Aalto University
School of Science

MS-E2135
Decision Analysis
Lecture 9

- *The Analytic Hierarchy Process*
- *Outranking methods*

Salo, Liesiö, Punkka, Vilkkumaa

Motivation

- When alternatives are evaluated w.r.t. multiple attributes / criteria, decision-making can be supported by methods of
  - Multiattribute value theory MAVT (certain attribute-specific performances)
  - Multiattribute utility theory MAUT (uncertain attribute-specific performances)

- Both MAVT and MAUT have a solid axiomatic basis
  - Characterization of preferences ➔ Representation theorems

- But there are many other multicriteria methods, too

Analytic Hierarchy Process (AHP)

- Thomas L. Saaty (1977, 1980)

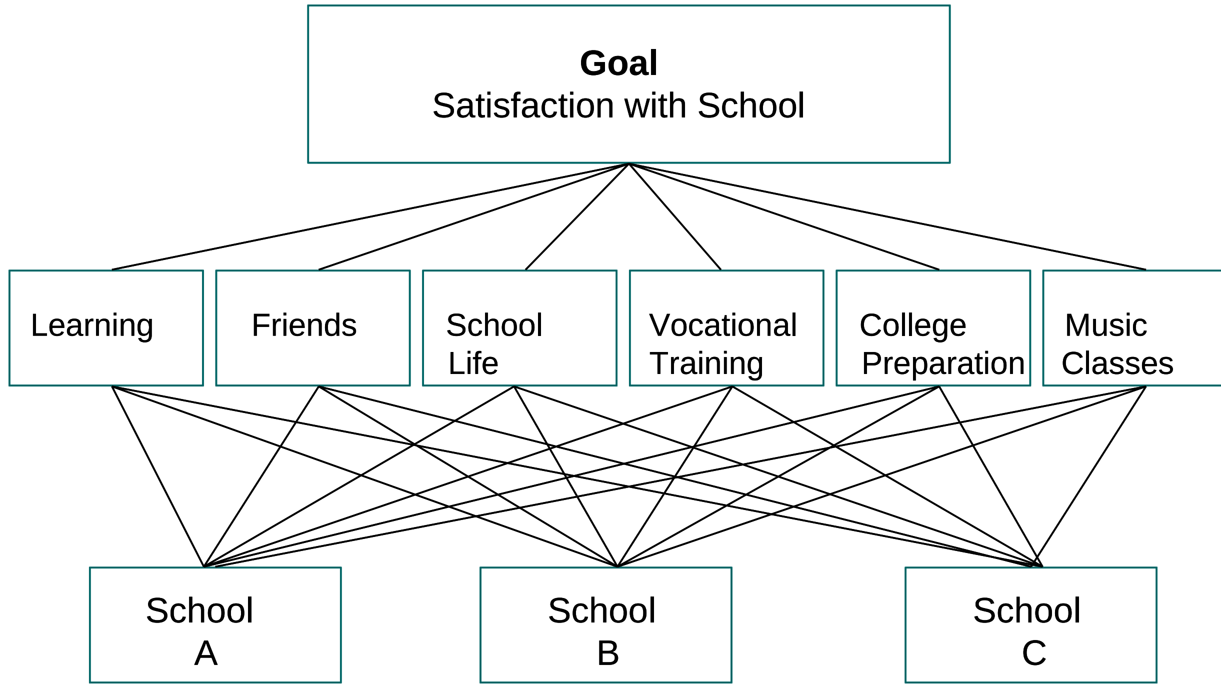
- Has gained much popularity
  - Thousands of reported applications
  - Dedicated conferences and scientific journals
  - Is rather straightforward to apply

- Implemented in many software tools
  - Expert Choice, WebHipre etc.

