ELEC-A7200

Signals and Systems

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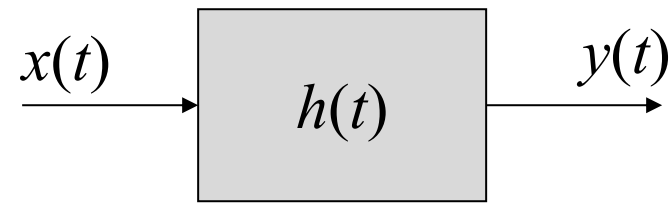
Fall 2021

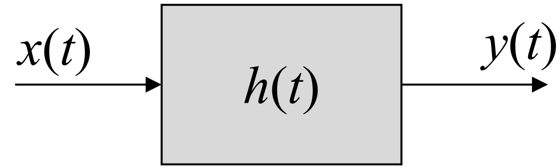
Aalto University
School of Electrical
Engineering



Lecture 9
Linear Time Invariant Systems – Part II

Continuous LTI systems

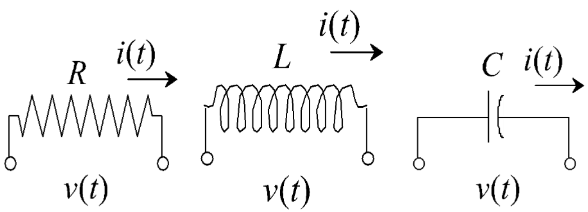
Are described in time domain by a linear differential equation with constant parameters



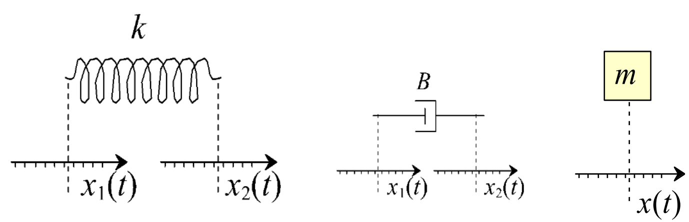
$$\frac{d^n}{dt^n}y(t) = -a_1\frac{d^{n-1}}{dt^{n-1}}y(t) - \cdots - a_n y(t) + b_0\frac{d^m}{dt^m}x(t) + b_1\frac{d^{m-1}}{dt^{m-1}}x(t) + \cdots + b_m x(t)$$

n order of the system

Proper system $m \leq n$
Strictly proper system $m < n$



$v(t) = Ri(t)$ $\quad v(t) = L\frac{di(t)}{dt}$ $\quad i(t) = C\frac{dv(t)}{dt}$



$$F_k(t) = k(x_1(t) - x_2(t)) = k\Delta x(t)$$

$$F_b(t) = B\frac{d\Delta x(t)}{dt} \quad F_m(t) = m\frac{d^2x(t)}{dt^2}$$

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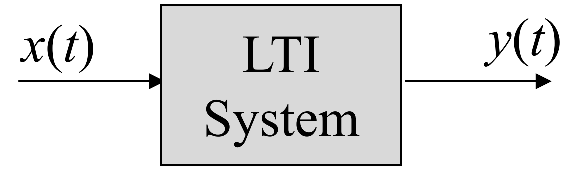
Frequency response



A generic n^{th} order LTI system is described by differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_0 = b_m\frac{d^m x(t)}{dt^n} + b_{m-1}\frac{d^{m-1}x(t)}{dt^{n-1}} + \cdots + b_0 x(t)$$

Let us use input $x(t) = e^{j2\pi f t}$ and guess that the response is of the form $y(t) = H(f)e^{j2\pi f t}$. It follows that

$$(j2\pi f)^n H(f)e^{j2\pi f t} + a_{n-1}(j2\pi f)^{n-1}H(f)e^{j2\pi f t} + \cdots + a_0 H(f)e^{j2\pi f t} = b_m(j2\pi f)^m e^{j2\pi f t} + b_{m-1}(j2\pi f)^{m-1}e^{j2\pi f t} + \cdots + b_0 e^{j2\pi f t}$$

Solving for $H(f)$ gives the frequency response function

$$H(f) = \frac{b_m(j2\pi f)^m + b_{m-1}(j2\pi f)^{m-1} + \cdots + b_0}{(j2\pi f)^n + a_{n-1}(j2\pi f)^{n-1} + \cdots + a_0}$$

