Mathematics for Economists

Mitri Kitti

Aalto University

Difference Equations

Introduction

Modeling economic phenomena that evolve in time

- economic growth, how economic conditions evolve (predictions)

Difference equations

- models for discrete time dynamics

Objective

- linear difference equations
- solutions by eigenvalues

First-order linear difference equations: Example

- **Example.** Suppose you deposit y_0 euros in a savings account
- The interest rate r is compounded in each time period $t = 0, 1, 2, \ldots$
- The value of your initial deposit in each time period is given by

$$y_{t+1} = (1 + r)y_t, \quad t = 0, 1, 2, \ldots \tag{1}$$

- Equation (1) is a *first-order linear difference equation*
  - First-order: for every t, y_t affects (directly) only y_{t+1}
  - Linear: for every t, (1) is a linear equation

First-order linear difference equations: Example

- **Example (cont'd).** How can we solve (1)? That is, how can we find an expression for y_t that, for every t, depends only on the constant parameter r, the initial condition y_0, and the time index t?

- We have:

$$\begin{aligned} y_1 &= (1+r)y_0 \\ y_2 &= (1+r)y_1 = (1+r)^2 y_0 \\ y_3 &= (1+r)y_2 = (1+r)^3 y_0 \\ y_4 &= (1+r)y_3 = (1+r)^4 y_0 \\ &\dots \end{aligned}$$

  and so on

- The solution we are looking for is $y_t = (1+r)^t y_0$, $t = 0, 1, 2, \dots$

First-order linear difference equations

- Consider the first-order linear difference equation

$$x_{t+1} = ax_t + b_t, \quad t = 0, 1, 2, \ldots,$$

where a is a constant

- Starting with an initial condition x_0, we can calculate x_t as follows:

$$\begin{aligned}
x_1 &= ax_0 + b_0 \\
x_2 &= ax_1 + b_1 = a(ax_0 + b_0) + b_1 = a^2 x_0 + ab_0 + b_1 \\
x_3 &= ax_2 + b_2 = a(a^2 x_0 + ab_0 + b_1) + b_2 = a^3 x_0 + a^2 b_0 + ab_1 + b_2 \\
x_4 &= ax_3 + b_3 = a(a^3 x_0 + a^2 b_0 + ab_1 + b_2) + b_3 = a^4 x_0 + \sum_{k=1}^{4} a^{4-k} b_{k-1} \\
&\ldots
\end{aligned}$$

and so on

First-order linear difference equations

- In general, we have

$$x_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_{k-1}, \quad t = 0, 1, 2, \ldots \tag{2}$$

- When $b_k = b$ for all $k = 0, 1, 2, \ldots$, we have

$$\sum_{k=1}^{t} a^{t-k} b_{k-1} = b \sum_{k=1}^{t} a^{t-k} = b(a^{t-1} + a^{t-2} + \cdots + a + 1)$$

- The term $(a^{t-1} + a^{t-2} + \cdots + a + 1)$ is the sum of the first t terms of a geometric series. When $a \neq 1$,

$$(a^{t-1} + a^{t-2} + \cdots + a + 1) = \frac{1 - a^t}{1 - a}$$

First-order linear difference equations

- Therefore, when $b_k = b$ for all $k = 0, 1, 2, \dots$ and $a \neq 1$, the solution (2) reduces to

$$x_t = a^t \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a}, \quad t = 0, 1, 2, \dots \tag{3}$$

- When $b_k = b$ for all $k = 0, 1, 2, \dots$ and $a = 1$, we have that $(a^{t-1} + a^{t-2} + \cdots + a + 1) = t$. Thus (2) simplifies to

$$x_t = x_0 + tb, \quad t = 0, 1, 2, \dots$$

First-order linear difference equations

- Consider the solution (3) and suppose that $|a| < 1$, i.e. $-1 < a < 1$. It is then easy to verify that

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} a^t \left(x_0 - \frac{b}{1-a} \right) + \frac{b}{1-a} = \frac{b}{1-a}$$

- If $x_s = \frac{b}{1-a}$ for some $s \geq 0$, then $x_{s+k} = \frac{b}{1-a}$ for all $k = 0, 1, 2, \ldots$

- We say that the constant $x^* = \frac{b}{1-a}$ is the **equilibrium** (or stationary state) of the difference equation

- When $|a| < 1$, the solution (3) converges to the equilibrium state $x^* = \frac{b}{1-a}$. In this case, we say that the difference equation is **globally asymptotically stable**

