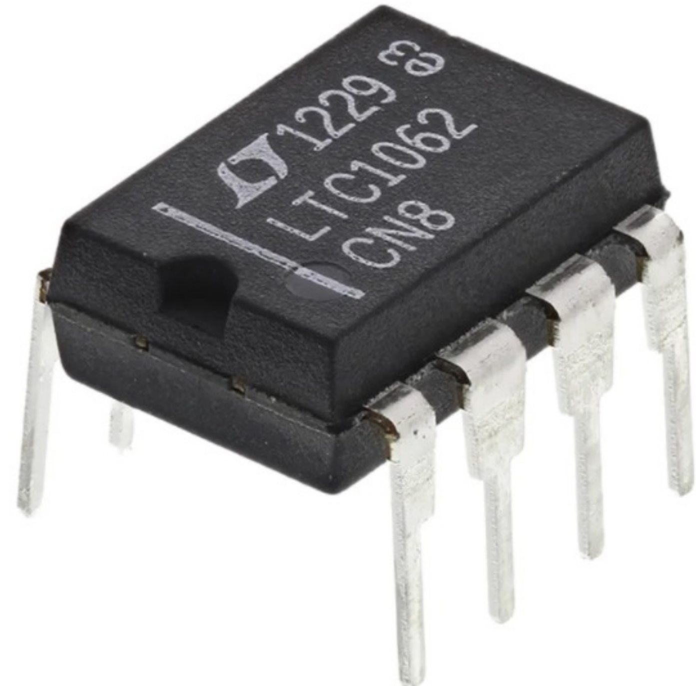
ELEC-A7200

Signals and Systems

Professor Riku Jäntti
Fall 2022

Aalto University
School of Electrical
Engineering

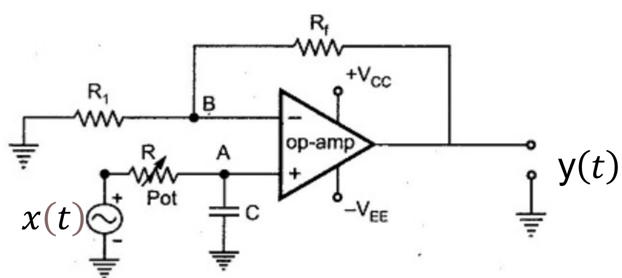


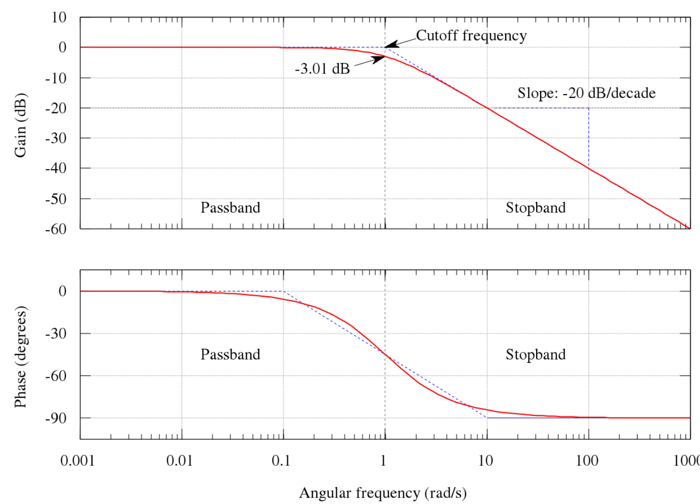
Lecture 10
Linear filters

Linear filters

The term "filter" typically refer to a continuous-time circuit that is designed to remove certain frequencies and allow others to pass.

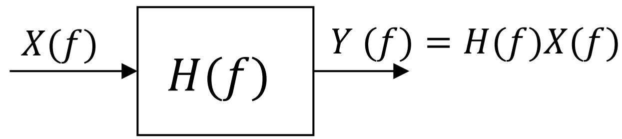
First order Butterworth filter and its Bode plot

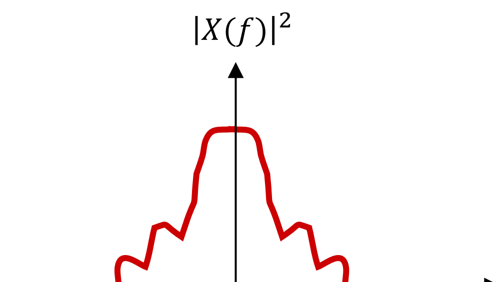




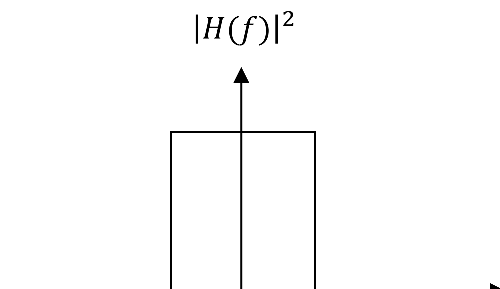
Aalto University
School of Electrical
Engineering

