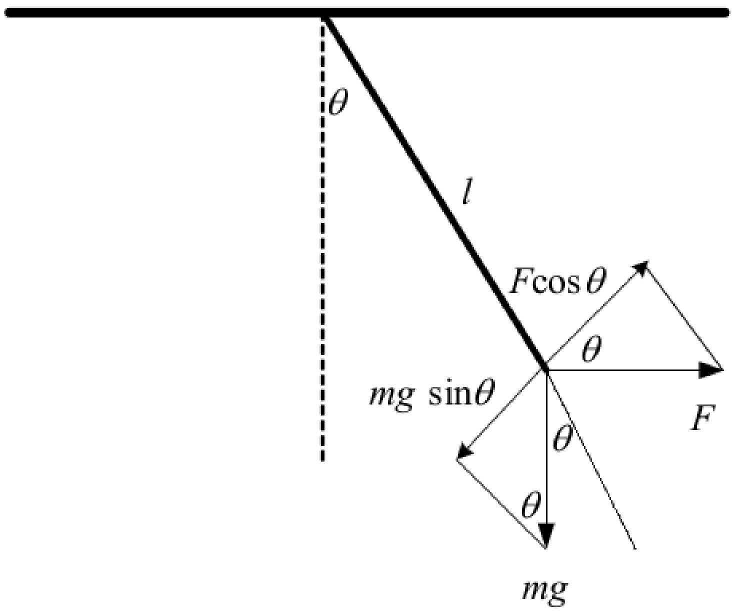
ELEC-E8101 Digital and Optimal Control
Exercise 9 - solution
Autumn 2022

1



For rotational motion, the sum of torques acting on the system is equal to the product of the moment of inertia *I* and angular acceleration $\ddot{\theta}$. For this system the torques are caused by the external force *F* and the gravity of the end of the pendulum (assume that the stick is massless).

Thus,

$$I\ddot{\theta} = \sum M$$

$$ml^2\ddot{\theta} = -mgl\sin\theta + Fl\cos\theta \Rightarrow$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + F\frac{1}{ml}\cos\theta$$

Using state variables $x_1 = \theta, x_2 = \dot{\theta}, u = F,\ y = \theta = x_1$ gives the state-space form

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 + \frac{1}{ml}u\cos x_1$$

$$y = x_1$$

2

Linearization of a function $f(x)$ around a point x_0 is equivalent to approximating the function around the point with a first-order Taylor series, given by

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

The nonlinear functions in the state equations are the trigonometric functions, which only appear in the terms for $\dot{x}_2$. Their linearizations around zero are then

$$\begin{aligned}\sin x &\approx \sin 0 + \left(\frac{\mathrm{d}\sin x}{\mathrm{d}x}\,\middle|\, x = 0\right) \times x \\ &\approx 0 + \cos 0 \times x \\ &\approx x \\ \cos x &\approx \cos 0 + \left(\frac{\mathrm{d}\cos x}{\mathrm{d}x}\,\middle|\, x = 0\right) \times x \\ &\approx 1 + (-\sin 0) \times x \\ &\approx 1\end{aligned}$$

The linearization can then be written

$$\begin{aligned}\dot{x}_2 &= -\frac{g}{l}\sin x_1 + \frac{1}{ml} u \cos x_1 \\ &\approx -\frac{g}{l} x_1 + \frac{1}{ml} u\end{aligned}$$

with all other parts of the model remaining original.

In the standard form this gives

$$\begin{aligned}\begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} &= A\begin{pmatrix}x_1 \\ x_2\end{pmatrix} + Bu = \begin{pmatrix}0 & 1 \\ -\frac{g}{l} & 0\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} + \begin{pmatrix}0 \\ \frac{1}{ml}\end{pmatrix}u \\ y &= C\begin{pmatrix}x_1 \\ x_2\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix}\end{aligned}$$

3

A state space model

$$\dot{x} = Ax + Bu,\; y = Cx + Du$$

can be discretized with ZOH with sample time T_s as

$$x[k+1] = A_d x[k] + B_d u[k],\; y[k] = Cx[k] + Du[k]$$

where

$$A_d = e^{AT_s}, B_d = A^{-1}(A_d - I)B.$$

The matrices can be calculated using Matlab symbolic toolbox as

```
syms g l Ts m positive
A=[0 1;-g/l 0]
B=[0 ; 1/(m*l)]
Ad=simplify(expm(A*Ts))
Bd=simplify(inv(A)*(Ad-eye(2))*B)
```

giving

$$A_d = \begin{pmatrix} \cos\left(T_s\sqrt{\frac{g}{l}}\right) & \sqrt{\frac{l}{g}}\sin\left(T_s\sqrt{\frac{g}{l}}\right) \\ -\sqrt{\frac{g}{l}}\sin\left(T_s\sqrt{\frac{g}{l}}\right) & \cos\left(T_s\sqrt{\frac{g}{l}}\right) \end{pmatrix}$$

$$B_d = \begin{pmatrix} \left(1-\cos\left(T_s\sqrt{\frac{g}{l}}\right)\right)/(mg) \\ \sin\left(T_s\sqrt{\frac{g}{l}}\right)/\left(m\sqrt{gl}\right) \end{pmatrix}$$