- Not based on a well-founded axiomatization of preferences
  - Is viewed as controversial by rigorous decision theorists

Problem structuring in the AHP

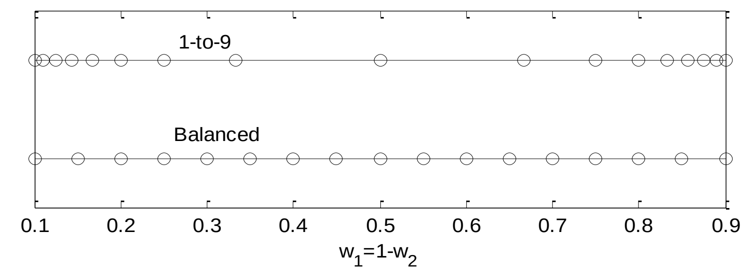
- Objectives, sub-objectives / criteria, and alternatives are represented as a hierarchy of elements (cf. value tree)



Local priorities

- For each objective / sub-objective, a local priority vector is determined
- This vector reflects the relative importance of the elements (either sub-objectives or alternatives) that are placed immediately below the chosen objective / sub-objective
- Pairwise comparisons:
  - For (sub-)**objectives**: "Which sub-objective / criterion is more important for the attainment of the objective? How much more important is it?"
  - For **alternatives**: "Which alternative contributes more to the attainment of the criterion? How much more does it contribute?"
- Responses on a verbal scale correspond to weight ratios

Verbal statement	Scale: 1-to-9	Scale: Balanced
Equally important	1	1.00
-	2	1.22
Slightly more important	3	1.50
-	4	1.86
Strongly more important	5	2.33
-	6	3.00
Very strongly more important	7	4.00
-	8	5.67
Extremely more important	9	9.00



The balanced scale presented in: Salo and Hämäläinen, On the Measurement of Preferences in the Analytic Hierarchy Process, *Journal of Multi-Criteria Decision Analysis* 6/6 (1997) 309–319.

Pairwise comparison matrix

- Ratios $r_{ij} = \frac{w_i}{w_j}$ give the pairwise comparison matrix A (the more important on the row i)

$$A = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} = 1/r_{1n} & \cdots & r_{nn} \end{bmatrix}$$

	L	F	SL	VT	CP	MC
Learning	1	4	3	1	3	4
Friends	1/4	1	7	3	1/5	1
School life	1/3	1/7	1	1/5	1/5	1/6
Voc. training	1	1/3	5	1	1	1/3
College prep.	1/3	5	5	1	1	3
Music classes	1/4	1	$\boxed{6}$	3	1/3	1

Music classes contribute strongly/very strongly more important than school life

	Learning		
	A	B	C
A	1	1/3	½
B	3	1	3
C	2	1/3	1

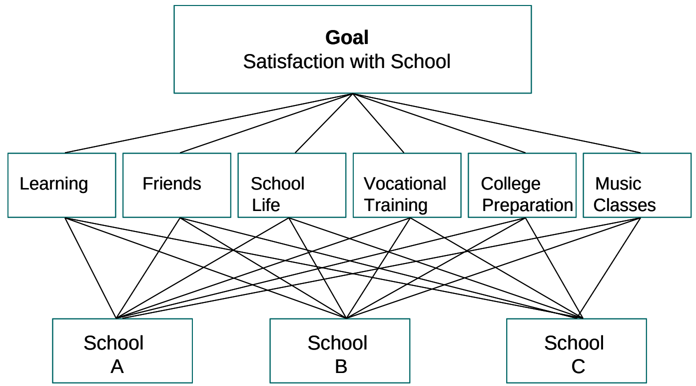
	Friends		
	A	B	C
A	1	1	1
B	1	1	1
C	1	1	1

	School life		
	A	B	C
A	1	5	1
B	1/5	1	1/5
C	1	5	1

	Voc. training		
	A	B	C
A	1	9	7
B	1/9	1	5
C	1/7	1/5	1

	College prep.		
	A	B	C
A	1	1/2	1
B	2	1	2
C	1	1/2	1

	Music classes		
	A	B	C
A	1	6	4
B	1/6	1	1/3
C	1/4	3	1



Inconsistency in pairwise comparison matrices

- **Problem:** Pairwise comparisons are not necessarily consistent
  - Consistency: $r_{ij} = \frac{w_i}{w_j}$ and $r_{jk} = \frac{w_j}{w_k}$ imply that $r_{ik} = \frac{w_i}{w_k} = \frac{w_i}{w_j} \times \frac{w_j}{w_k} = r_{ij} \times r_{jk}$