Frequency response

Input $x(t) = e^{j2\pi ft}$ **yields output** $y(t) = H(f)e^{j2\pi ft}$

Input $x(t) = e^{-j2\pi ft}$ **yields output** $y(t) = H^*(f)\, e^{-j2\pi ft}$

Input $x(t) = \cos(2\pi ft)$ **yields output**

$$y(t) = H(f)\frac{1}{2}e^{j2\pi ft} + H^*(f)\frac{1}{2}e^{-j2\pi ft} = |H(f)|\cos(2\pi ft + \arg[H(f)])$$

Amplitude: $|H(f)|$

Phase: $\arg[H(f)]$

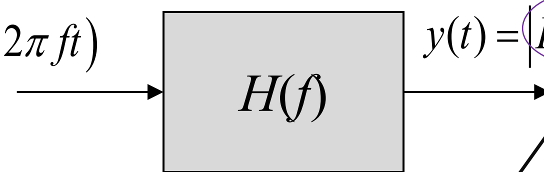
$$H(f) = |H(f)|\, e^{-j\arg[H(f)]}$$

$$\cos(2\pi ft) = \frac{1}{2}e^{j2\pi ft} + \frac{1}{2}e^{-j2\pi ft}$$

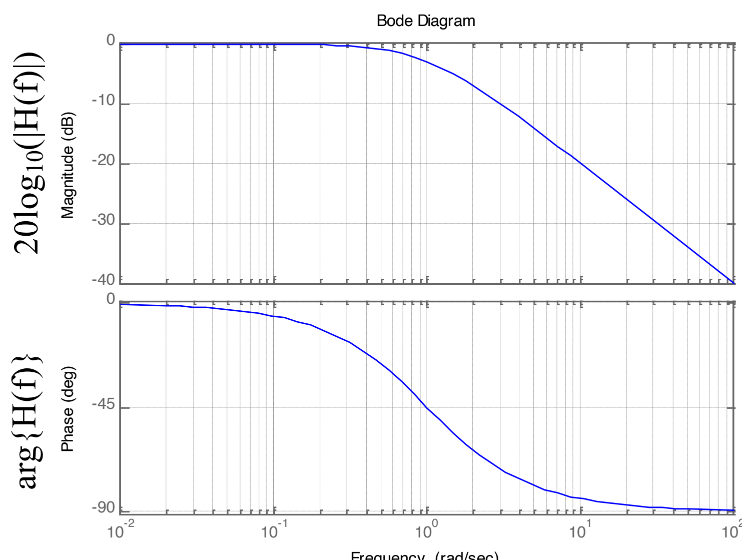
LTI system have an impact on the signal amplitude and phase, but it does not change the frquency.

Frequency response/ Bode plot

$x(t) = \cos\left(2\pi f t\right)$



$y(t) = \left|H\left(f\right)\right| \cos\left(2\pi f t + \arg\left\{H\left(f\right)\right\}\right)$



$\omega = 2\pi f$

Power attenuation

Phase shift

Frequency response function vs transfer function

- **Frequency response function**
- **Transfer function**

$$H(f) = \frac{b_m(j2\pi f)^m + b_{m-1}(j2\pi f)^{m-1} + \cdots + b_0}{(j2\pi f)^n + a_{n-1}(j2\pi f)^{n-1} + \cdots + a_0}$$

$$\hat{H}(s) = \frac{\hat{Y}(s)}{\hat{X}(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

$s = j2\pi f$

Discrete time LTI systems

Discrete time systems are described by difference equations

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

Continuous time system

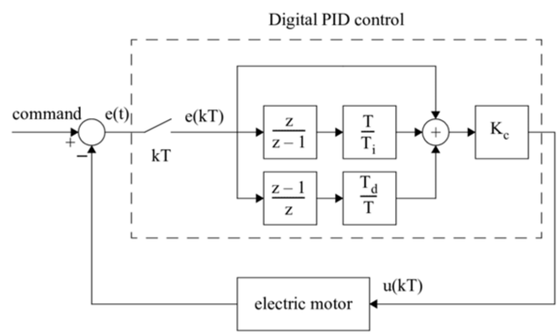
$$\frac{dy(t)}{dt} = -ay(t) + b \quad x(t)$$