Operation principle of a filter

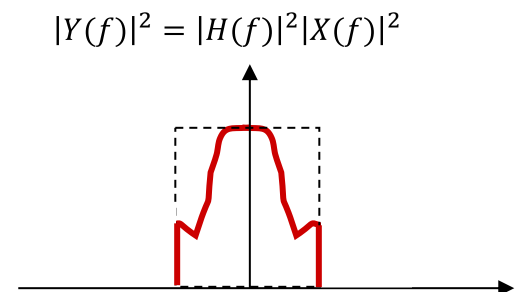




Input signal



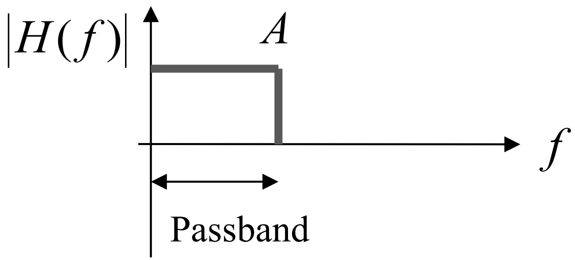
Ideal low-pass Filter



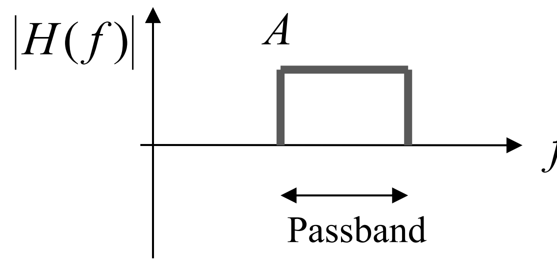
Output signal

Ideal filters

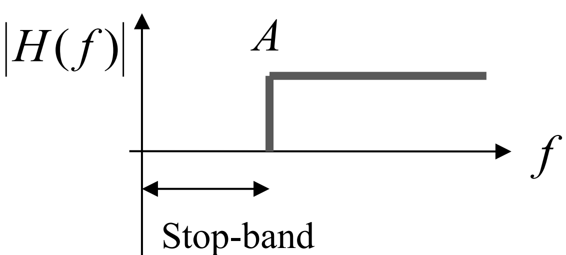
- Low-pass filter



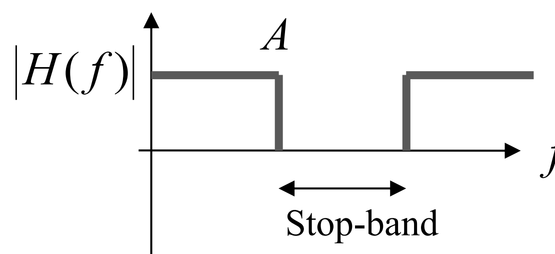
- Band-pass filter



- High-pass filter



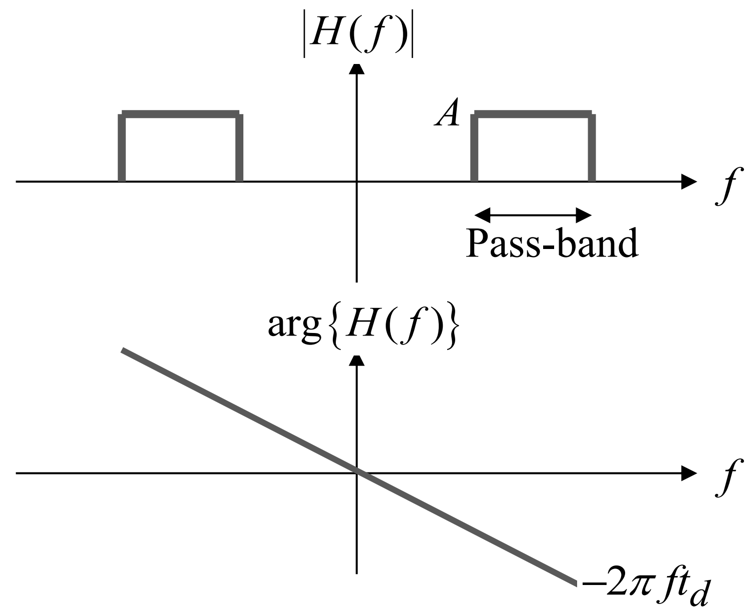
- Band-stop filter



Ideal filters

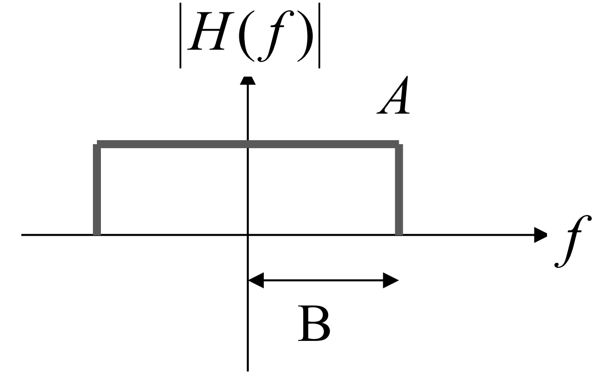
Ideal filter would not cause phase and amplitude distortions on the pass-band and completely reject the stop-band:

- **Pass-band:** A(f)=A **and** $\phi(f)=2\pi f t_d$
- **Stop-band:** A(f)=0



Ideal band-pass filter

Ideal filters



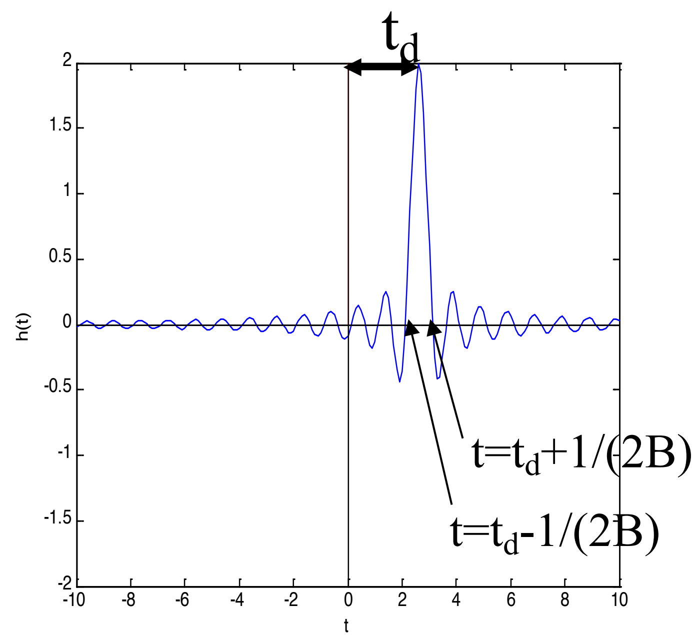
Ideal low-pass filter

$$H(f) = \mathrm{rect}\left(\frac{f}{2B}\right)e^{-i2\pi f t_d}$$

Impulse response

$$h(t) = F^{-1}\{H(f)\} = 2BA\,\mathrm{sinc}\big(2B(t - t_d)\big)$$

Is not causal unless the delay caused by the filter becomes infinite $t_d \to \infty$



Uncertainty principle

A function [signal] and its Fourier transform cannot both be localized.

Consider an energy signal x(t) with unit energy $\int_{-\infty}^{\infty} |x(t)|^2 \, dt = 1$

Its dispersion in the time and frequency domain fulfill

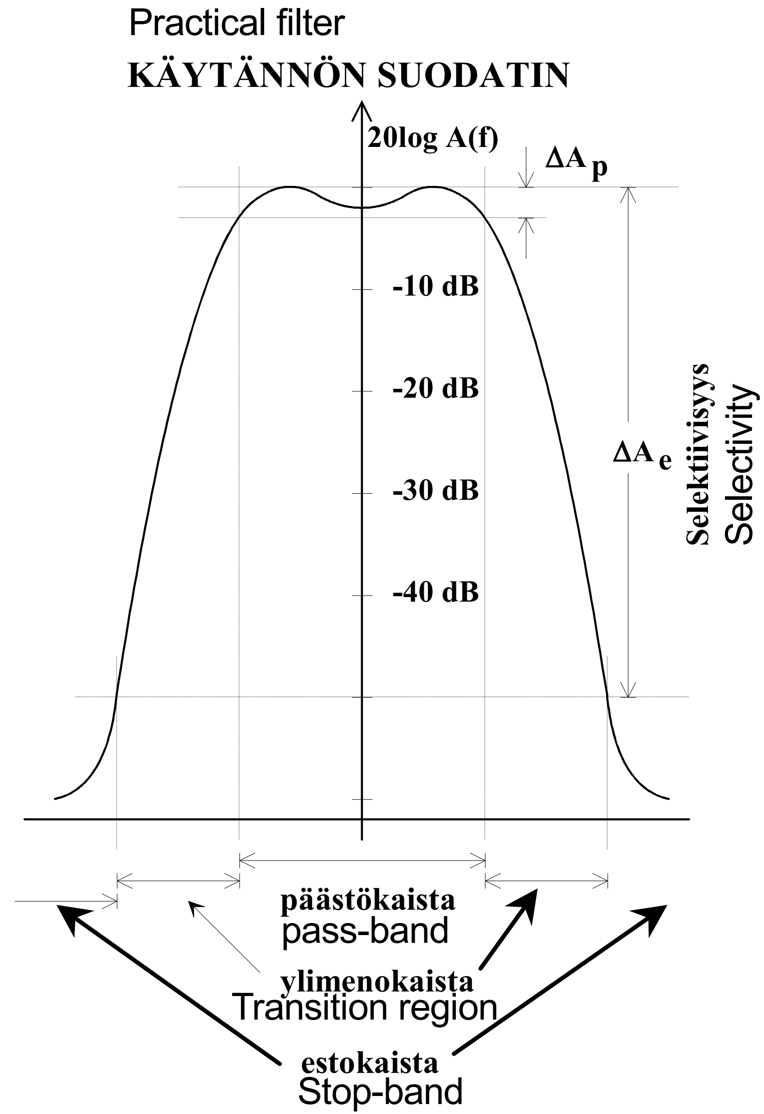
$$\int_{-\infty}^{\infty} t^2 |x(t)|^2 \, dt \int_{-\infty}^{\infty} f^2 |X(f)|^2 \, df \geq \frac{1}{16\pi^2}$$

This bound resembles the Heisenberg's uncertainty principle in quantum physics.

