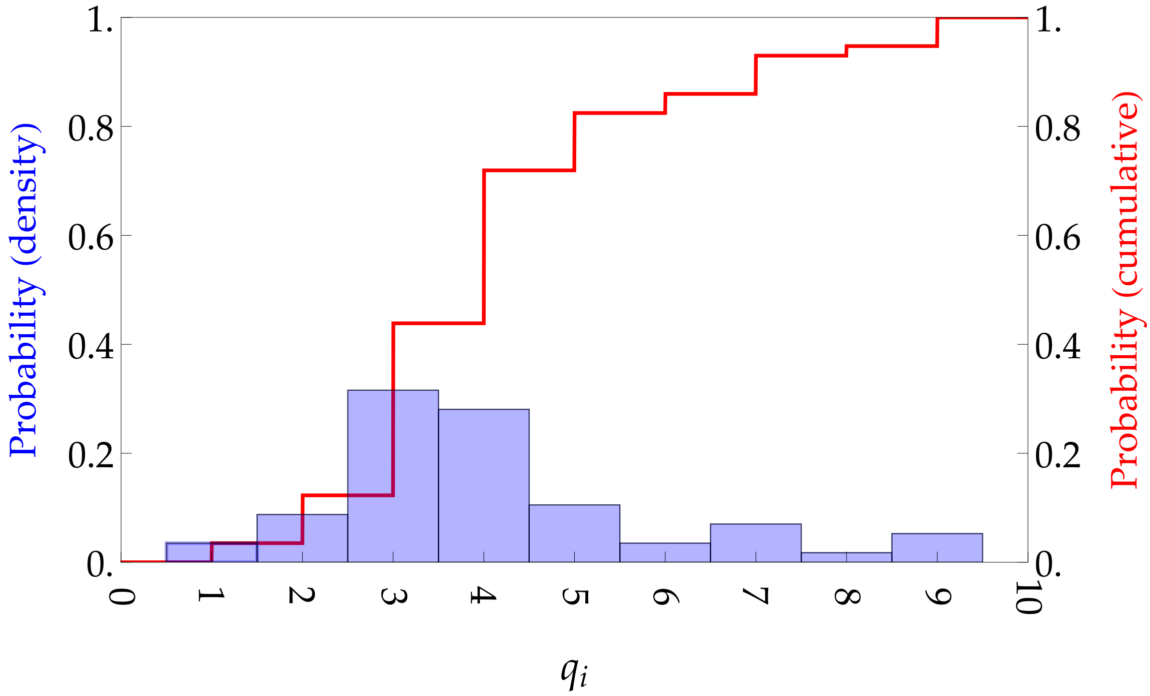


Figure 3: Expected value conditional on own choice in the stopping game when playing against a randomly drawn opponent from the classroom.

6.5b) (9p) Duopoly with simultaneous quantity choice

In this game the demand curve was $P^d(Q) = 10 - Q$, and all firms had $\mathrm{MC} = 1$ and $\mathrm{FC} = 1$. Had the choice been continuous, the best response of player 1 would be the maximizer of $P^d(q_1+q_2)(q_1-\mathrm{MC}) = (10-q_1-q_2-1)q_1$, so the f.o.c. is $9-2q_1-q_2 = 0$ and $\mathrm{BR}(q) = 4.5-0.5q$. Nash-equilibrium is $q^* = 3$. In the classroom choices were restricted to integers.