Discretized (sampled) system

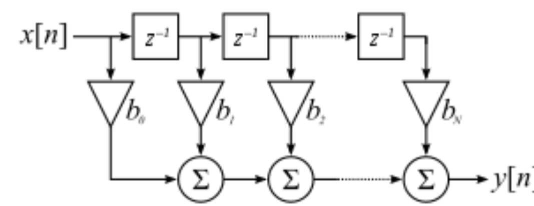
$$y[n] = y(nT_s),\ x[n] = x(nT_s)$$

$$y[n+1] = e^{aT_s}\mathrm{y[n]} + \frac{e^{aT_s}-1}{a} b \quad \mathrm{x[n]}$$

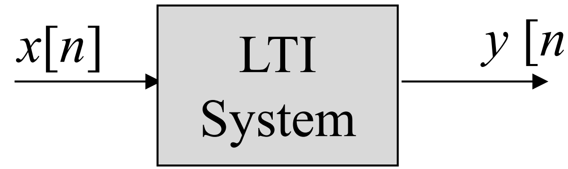
Digital control systems



Digital filters



Frequency response



A generic n^{th} order LTI system is described by difference equation

$$y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

Let us use input $x[n] = e^{j2\pi fnT}$ and guess that the response is of the form $y[n] = H(f)e^{j2\pi fnT}$. It follows that

$$H(f)e^{j2\pi fnT} + a_1 H(f)e^{-j2\pi fT}e^{j2\pi fnT} + \cdots + a_N H(f)e^{-j2\pi fNT}e^{j2\pi fnT} = b_0 e^{j2\pi fnT} + b_1 e^{-j2\pi fT}e^{j2\pi fnT} + \cdots + b_M e^{-j2\pi fMT}e^{j2\pi fT}$$

Solving for $H(f)$ gives the frequency response function

$$H(f) = \frac{b_0 + b_1 e^{-j2\pi fT} + \cdots + b_M e^{-j2\pi fMT}}{1 + a_1 e^{-j2\pi fT} + \cdots + a_N e^{-j2\pi fNT}}$$

Frequency response

Input $x[n] = e^{j2\pi fnT}$ **yields output** $y[n] = H(f)e^{j2\pi fnT}$

Input $x[n] = e^{-j2\pi fnT}$ **yields output** $y[n] = H^*(f)\, e^{-j2\pi fnT}$

Input $x[n] = \cos(2\pi fnT)$ **,** $n = \cdots, -1,0,1, \ldots$ **yields output**

$$y[n] = H(f)\frac{1}{2}e^{j2\pi fnT} + H^*(f)\frac{1}{2}e^{-j2\pi fnT} = |H(f)|\cos(2\pi fnT + \arg[H(f)])$$