- E.g., if learning is slightly more important (3) than college preparation, which is strongly more important (5) than school life, then learning should be $3 \times 5 = 15$ times more important than school life … but this is impossible due to the scale upper bound 9

$\rightarrow$ Weights need to be estimated

Local priority vector

	Learning			W
	A	B	C	
A	1	1/3	1/2	0.16
B	3	1	3	0.59
C	2	1/3	1	0.25

- The local priority vector w (=estimated weights) is obtained by normalizing the eigenvector corresponding to the largest eigenvalue of matrix A

$$Aw = \lambda_{max} w,$$

$$w := \frac{1}{\sum_{i=1}^{n} w_i} w$$

- If A is consistent, then λ_{max} = n, the number of rows/colums of A
- Matlab:
  - [v,lambda]=eig(A) returns the eigenvectors and eigenvalues of A

```
>> real(v(:,1))/sum(real(v(:,1)))

ans =

    0.1571
    0.5936
    0.2493
```

Only one eigenvector with all real elements: (0.237, 0.896, 0.376) → normalized eigenvector *w*=(0.16, 0.59, 0.25).

```
>> A=[1 1/3 .5; 3 1 3; 2 1/3 1]

A =

    1.0000    0.3333    0.5000
    3.0000    1.0000    3.0000
    2.0000    0.3333    1.0000

>> [v,l]=eig(A)

v =

   0.2370 + 0.0000i   0.1185 + 0.2052i   0.1185 - 0.2052i
   0.8957 + 0.0000i  -0.8957 + 0.0000i  -0.8957 + 0.0000i
   0.3762 + 0.0000i   0.1881 - 0.3258i   0.1881 + 0.3258i

l =

   3.0536 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.0268 + 0.4038i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0268 - 0.4038i
```

Local priority vectors = “weights”

	Learning			W
	A	B	C	
A	1	1/3	1/2	0.16
B	3	1	3	0.59
C	2	1/3	1	0.25

	Friends			W
	A	B	C	
A	1	1	1	0.33
B	1	1	1	0.33
C	1	1	1	0.33

	School life			W
	A	B	C	
A	1	5	1	0.45
B	1/5	1	1/5	0.09
C	1	5	1	0.46

	Voc. training			W
	A	B	C	
A	1	9	7	0.77
B	1/9	1	5	0.05
C	1/7	1/5	1	0.17

	College prep.			W
	A	B	C	
A	1	1/2	1	0.25
B	2	1	2	0.50
C	1	1/2	1	0.25

	Music classes			W
	A	B	C	
A	1	6	4	0.69
B	1/6	1	1/3	0.09
C	1/4	3	1	0.22

	L	F	SL	VT	CP	MC	W
Learning	1	4	3	1	3	4	0.32
Friends	1/4	1	7	3	1/5	1	0.14
School life	1/3	1/7	1	1/5	1/5	1/6	0.03
Voc. Training	1	1/3	5	1	1	1/3	0.13
College prep.	1/3	5	5	1	1	3	0.24
Music classes	1/4	1	6	3	1/3	1	0.14

Consistency checks

- The consistency of the pairwise comparison matrix *A* is assessed by comparing the *consistency index* (*CI*) of *A* to the *average consistency index RI* of a random pairwise comparison matrix:

$$CI = \frac{\lambda_{max} - n}{n-1}, \qquad CR = \frac{CI}{RI}$$

n	3	4	5	6	7	8	9	10
RI	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

- Rule of thumb: if *CR*>0.10, comparisons are so inconsistent that they should be revised

Three alternatives, *n=3:*

- Learning: $\lambda_{max}= 3.05$, $CR = 0.04$
- Friends: $\lambda_{max}= 3.00$, $CR = 0$
- School life: $\lambda_{max}= 3.00$, $CR = 0$
- Voc. training $\lambda_{max}= 3.40$, $CR = 0.34$
- College prep: $\lambda_{max}= 3.00$, $CR = 0$
- Music classes: $\lambda_{max}= 3.05$, $CR = 0.04$