Practical filters

Practical filter is characterized by

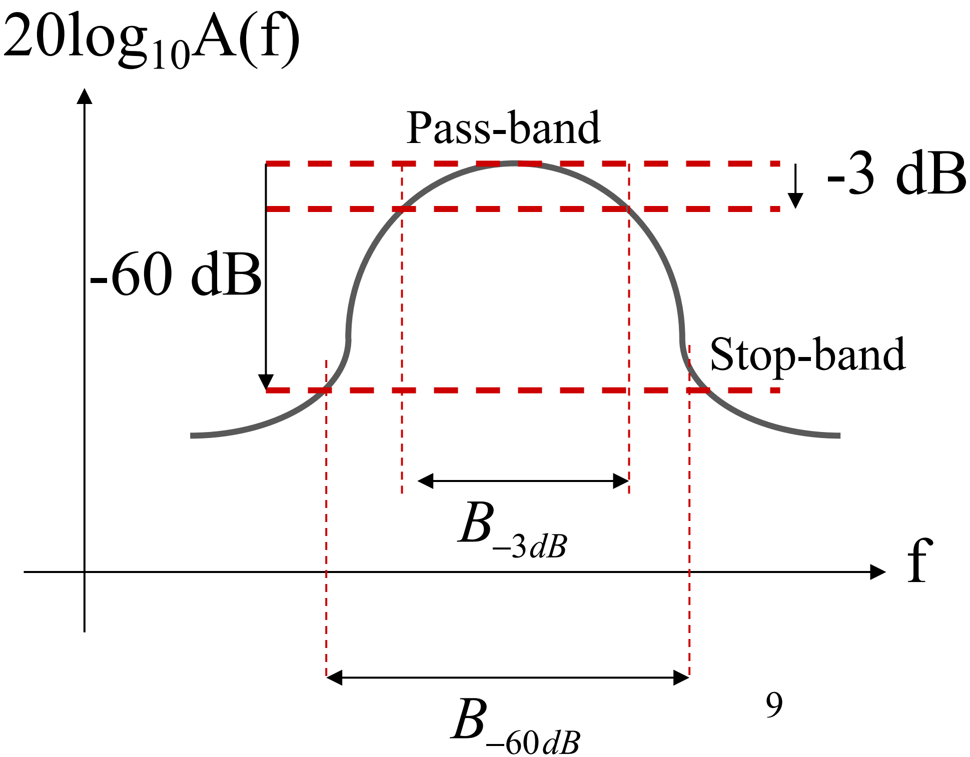
- **Filter order (order of the LTI system realizing the filter)**
- **Pass-band and pass band ripple ΔA_p**
- **Transition region, roll of rate, and shape factor**
- **Stop-band and selectivity ΔA_e**



Filter shape factor

- **Typically we want the transition region to be narrow.**
- **Filter shape factor describes the relative width of the transition band**

$$r_{shape} = \frac{B_{-60dB}}{B_{-3dB}}$$



Example: RC Filter

Frequency transfer function

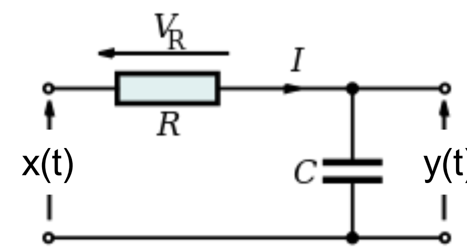
$$H(f) = \frac{1}{j2\pi f T + 1}$$

Amplitude function

$$A(f) = |H(f)| = \frac{1}{\sqrt{(2\pi f T)^2 + 1}}$$

Half-power (-3 dB) bandwidth :

$$A^2(f) = |H(f)|^2 = \frac{1}{(2\pi f T)^2+1} = \frac{1}{2} \Rightarrow f = \frac{1}{2\pi T} = B_{-3\ dB}$$



Filter time constant $T = RC$

-60 dB bandwidth:

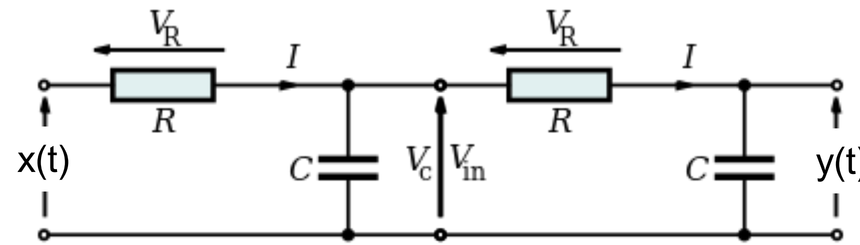
$$A^2(f) = |H(f)|^2 = \frac{1}{(2\pi f T)^2+1} = 10^{-6} \Rightarrow f = \frac{\sqrt{10^6-1}}{2\pi T} = B_{-60\ dB}$$

Selectivity

$$r = \frac{B_{-60\ dB}}{B_{-3\ dB}} = \sqrt{10^6 - 1} \approx 1000$$

Aalto University
School of Electrical
Engineering

Example 2: Two RC Filters in series



Filter time constant $T = RC$

Frequency transfer function

$$H(f) = \frac{1}{(j2\pi fT + 1)^2}$$

Amplitude function

$$A(f) = |H(f)| = \frac{1}{(2\pi fT)^2 + 1}$$

Half-power (-3 dB) bandwidth :

$$A^2(f) = |H(f)|^2 = \frac{1}{\left((2\pi fT)^2+1\right)^2} = \frac{1}{2} \Rightarrow f = \frac{\sqrt{\sqrt{2}-1}}{2\pi T} = B_{-3\ dB}$$