Difference equations

A sequence $\{\mathbf{x}_k\}_{k=0}^{\infty} \subset \mathbb{R}^n$ satisfies a difference equation of order T if $G(\mathbf{x}_k, \mathbf{x}_{k-1}, \ldots, \mathbf{x}_{k-T}) = \mathbf{0}$ (for all $k \geq T$), where $G : \mathbb{R}^{Tn} \mapsto \mathbb{R}^n$

- often difference equations are transformed into recursions $x_{k+1} = F(x_k)$
- in this case the solution is a sequence $\mathbf{x}_1$, $\mathbf{x}_2$, and so on corresponding to the intial value $\mathbf{x}_0$
- other names for the recursion: *state transition equation*, *law of motion*, or *dynamic system*, *discrete time process*

Example: a model of national income

- In period t national income Y_t satisfies $Y_t = C_t + I_t + G_t$
  - C_t is of private consumption, I_t is investments and G_t is public spending
  - Assume that $C_t = \alpha Y_t$, $G_t = G_0$ for all t, $I_t = \beta(C_t - C_{t-1})$
- This yields a difference equation
  $Y_t = \alpha Y_t + \beta(C_t - C_{t-1}) + G_0 = \alpha(1+\beta)Y_{t-1} - \alpha\beta Y_{t-2} + G_0$
  - difference equation of degree 2
  - choosing $x_t = Y_t$ and $z_t = Y_{t-1}$ yields a recursion:

$$
\begin{aligned}
x_k &= \alpha(1+\beta)x_{k-1} - \alpha\beta z_{k-1} + G_0 \\
z_k &= z_{k-1}
\end{aligned}
$$

Other examples

- A growth model $K_{t+1} = f(K_t, L_t) + (1-\delta)K_t - C_t$
  - K is capital, L is labor, C is consumption, δ is the capital depreciation rate
  - production function $f(K_k, L_k)$
- Harvesting of a natural resource: $s_{t+1} = f(s_t) - x_t$
  - resource stock s_t, harvest x_t, growth $f(s)$

Linear difference equations

Dynamical system given by $z^{k+1} = Az^k$, $k = 0, 1, \ldots$, where $A \in \mathbb{R}^{n \times n}$ and $z^0 \in \mathbb{R}^n$ is given

- model for a discrete time process
- this kind of systems can be obtained by linearizing nonlinear difference equations
- solution by brute force: find z^N corresponding to z^0 by iterating the system, i.e., $z^N = A^N z^0$, which means that we need A^N (isn't this a good enough solution?)

Example: an uncoupled system $z^{k+1} = Az^k$, where $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

- what is the solution?

Systems of first-order linear difference equations: Example

- Suppose we have the following system:

$$x_{t+1} = x_t + 4y_t \tag{4}$$

$$y_{t+1} = \frac{1}{2}x_t, \tag{5}$$

with $t = 0, 1, 2, \ldots$

- We can solve the system through a *change of variables*. More specifically, suppose we use the transformation

$$X = \frac{1}{6}x + \frac{1}{3}y \tag{6}$$

$$Y = -\frac{1}{6}x + \frac{2}{3}y \tag{7}$$

and the inverse transformation

$$x = 4X - 2Y \tag{8}$$

$$y = X + Y \tag{9}$$

Systems of first-order linear difference equations: Example

- By the transformation (6)-(7) and the system (4)-(5) we get

$$X_{t+1} = \frac{1}{6}x_{t+1} + \frac{1}{3}y_{t+1} = \frac{1}{6}(x_t + 4y_t) + \frac{1}{3}\left(\frac{1}{2}x_t\right) = \frac{1}{3}x_t + \frac{2}{3}y_t$$

$$Y_{t+1} = -\frac{1}{6}x_{t+1} + \frac{2}{3}y_{t+1} = -\frac{1}{6}(x_t + 4y_t) + \frac{2}{3}\left(\frac{1}{2}x_t\right) = \frac{1}{6}x_t - \frac{2}{3}y_t$$

- Now, using the inverse transformation (8)-(9) yields

$$X_{t+1} = \frac{1}{3}x_t + \frac{2}{3}y_t = \frac{1}{3}(4X_t - 2Y_t) + \frac{2}{3}(X_t + Y_t) = 2X_t$$

$$Y_{t+1} = \frac{1}{6}x_t - \frac{2}{3}y_t = \frac{1}{6}(4X_t - 2Y_t) - \frac{2}{3}(X_t + Y_t) = -Y_t$$