Amplitude

Phase

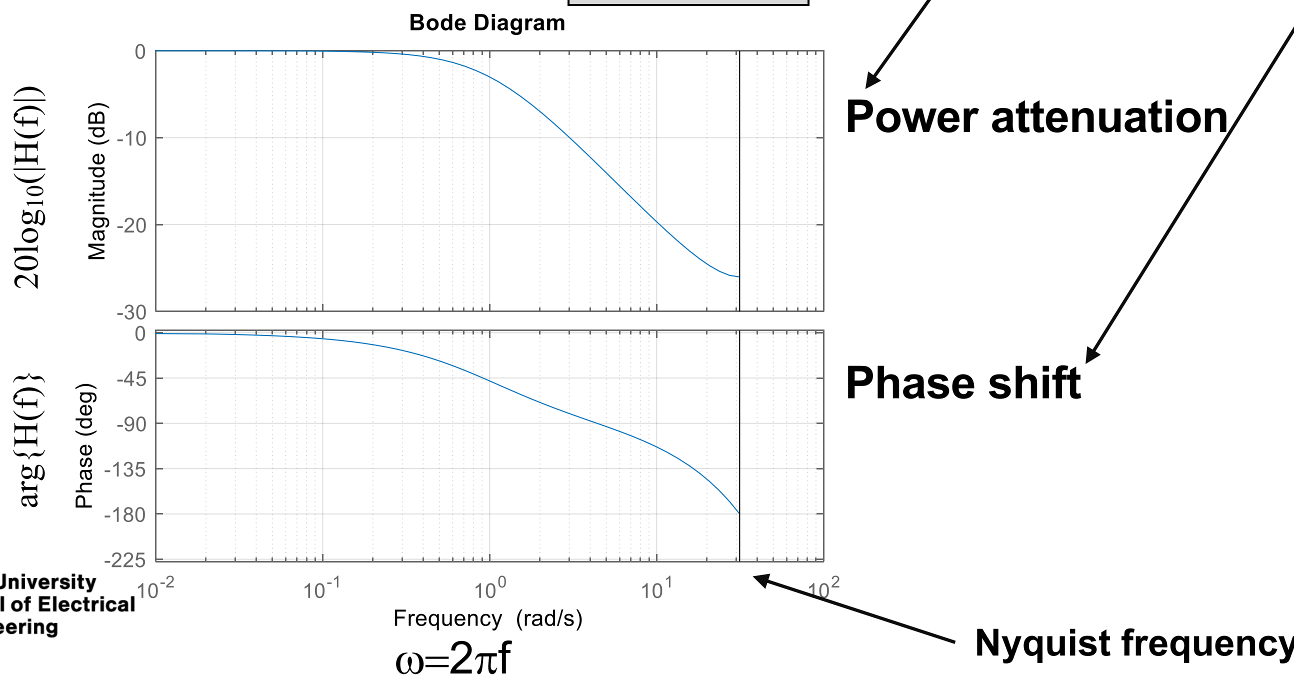
$H(f) = |H(f)|\, e^{-j\arg[H(f)]}$

$\cos(x) = \frac{1}{2}e^{jx} + \frac{1}{2}e^{-jx}$

LTI system have an impact on the signal amplitude and phase, but it does not change the frquency.

Frequency response/ Bode plot

$$x[n] = \cos(2\pi f n T) \rightarrow \boxed{H(f)} \rightarrow y[n] = |H(f)| \cos(2\pi f n T + \arg[H(f)])$$



Frequency response function vs transfer function

- **Frequency response function**

$$H(f) = \frac{b_0 + b_1 e^{-j2\pi fT} + \cdots + b_M e^{-j2\pi fMT}}{1 + a_1 e^{-j2\pi fT} + \cdots + a_N e^{-j2\pi fNT}}$$