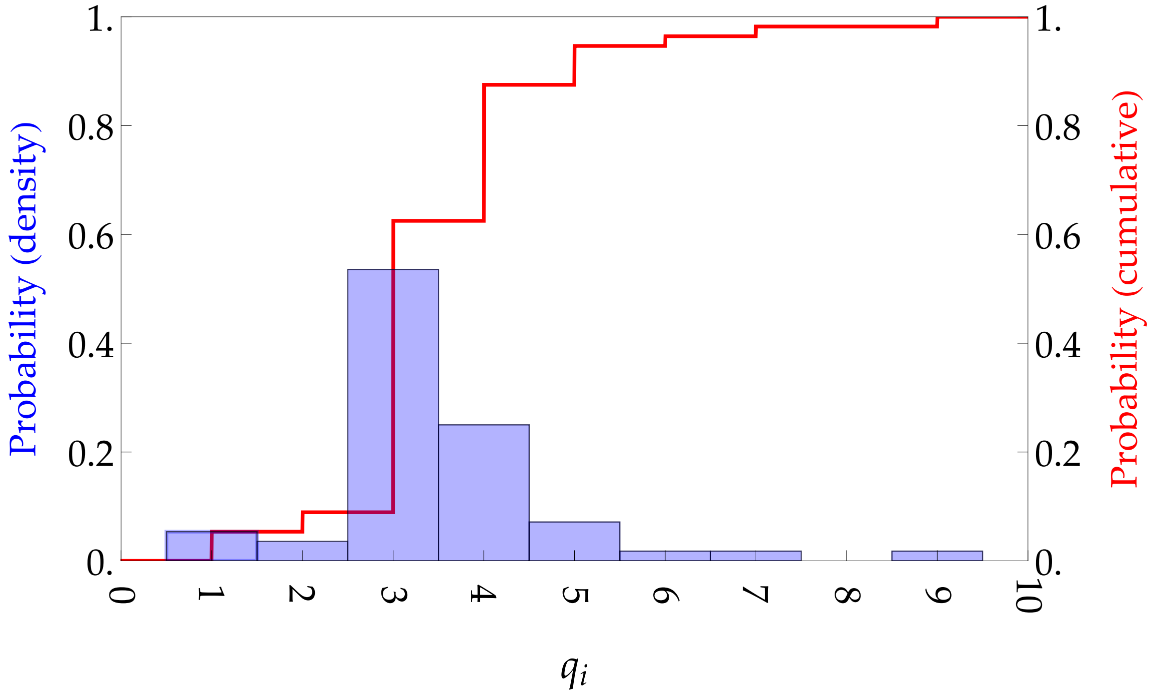


Figure 4: Distribution of choices in Rounds 1 and 2 (top and bottom).

A continuos-choice monopolist would produce BR(0) = 4.5, which is also quick to see by setting MR $= 10 - 2q$ equal to MC $= 1$. With discrete choice, any output choice higher than $q > 5$ is clearly a dominated strategy. Using a non-dominated strategy $q \in \{0, \ldots, 5\}$ earned (at least) 3 points in each round. Dominated strategies earned negative profits and earned 2 points per round.