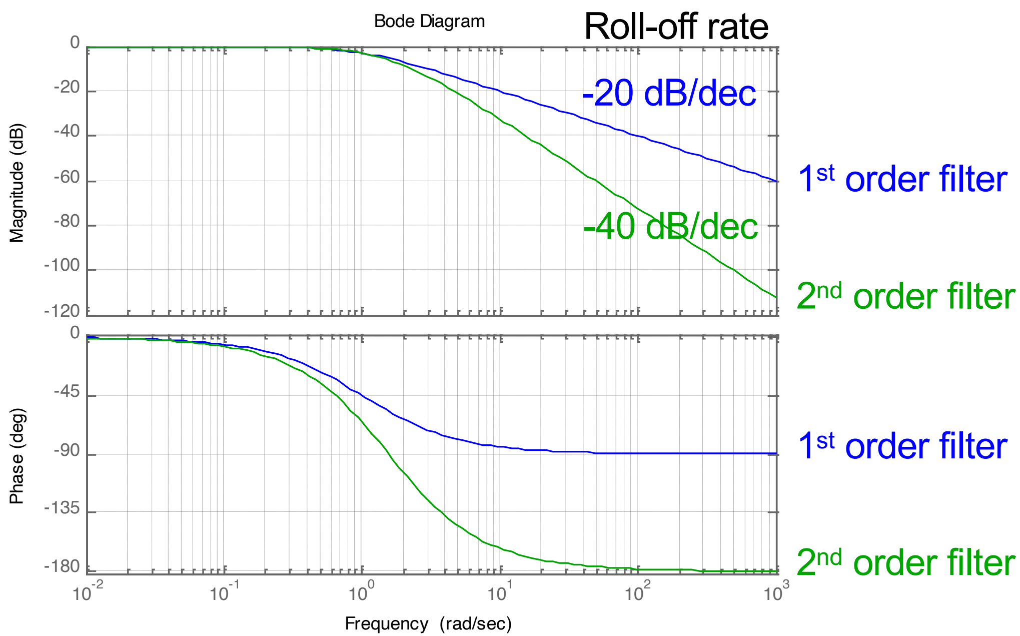
-60 dB bandwidth:

$$A^2(f) = |H(f)|^2 = \frac{1}{\left((2\pi fT)^2+1\right)^2} = 10^{-6} \Rightarrow f = \frac{\sqrt{10^3-1}}{2\pi T} = B_{-60\ dB}$$

Selectivity

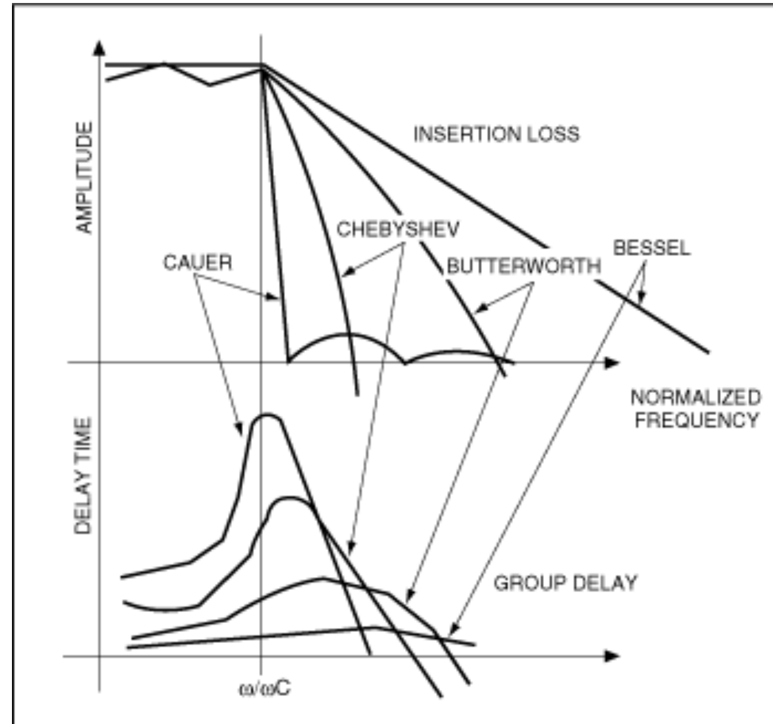
$$r = \frac{B_{-60\ dB}}{B_{-3\ dB}} = \frac{\sqrt{10^3-1}}{\sqrt{\sqrt{2}-1}} \approx 49.1$$

Example: 1st and 2nd order RC filters



Filter families

- **There are four classic analogue filter types:**
  - Butterworth: Flattest pass-band but a poor roll-off rate.
  - Chebyshev: Some pass-band ripple but a better (steeper) roll-off rate.
  - Elliptic (aka Cauer): Some pass- and stop-band ripple but with the steepest roll-off rate.
  - Bessel: Worst roll-off rate of all four filters but the best phase response. Filters with a poor phase response will react poorly to a change in signal level.



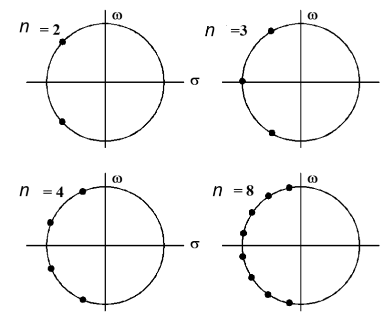
Butterworth filter

Transfer function of nth order Butterworh filter

$$\hat{H}(s) = \frac{1}{\prod_{k=1}^{n}(s - p_k)}$$

$p_k = \sqrt[n]{1} = \sqrt[n]{e^{j2\pi k}} = e^{jk2\pi\frac{k}{n}}$

Re$\{s_k\}$<0



Rescale to get desirable bandwith W

s:=s/(2πW)