Simplifying this by substituting

$$\omega = \sqrt{\frac{g}{l}}$$

gives

$$A_d = \begin{pmatrix} \cos(T_s\omega) & \omega^{-1}\sin(T_s\omega) \\ -\omega\sin(T_s\omega) & \cos(T_s\omega) \end{pmatrix}$$

$$B_d = \begin{pmatrix} (1-\cos(T_s\omega))/(mg) \\ \sin(T_s\omega)/(ml\omega) \end{pmatrix}$$

The quantity ω is the natural frequency of the pendulum.

4

Assume $g = 9.81,\ l = 9.81, m = 1/9.81$. Thus $\omega = 1$.

The discretized system is stable if the eigenvalues of A_d are in the unit circle. Finding them using Matlab symbolic toolbox

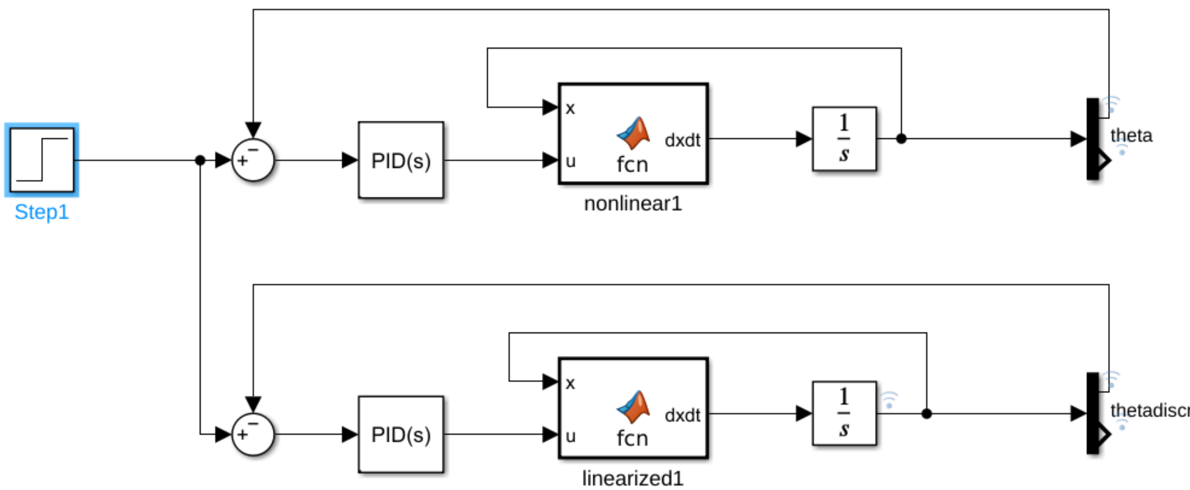
```
eig(Ad)
```

gives

$$z = \cos(T_s\omega) \pm j\sin(T_s\omega)$$

for which $|z| = 1$. The system (without a controller) is thus marginally stable (oscillation continues forever).
5 *