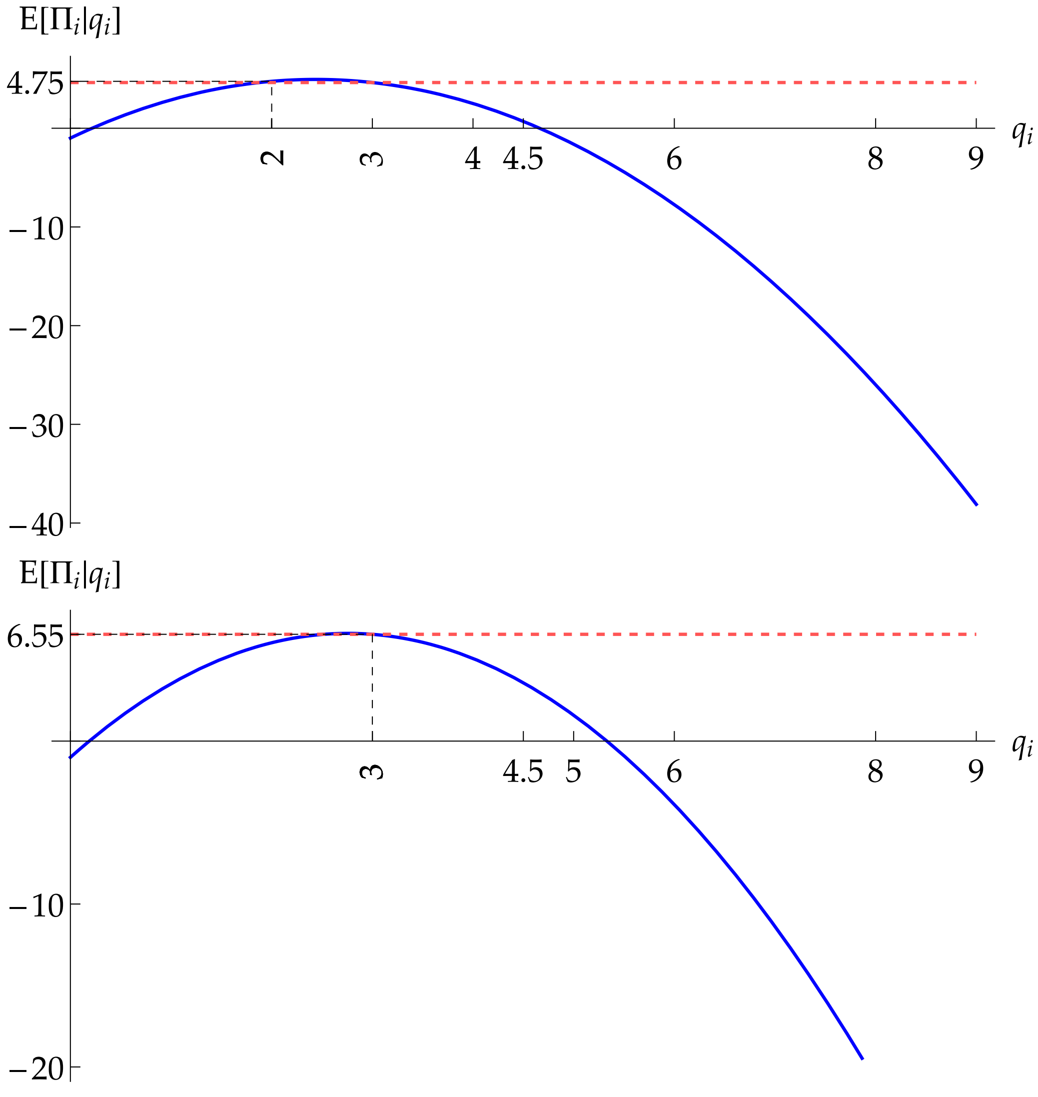


Figure 5: Expected profits conditional on own choice in Rounds 1 and 2 (top and bottom), when playing against a randomly drawn opponent from the classroom.

In Round 1 the highest expected profits against a randomly selected player in this classroom were $E[\Pi|q = 2] = 4.75$ and $E[\Pi|q = 3] = 4.63$. Choosing $q \in \{2, 3\}$ in the first round earned the full 4 points from that round. Producing less than Nash-output was the best choice because it limits the losses from those cases when you are randomly drawn against someone who is producing way too much.

By Round 2 there had clearly been some learning and fewer players were choosing very high output levels. Now the highest expected profits were obtained by using the Nash-equilibrium

strategy: $E[\Pi|q = 3] = 6.55$, but choosing $q = 2$ did quite well too: $E[\Pi|q = 2] = 6.03$. Choosing $q = 2$ in Round 2 earned 4 points and $q = 3$ the full 5 points from that round.
The maximum choice made by anyone, $q = 9$, resulted in a whopping expected loss, -38.1, in Round 1. In any single match-up facing such a competitor would have caused the other firm also to suffer losses (unless they chose $q = 0$).
One lesson here is that competitors who make irrational choices can cause you to suffer losses (while causing themselves even larger losses) when you are unlucky in the drawing of an opponent. On average and in the long run such problems tend to smooth out.