Six attributes, *n=6:*

- All attributes: $\lambda_{max}= 7.42$, $CR = 0.23$

Total priorities

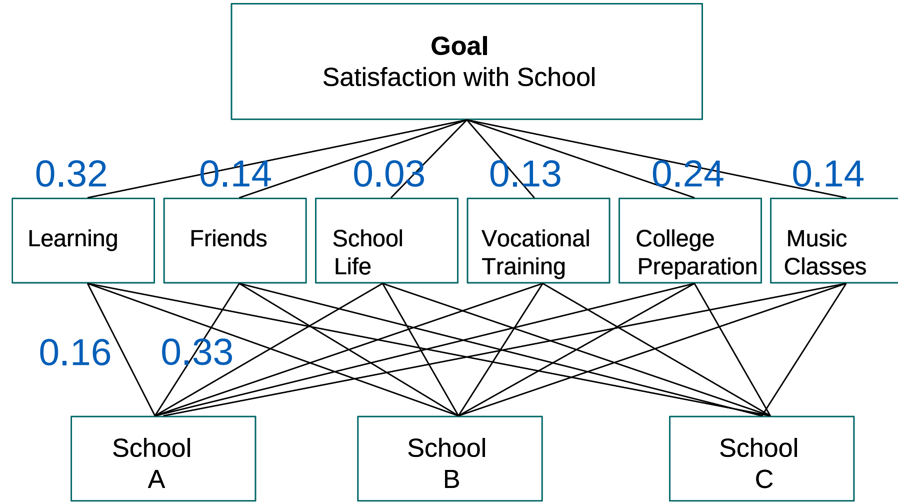
❑ The total (overall) priorities are obtained recursively:

$$w_k = \sum_{i=1}^{n} w_i \, w_k^i,$$

where

- w_i is the total priority of criterion i,
- w_k^i is the local priority of criterion / alternative k with regard to criterion i,
- The sum is computed over all criteria i below which criterion / alternative k is positioned in the hierarchy

	L	F	SL	VT	CP	MC	W
Learning	1	4	3	1	3	4	0.32
Friends	1/4	1	7	3	1/5	1	0.14
School life	1/3	1/7	1	1/5	1/5	1/6	0.03
Voc. Training	1	1/3	5	1	1	1/3	0.13
College prep.	1/3	5	5	1	1	3	0.24
Music classes	1/4	1	6	3	1/3	1	0.14



	Learning			W
	A	B	C	
A	1	1/3	1/2	$\boxed{0.16}$
B	3	1	3	0.59
C	2	1/3	1	0.25

	Friends			w
	A	B	C	
A	1	1	1	$\boxed{0.33}$
B	1	1	1	0.33
C	1	1	1	0.33

Aalto University
School of Science

$$w_A = \sum_{i=1}^{6} w_i \, w_k^i = 0.32 \cdot 0.16 + 0.14 \cdot 0.33 + \ldots$$

Total priorities

	Learning			w
	A	B	C	
A	1	1/3	1/2	0.16
B	3	1	3	0.59
C	2	1/3	1	0.25

	Friends			w
	A	B	C	
A	1	1	1	0.33
B	1	1	1	0.33
C	1	1	1	0.33

	School life			w
	A	B	C	
A	1	5	1	0.45
B	1/5	1	1/5	0.09
C	1	5	1	0.46

	Voc. training			w
	A	B	C	
A	1	9	7	0.77
B	1/9	1	5	0.05
C	1/7	1/5	1	0.17

	College prep.			w
	A	B	C	
A	1	1/2	1	0.25
B	2	1	2	0.50
C	1	1/2	1	0.25

	Music classes			w
	A	B	C	
A	1	6	4	0.69
B	1/6	1	1/3	0.09
C	1/4	3	1	0.22

	L	F	SL	VT	CP	MC	w
Learning	1	4	3	1	3	4	0.32
Friends	1/4	1	7	3	1/5	1	0.14
Schoo life	1/3	1/7	1	1/5	1/5	1/6	0.03
Voc. Training	1	1/3	5	1	1	1/3	0.13
College prep.	1/3	5	5	1	1	3	0.24
Music classes	1/4	1	6	3	1/3	1	0.14