Discrete Time Fourier Transform

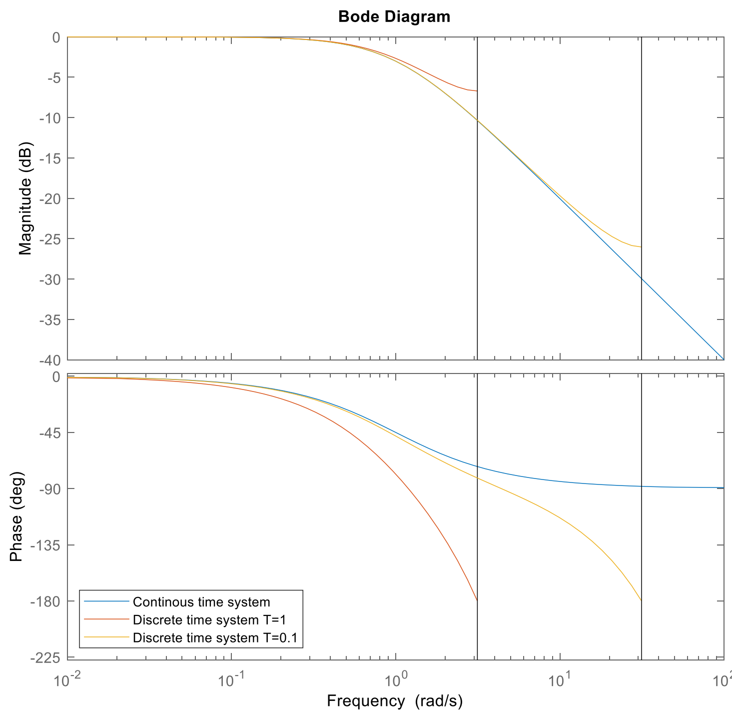
- **Transfer function**

$$\hat{H}(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-M}}$$

z-transform

$$z = e^{j2\pi fT}$$

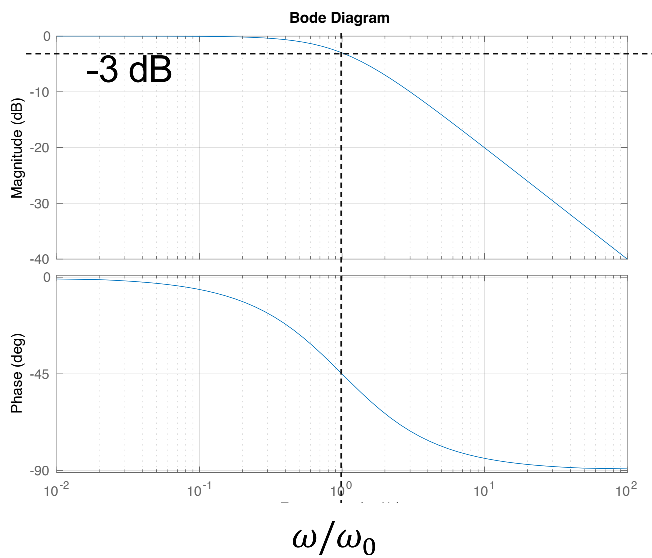
Continuous vs discrete time system

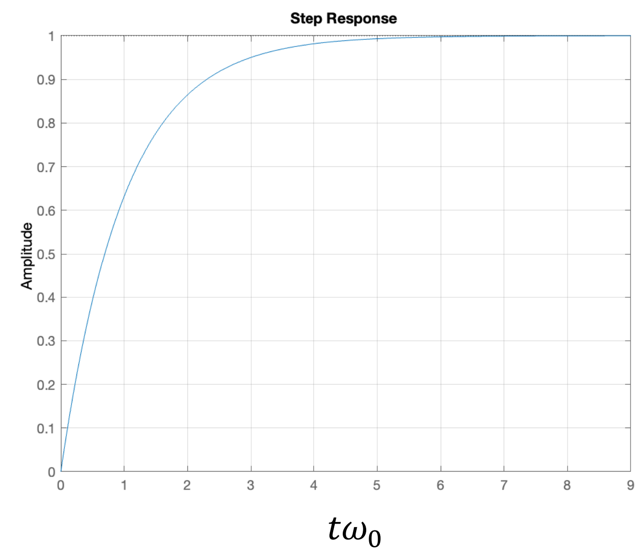


Time and frequency response

First order system

$$H(s) = \frac{\omega_0}{s + \omega_0}$$





ω_0 large => wide bandwith => fast response

ω_0 small => narrow bandwith => slow response

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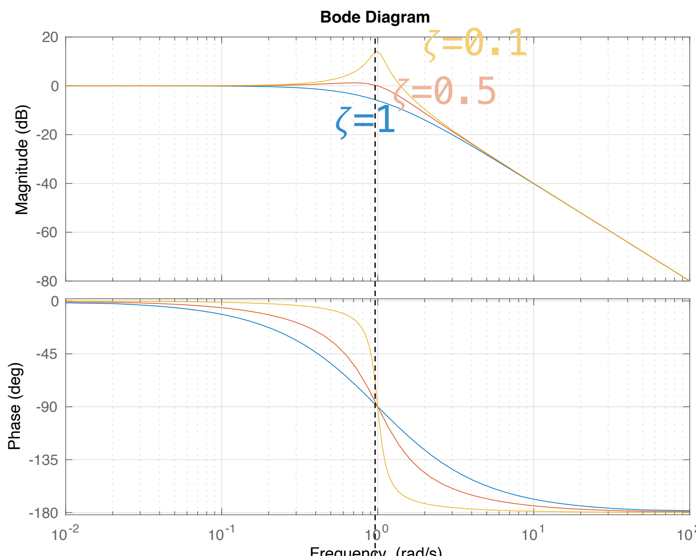
Time and frequency response