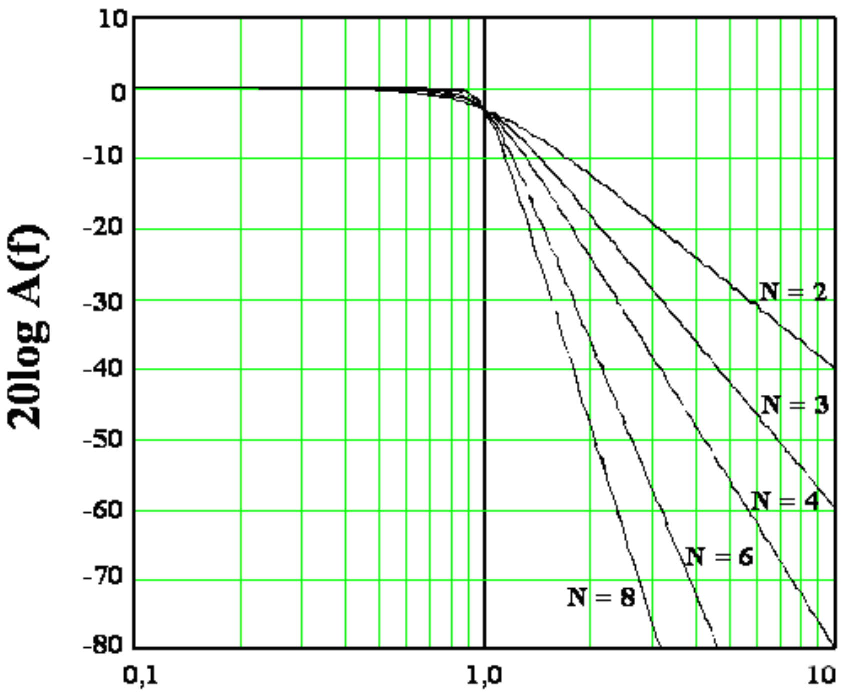
Amplitude function

$$A(f) = \frac{1}{\sqrt{\left(\frac{f}{W}\right)^{2n} + 1}}$$

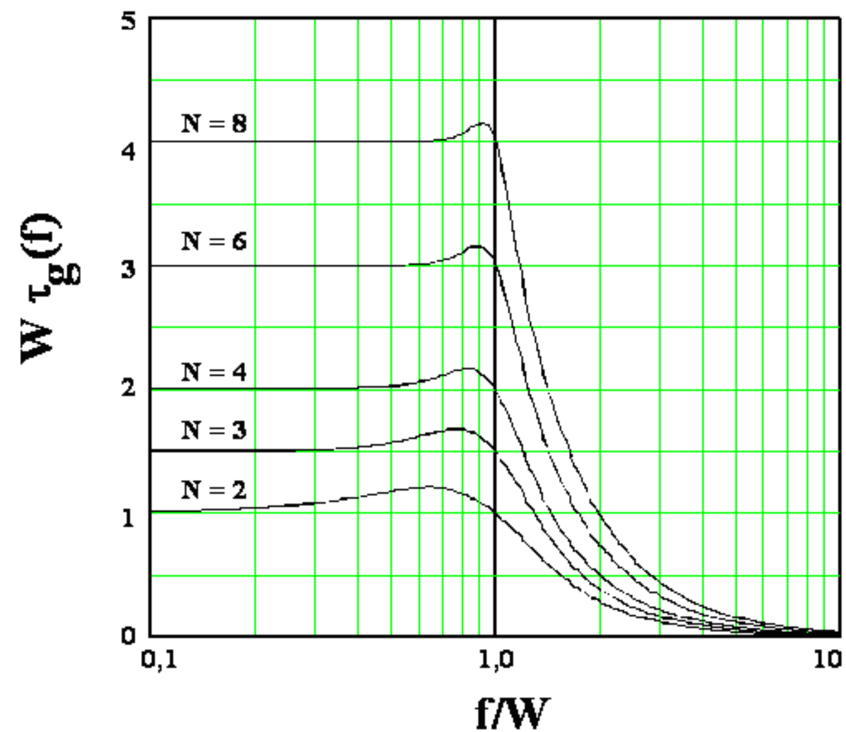
N	Transfer Function
2	$\frac{1}{s^2 + 1.414s + 1}$
3	$\frac{1}{(s+1)(s^2+s+1)}$
4	$\frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)}$
5	$\frac{1}{(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)}$
6	$\frac{1}{(s^2+0.5176s+1)(s^2+1.4142s+1)(s^2+1.9318s+1)}$
7	$\frac{1}{(s+1)(s^2+0.4450s+1)(s^2+1.2480s+1)(s^2+1.8019s+1)}$
8	$\frac{1}{(s^2+0.3902s+1)(s^2+1.1111s+1)(s^2+1.6629s+1)(s^2+1.9616s+1)}$