	0.32	0.14	0.03	0.13	0.24	0.14	
	L	F	SL	VT	CP	MC	Total w
A	0.16	0.33	0.45	0.77	0.25	0.69	0.37
B	0.59	0.33	0.09	0.05	0.50	0.09	**0.38**
C	0.25	0.33	0.46	0.17	0.25	0.22	0.25

E.g., $w_B = 0.32\times0.59 + 0.14\times0.33 + 0.03x0.09 +$
$0.13\times0.05 + 0.24\times0.50 + 0.14\times0.09 = 0.38$

Problems with AHP

- **Rank reversals**: The introduction of an additional alternative may change the relative ranking of other, previously introduced alternatives
  - This means that the preferences between two alternatives do not depend on these alternatives only, but on the other alternatives as well, even if these other ones are less preferred

	C_1	C_2
A	1	5
B	4	1
C	**1**	**5**

- Example:
  – Alternatives A and B are compared w.r.t. two "equally important" criteria C_1 and C_2 ($w_{C1} = w_{C2} = 0.5$)
  – A is better than B:

$$w_A = \frac{1}{2}\times\frac{1}{5}+\frac{1}{2}\times\frac{5}{6}\approx 0.517, \qquad w_B = \frac{1}{2}\times\frac{4}{5}+\frac{1}{2}\times\frac{1}{6}\approx 0.483$$

  – Add C which is **identical to A** in terms of its evaluations:

$$w_A = w_C = \frac{1}{2}\times\frac{1}{6}+\frac{1}{2}\times\frac{5}{11}\approx 0.311, \qquad w_B = \frac{1}{2}\times\frac{4}{6}+\frac{1}{2}\times\frac{1}{11}\approx 0.379$$

  – Now B is better than A!

Outranking methods*

- Basic question: is there enough preference information / evidence to state that an alternative is at least as good as another alternative?

- I.e., does an alternative *outrank* some other alternative?

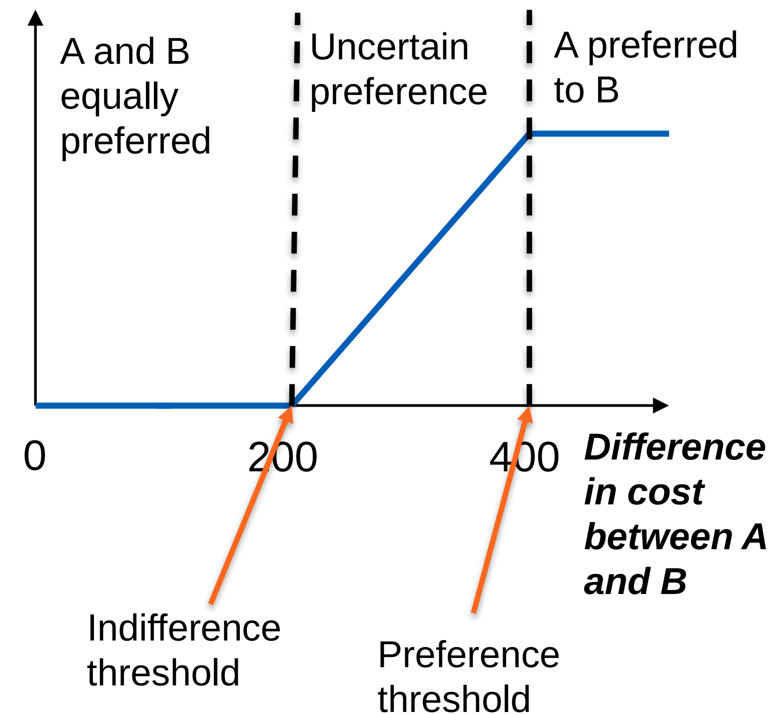
* For an overview of these methods (not required), see, e.g., B. Roy. The outranking approach and the foundations of ELECTRE methods. *Theory and Decision*, 31:49–73, 1991.