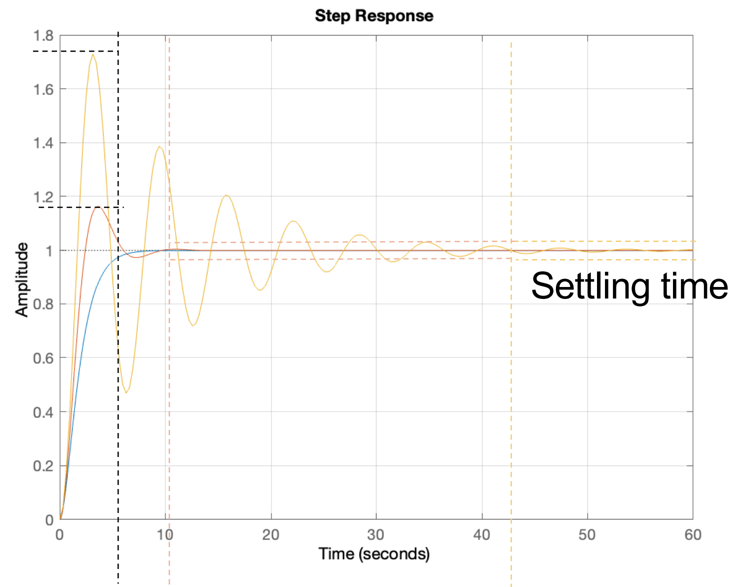
2nd order system

$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$



$\omega_0\sqrt{1-\zeta^2}$

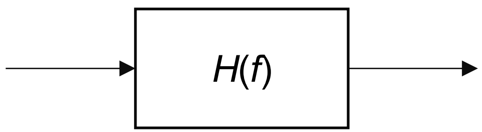
Resonant frequency



Overshoot

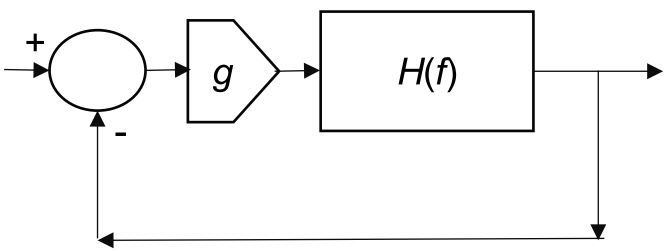
Frequency domain stability analysis

Stable open loop system



$$Y(f) = H(f)X(f)$$

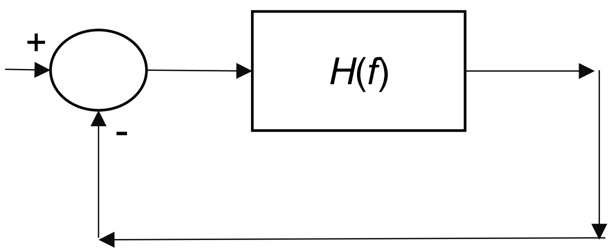
Is the closed loop negative feedback system stable?



$$Y(f) = \frac{gH(f)}{1 + gH(f)}X(f)$$

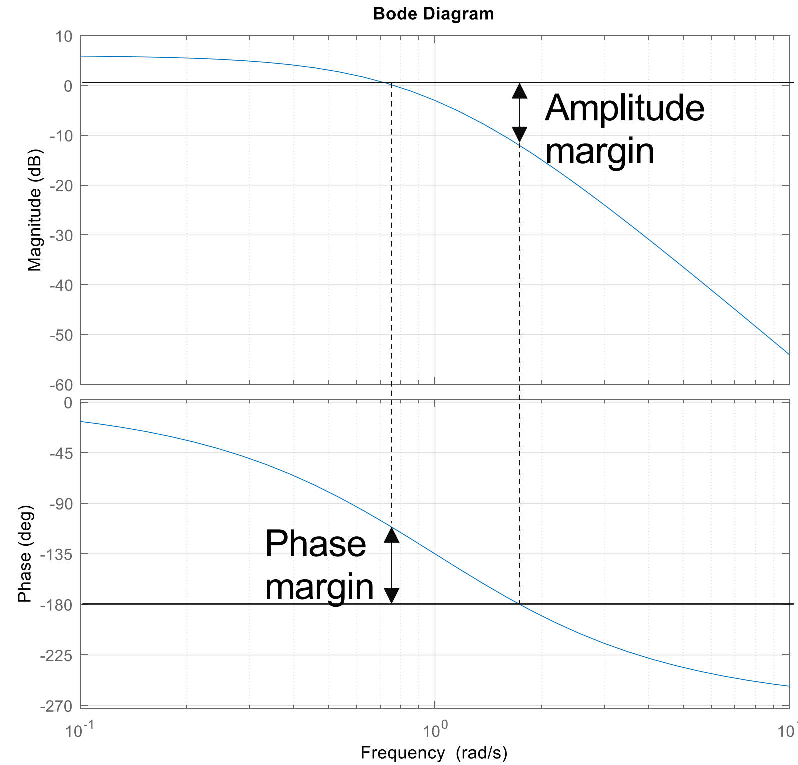
Frequency domain stability analysis

Closed loop system



Stable if

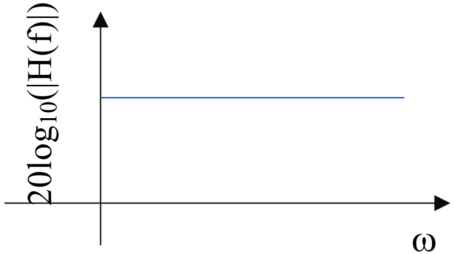
- |H(f)|<1 when arg{H(f)}=180°
- arg{H(f)} < 180° when |H(f)|>1

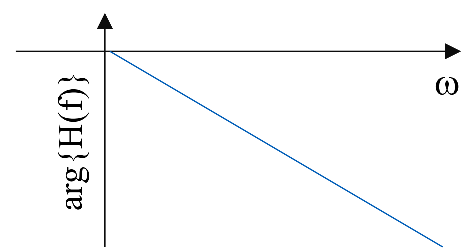


Passing signals through LTI systems

In many applications we would like to pass a pulse x(t) through an LTI system such as cable such that the pulse shape would not change.