Butterworth filters

Bode amplitude plot

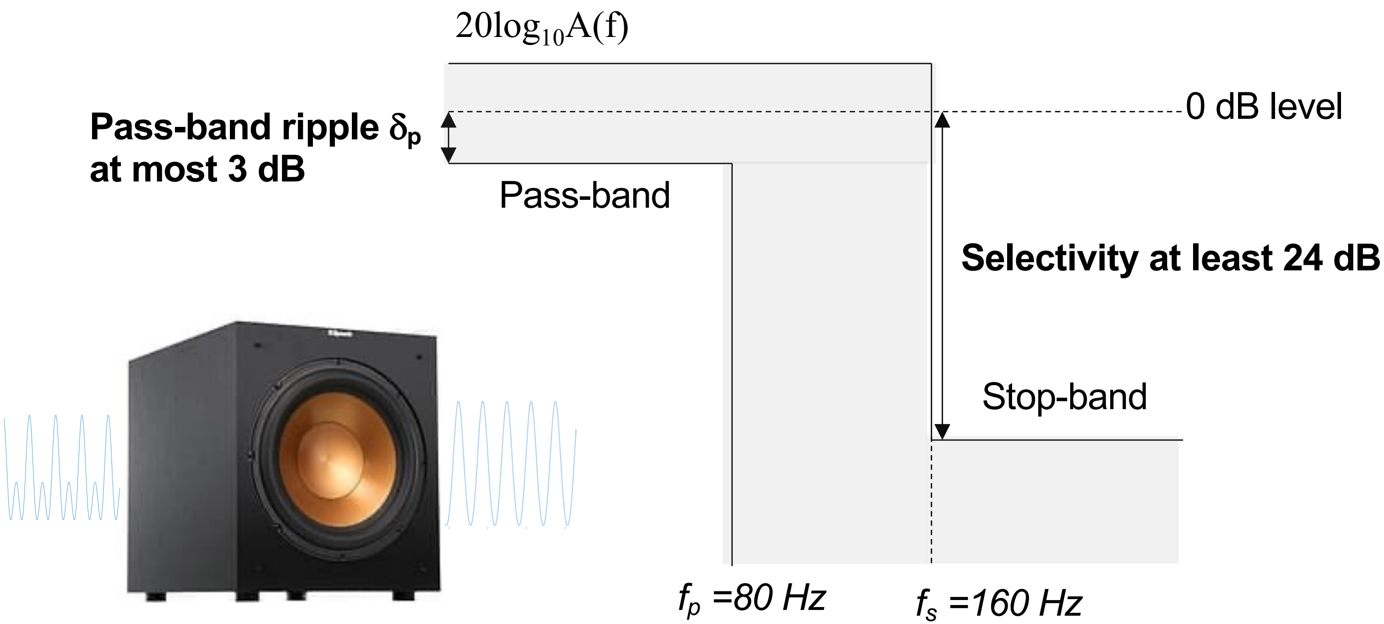


Group delay



Filter design example

Filter specifications for a subwoofer Butterworth filter



Filter design example

1. Fix pass-band attenuation

$20\log_{10}A(80)$=-3 dB

$\Rightarrow A(80)=\frac{1}{\sqrt{\left(\frac{80}{W}\right)^{2n}+1}} = 10^{-\frac{3}{20}} \Rightarrow \left(\frac{80}{W}\right)^{2n} = 10^{\frac{3}{10}} - 1$

- 2. Fix stop-band attenuation

$20\log_{10}A(160)$=-24 dB

$\Rightarrow A(160)=\frac{1}{\sqrt{\left(\frac{160}{W}\right)^{2n}+1}} = 10^{-\frac{24}{20}} \Rightarrow \left(\frac{160}{W}\right)^{2n} = 10^{\frac{24}{10}} - 1$

3. Solve filter order

$$\frac{\left(\frac{80}{W}\right)^{2n}}{\left(\frac{160}{W}\right)^{2n}} = \left(\frac{80}{160}\right)^{2n} = \frac{10^{\frac{3}{10}} - 1}{10^{\frac{24}{10}} - 1}$$

$$\Rightarrow n = \frac{1}{2}\frac{\ln\left(\frac{10^{\frac{3}{10}} - 1}{10^{\frac{24}{10}} - 1}\right)}{\ln\left(\frac{80}{160}\right)} \approx 3.42 \Rightarrow n = 4$$

Round up

- 4. Make either pass-band cutoff frequency exact and solve for W.

$$W = \frac{80}{\left(10^{\frac{3}{10}}-1\right)^{\frac{1}{8}}} \approx 80.05$$

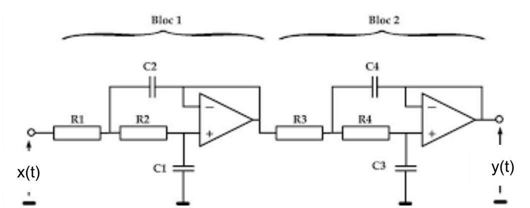
or

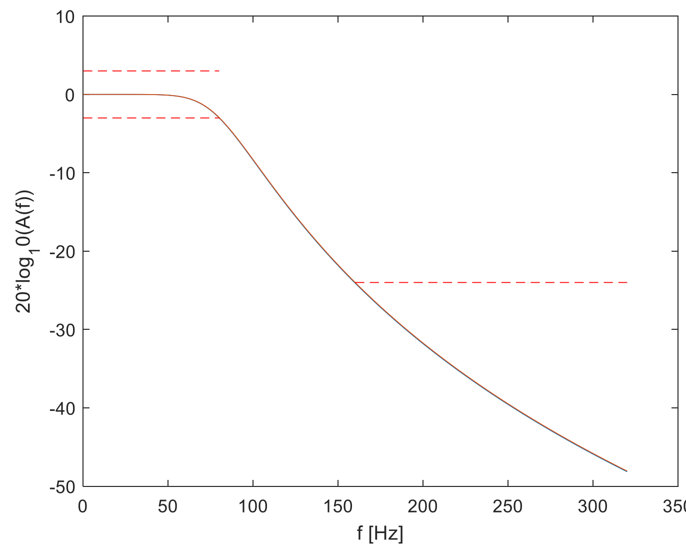
$$W = \frac{160}{\left(10^{\frac{24}{10}} - 1\right)^{\frac{1}{8}}} \approx 80.23$$