Indifference and preference thresholds divide the measurement scale into three parts

- If the difference between the criterion-specific performances of A and B is below a pre-defined **indifference threshold**, then A and B are "equally good" w.r.t. this criterion

- If the difference between the criterion-specific performances of A and B is above a pre-defined **preference threshold**, then A is preferred to B w.r.t this criterion

- Between indifference and preference thresholds, the DM is uncertain about preference



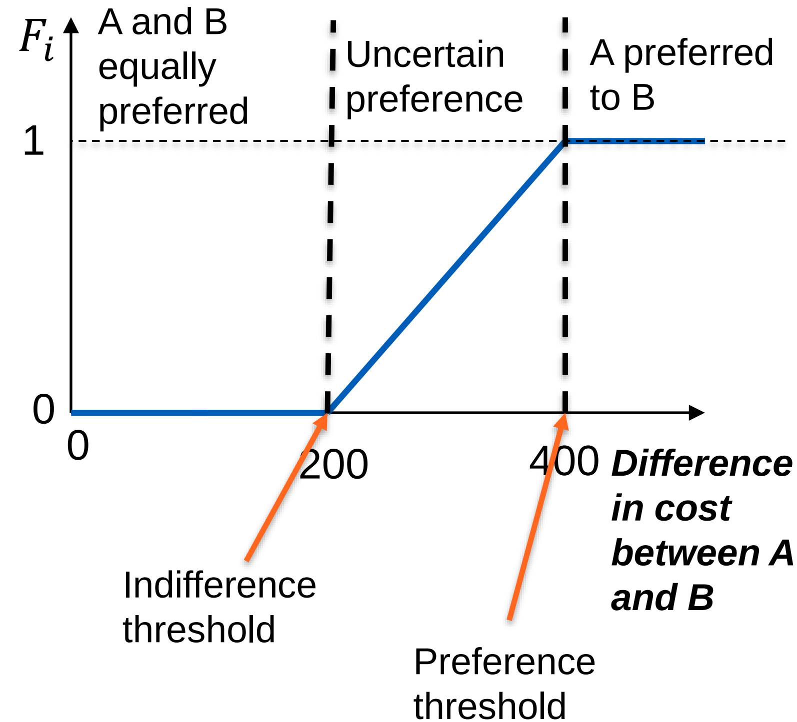
PROMETHEE I & II

- In PROMETHEE methods, the degree to which alternative *k* is preferred to *l* is

$$\sum_{i=1}^{n} w_i F_i(k,l) \geq 0,$$

where

- w_i is the weight of criterion i
- $F_i(k,l)$ =1, if k is preferred to l w.r.t. criterion i,
- $F_i(k,l)$ =0, if the DM is indifferent between k and l w.r.t. criterion i, or l is preferred to k
- $F_i(k,l) \in (0,1)$, if preference between k and l w.r.t. criterion i is uncertain



PROMETHEE I & II

- PROMETHEE I: k is preferred to k', if

$$\sum_{l \neq k} \sum_{i=1}^{n} w_i F_i(k,l) > \sum_{l \neq k'} \sum_{i=1}^{n} w_i F_i(k',l)$$

$$\sum_{l \neq k} \sum_{i=1}^{n} w_i F_i(l,k) < \sum_{l \neq k'} \sum_{i=1}^{n} w_i F_i(l,k')$$

There is more evidence in favor of k than k'

There is less evidence against k than k'

  - The resulting relation is not necessarily complete – it may be that k is not preferred to k' <u>and</u> k' is not preferred to k