- **Ideal case** $y(t) = ax(t - \tau_d)$, where the attenuation $a > 0$ is a constant and delay τ_d is the same for all frequencies. This corresponds to system $H(f) = ae^{-j2\pi f \tau_d}$





Passing signals through LTI systems

LTI response to sinusoidal input $x(t) = \cos(2\pi f t)$:

$$y(t) = A(f)\cos\big(2\pi f t - \phi(f)\big) = A(f)\cos\big(2\pi f(t - t_d)\big)$$

- **Frequency response function $H(f)$**
- ***Amplitude function*** $A(f) = |H(f)|$
- ***Phase function*** $\phi(f) = -\arg\{H(f)\}$
- ***Phase delay*** $t_d = \dfrac{\phi(f)}{2\pi f}$

Passing signals through LTI systems

LTI response to narrowband amplitude modulated signal

- Input: $x(t) = a(t)\cos(2\pi f_0 t)$
- Output: $y(t) \approx a(t - t_g)\cos\left(2\pi f_0 (t - t_d)\right)$

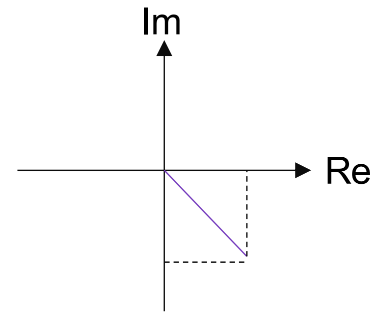
$$t_g = \frac{1}{2\pi}\frac{\mathrm{d}\phi(f)}{df}$$

Group delay

$$t_d = \frac{\phi(f)}{2\pi f}$$

Phase delay

Example: Group delay of a RC filter



1st order system such as RC filter

$$H(f) = \frac{1}{1+j2\pi f} = \frac{1-j2\pi f}{1+(2\pi f)^2}$$

Argument angle

$$\arg\{H(f)\} = \arg\left\{\frac{1}{1+j2\pi f}\right\} = \arg\left\{\frac{1-j2\pi f}{1+(2\pi f)^2}\right\} = \arg\{1 - j2\pi f\} = \text{atan}(-2\pi f)$$