Filter design example

Designed 4^{th} order filter

$$\hat{H}(s)=\frac{1}{\left(\left(\frac{s}{2\pi W}\right)^2+0.7654\left(\frac{s}{2\pi W}\right)+1\right)\left(\left(\frac{s}{2\pi W}\right)^2+1.8478\left(\frac{s}{2\pi W}\right)+1\right)}$$

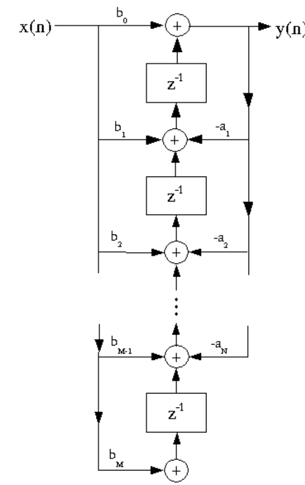




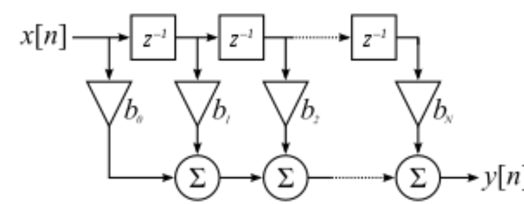
Digital filters

Digital Infinite impulse response (IIR) vs Finite Impulse Response (FIR) Filters

- **Analog filters can be translated to IIR filters by discretizing.**
- **IIR filters are less sensitive to quantization errors than FIR filters and use less memory (have lower delay).**
- **FIR filters can be designed to have linear phase response**



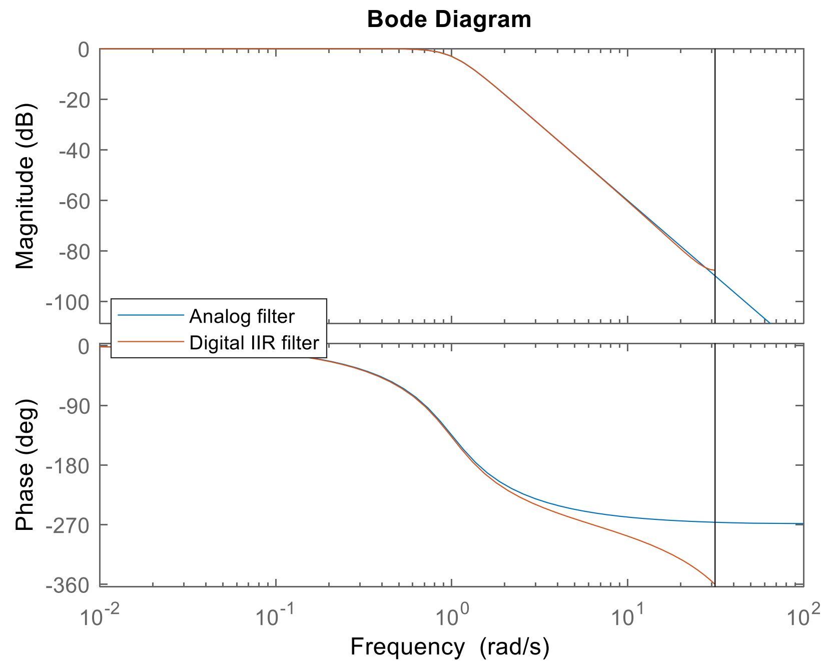
Infinite impulse response (IIR) filter



Finite impulse response (FIR) filter

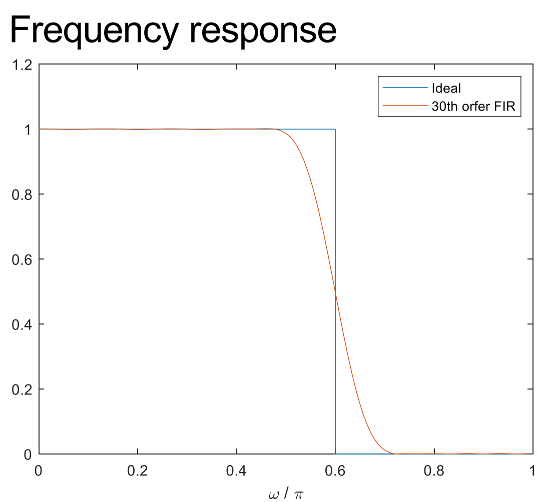
Digital IIR filter example

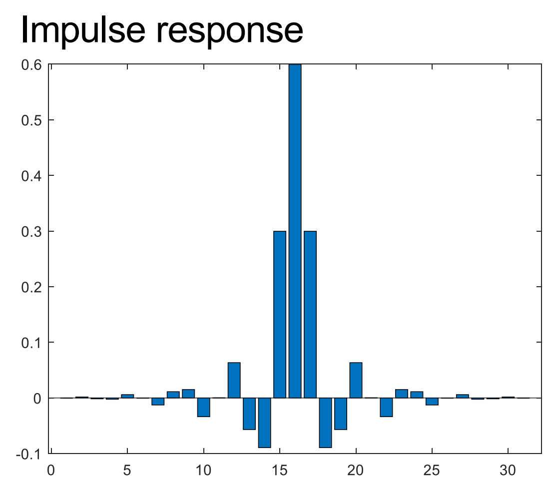
Example 3rd order Butterworth filter. Sampling frequency 10Hz



Digital FIR filter example

Several methods to design FIR filters exists. One is to approximate given ideal frequency response





aalto.fi

Aalto University
School of Electrical
Engineering