- PROMETHEE II: k is preferred to k', if

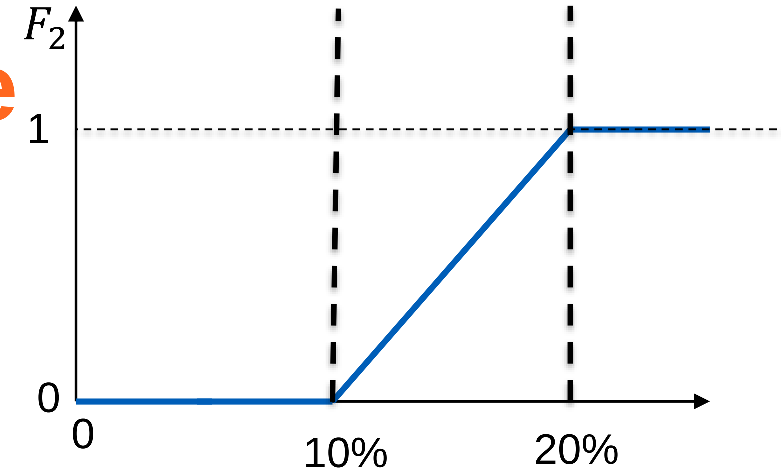
$$F_{net}(k) = \sum_{l \neq k} \sum_{i=1}^{n} w_i [F_i(k,l) - F_i(l,k)] > \sum_{l \neq k'} \sum_{i=1}^{n} w_i [F_i(k',l) - F_i(l,k')] = F_{net}(k')$$

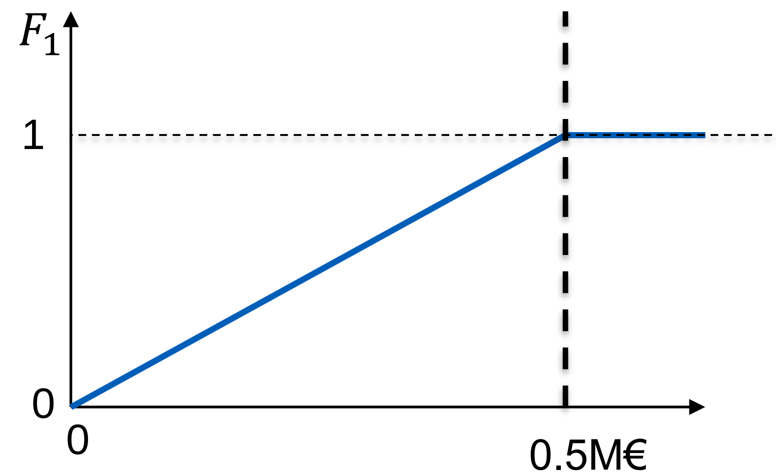
The "net evidence" for k is larger than for k'

PROMETHEE: Example

	Revenue	Market share
x^1	1M€	10%
x^2	0.5M€	20%
x^3	0	30%
Indiff. threshold	0	10%
Pref. threshold	0.5M€	20%
Weight	1	1

	Revenue F_1	Market share F_2	Weighted $F_w=w_1F_1+w_2F_2$
x^1, x^2	1	0	1
x^2, x^1	0	0	0
x^1, x^3	1	0	1
x^3, x^1	0	1	1
x^2, x^3	1	0	1
x^3, x^2	0	0	0





PROMETHEE I: Example

	F_1	F_2	F_w
x^1, x^2	1	0	1
x^2, x^1	0	0	0
x^1, x^3	1	0	1
x^3, x^1	0	1	1
x^2, x^3	1	0	1
x^3, x^2	0	0	0

- PROMETHEE I:
  - x^1 is preferred to x^2, if

$$\underbrace{\sum_{i=1}^{2}\left(F_i(x^1,x^2)+F_i(x^1,x^3)\right)}_{=1+1=2} > \underbrace{\sum_{i=1}^{2}\left(F_i(x^2,x^1)+F_i(x^2,x^3)\right)}_{=0+1=1}$$

$$\underbrace{\sum_{i=1}^{2}\left(F_i(x^2,x^1)+F_i(x^3,x^1)\right)}_{=0+1=1} < \underbrace{\sum_{i=1}^{2}\left(F_i(x^1,x^2)+F_i(x^3,x^2)\right)}_{=1+0=1}$$