Phase function

$$\phi(f) = -\arg\{H(f)\} = -\text{atan}(-2\pi f) = \text{atan}(2\pi f)$$

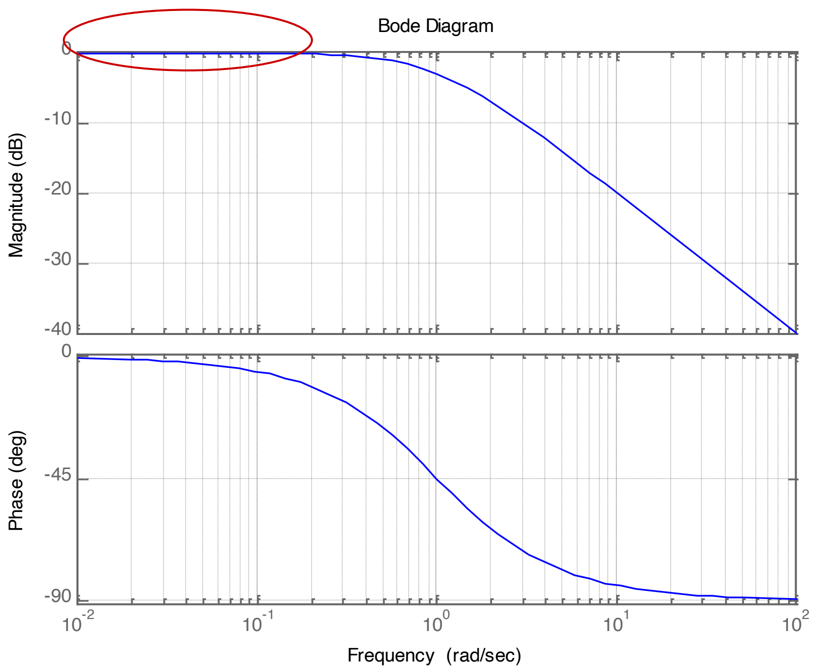
Group delay

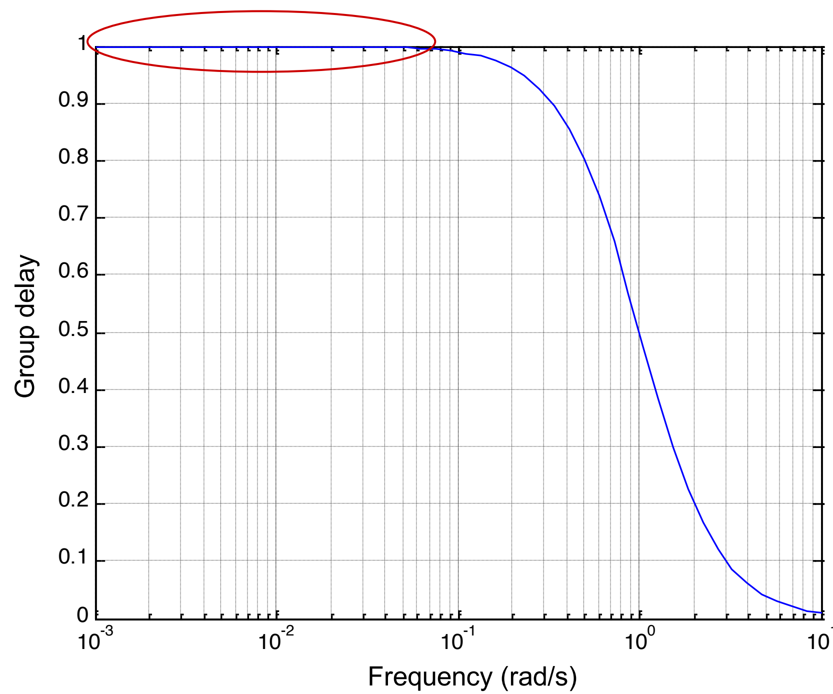
$$t_g = \frac{1}{2\pi}\frac{\mathrm{d}\phi(f)}{df} = \frac{1}{1 + (2\pi f)^2}$$

$$\frac{\text{datan}(x)}{dx} = \frac{1}{1 + x^2}$$

Example: Group delay of a RC filter

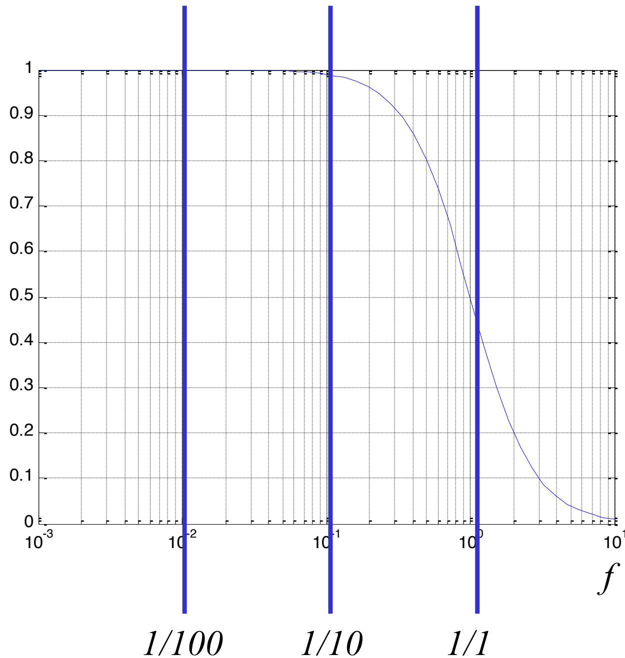
Narrowband signal sees constant group delay

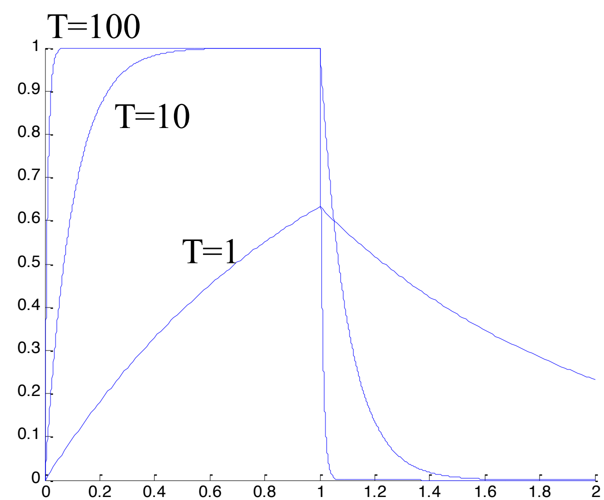




Example: Group delay of a RC filter

Pulse of length *T* passing the RC filter





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