Simulator model



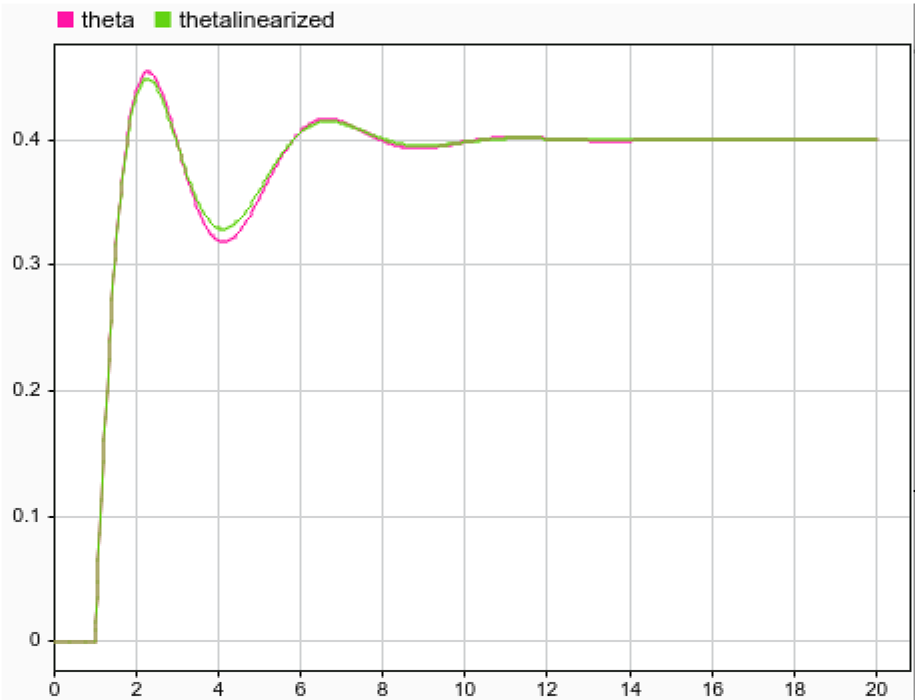
Matlab code for the non-linear plant

```
function dxdt = fcn(x,u)
theta=x(1);
dtheta=x(2);
F=u;
g=9.81;
l=9.81;
m=1/9.81;
dxdt=[dtheta;...
   -g/l*sin(theta)+1/(m*l)*cos(theta)*F];
```

and its linearized version

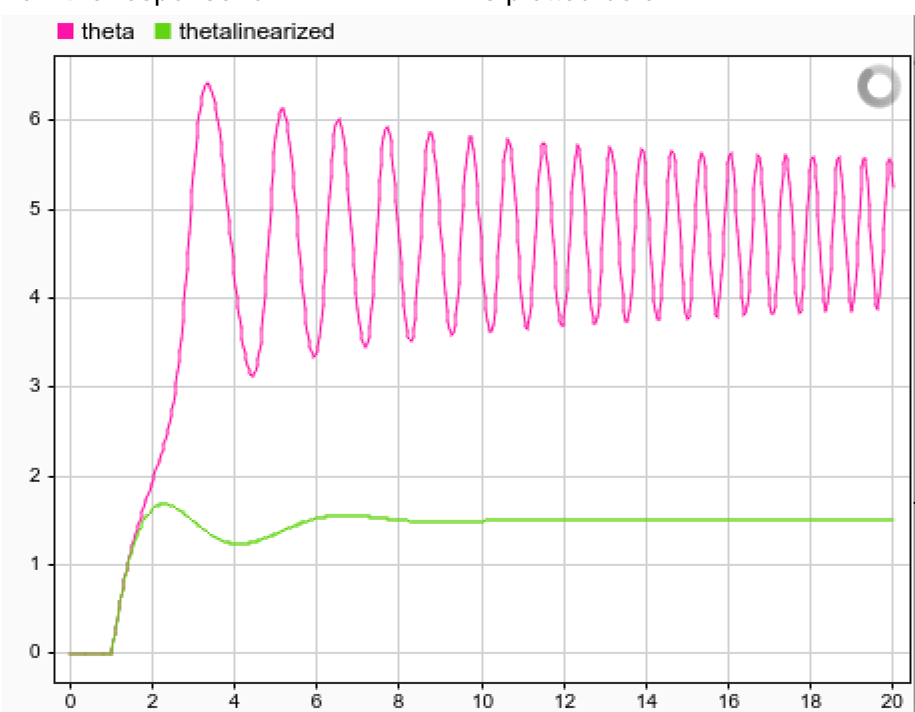
```
function dxdt = fcn(x,u)
theta=x(1);
dtheta=x(2);
F=u;
g=9.81;
l=9.81;
m=1/9.81;
dxdt=[dtheta;...
   -g/l*theta+1/(m*l)*F];
```

Using the PID parameters given, the response for $\theta_{target} = 0.4$ is shown below.



Looking at the figure, both the true non-linear system and its linear approximation behave in a similar fashion, with small differences.

Now the response for $\theta_{target} = 1.5$ is plotted below.



The linearized system behaves similarly to the earlier response, as expected for a linear system, where the set-point does not affect the system performance characteristics. However, the true non-linear system behaves in a very different way—it oscillates heavily and its steady state error appears to remain high. Also, unlike a linear system, the frequency of oscillation varies over time.

This experiment demonstrates a few important points: First, that the linearization is reasonably accurate only around the point in which it was performed. Thus, analysis and design of control based on a linearized model is valid only when the linearization error is small. Despite the shortcomings, linearization is an important tool for analyzing and designing controllers also for non-linear systems.