  - x^1 is not preferred to x^2 due to the latter condition
  - x^2 is not preferred to x^1 due to both conditions
  - x^1 is preferred to x^3
  - x^2 is not preferred to x^3 and vice versa
- Note: **preferences are not transitive**
  - $x^1 \succ x^3 \sim x^2 \nRightarrow x^1 \succ x^2$

PROMETHEE I: Example (Cont'd)

- PROMETHEE I is also prone to rank reversals:
  - Remove x^2
  - Then,

$$\underbrace{\sum_{i=1}^{2}\left(F_i(x^1,x^3)\right)}_{=1} \not> \underbrace{\sum_{i=1}^{2}\left(F_i(x^3,x^1)\right)}_{=1}$$

$$\underbrace{\sum_{i=1}^{2}\left(F_i(x^3,x^1)\right)}_{=1} \not< \underbrace{\sum_{i=1}^{2}\left(F_i(x^1,x^3)\right)}_{=1}$$

$\rightarrow x^1$ is no longer preferred to x^3

	F_1	F_2	F_w
~~x^1, x^2~~	~~1~~	~~0~~	~~1~~
~~x^2, x^1~~	~~0~~	~~0~~	~~0~~
x^1, x^3	1	0	1
x^3, x^1	0	1	1
~~x^2, x^3~~	~~1~~	~~0~~	~~1~~
~~x^3, x^2~~	~~0~~	~~0~~	~~0~~

PROMETHEE II: Example

- The "net flow" of alternative x^j

$$F_{net}(x^j) = \sum_{k \neq j} [F_w(x^j, x^k) - F_w(x^k, x^j)]$$

- $F_{net}(x^1) = (1-0) + (1-1) = 1$
- $F_{net}(x^2) = (0-1) + (1-0) = 0$
- $F_{net}(x^3) = (1-1) + (0-1) = -1$

$\rightarrow x_1 \succ x_2 \succ x_3$

	F_1	F_2	F_w
x^1, x^2	1	0	1
x^2, x^1	0	0	0
x^1, x^3	1	0	1
x^3, x^1	0	1	1
x^2, x^3	1	0	1
x^3, x^2	0	0	0

PROMETHEE II: Example (Cont'd)

- PROMETHEE II is also prone to rank reversals
  - Add two altrenatives that are equal to x^3 in both criteria. Then, x^2 becomes the most preferred:
    $$F_{net}(x^1) = (1-0) + 3 \times (1-1) = 1$$
    $$F_{net}(x^2) = (0-1) + 3 \times (1-0) = 2$$
    $$F_{net}(x^{3:5}) = (1-1) + (0-1) = -1$$
  - Add two alternatives that are equal to x^1 in both criteria. Then, x^2 becomes the least preferred:
    $$F_{net}(x^{1,4,5}) = (1-0) + (1-1) + 2 \times (0-0) = 1$$
    $$F_{net}(x^2) = 3 \times (0-1) + (1-0) = -2$$
    $$F_{net}(x^3) = 3 \times (1-1) + (0-1) = -1$$
  - Remove x^2. Then, x^1 and x^3 are equally preferred.
    $$F_{net}(x^1) = F_{net}(x^3) = (1-1) = 0$$

	F_1	F_2	F_w
x^1, x^2	1	0	1
x^2, x^1	0	0	0
x^1, x^3	1	0	1
x^3, x^1	0	1	1
x^2, x^3	1	0	1
x^3, x^2	0	0	0

Summary

- AHP and outranking methods are widely used to support multiattribute decision-making

- Unlike MAVT (and MAUT), these methods are not founded on a rigorous axiomatization of preferences →
  - Rank reversals
  - Preferences are not necessarily transitive

- Model parameters can be difficult to elicit
  - Weights have no clear interpretation
  - In outranking methods, statement "I prefer 2€ to 1€" and "I prefer 3€ to 1€" are both modeled with the same number (1); to make a difference, indifference and preference thresholds need to be carefully selected