Systems of first-order linear difference equations: Example

- In sum, we have just transformed the initial system (4)-(5) into

$$X_{t+1} = 2X_t$$
$$Y_{t+1} = -Y_t,$$

which is an *uncoupled* system of two difference equations

- The solution of the transformed system is

$$X_t = 2^t c_1$$
$$Y_t = (-1)^t c_2,$$

where c_1 and c_2 are constants determined by the initial conditions x_0 and y_0

Systems of first-order linear difference equations: Example

- Finally, we use again the transformation (6)-(7) to obtain the solution to the initial system:

$$x_t = 4X_t - 2Y_t = 4 \cdot 2^t c_1 - 2(-1)^t c_2$$
$$y_t = X_t + Y_t = 2^t c_1 + (-1)^t c_2$$

- If we are given initial conditions x_0 and y_0, we can also find the exact value of the two constants c_1 and c_2 by solving the following system of linear equations

$$x_0 = 4 \cdot 2^0 c_1 - 2(-1)^0 c_2 = 4c_1 - 2c_2$$
$$y_0 = 2^0 c_1 + (-1)^0 c_2 = c_1 + c_2$$

- You can verify that $c_1 = \frac{1}{6}x_0 + \frac{1}{3}y_0$ and $c_2 = -\frac{1}{6}x_0 + \frac{2}{3}y_0$

Systems of first-order linear difference equations

- The transformation we've used can be generalized to abstract systems of difference equations. Consider the following system of two equations (written in matrix form)
$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \quad t = 0, 1, 2, \ldots,$$
which we can also write in more compact form:
$$\boldsymbol{z}_{t+1} = A\boldsymbol{z}_t$$
- To make the change of variables, we choose a 2×2 (invertible) matrix P and its inverse P^{-1}, and then define
$$\boldsymbol{z} = P\boldsymbol{Z} \quad \text{and} \quad \boldsymbol{Z} = P^{-1}\boldsymbol{z} \tag{10}$$

Solution by diagonalization

- Then we have

$$
\begin{aligned}
\boldsymbol{Z}_{t+1} &= P^{-1}\boldsymbol{z}_{t+1} \\
&= P^{-1}(A\boldsymbol{z}_t) \\
&= (P^{-1}A)\boldsymbol{z}_t \\
&= (P^{-1}A)(P\boldsymbol{Z}_t) \\
&= P^{-1}AP\boldsymbol{Z}_t
\end{aligned}
$$

- We want to choose P in such a way that the coefficient matrix of the transformed system $P^{-1}AP$ is diagonal (so that the system is uncoupled and easy to solve)

Solution by diagonalization

- Let D be a diagonal matrix
$$D = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$
- Write the matrix P as $P = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{pmatrix}$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the two column vectors
- We want to choose P in such a way that
$$P^{-1}AP = D \iff AP = PD$$
- One can show that $AP = PD$ is equivalent to
$$A\boldsymbol{v}_1 = r_1\boldsymbol{v}_1 \quad \text{and} \quad A\boldsymbol{v}_2 = r_2\boldsymbol{v}_2$$
- The numbers r_1 and r_2 are the **eigenvalues** of A and $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the corresponding **eigenvectors**

Solution by diagonalization

- Since $P^{-1}AP = D$, the transformed system reduces to

$$\boldsymbol{Z}_{t+1} = D\boldsymbol{Z}_t \tag{11}$$

- The solution to (11) is

$$\boldsymbol{Z}_t = \begin{pmatrix} c_1 r_1^t \\ c_2 r_2^t \end{pmatrix}$$

- Therefore,

$$\boldsymbol{z}_t = P\boldsymbol{Z}_t = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{pmatrix} \begin{pmatrix} c_1 r_1^t \\ c_2 r_2^t \end{pmatrix} = c_1 r_1^t \boldsymbol{v}_1 + c_2 r_2^t \boldsymbol{v}_2$$

Solution by diagonalization

- Finally, given the initial conditions

$$\boldsymbol{z}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

  we can also determine the constants c_1 and c_2 as follows:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = P^{-1}\boldsymbol{z}_0$$

- In sum, we can solve the system of difference equations (4)-(5) by using the eigenvalues and the eigenvectors of the system's coefficient matrix A

Markov processes

- Finite number of states $i = 1, \dots, n$
- A stochastic process determined probabilities of moving from one state to another in each time instant
- Markov process: probability of state i in period $k+1$ depends only on the state in period k
- state transition probabilities $m_{ij} =$ prob. of state i in period $k+1$ for initial state j
- State transition matrix (Markov matrix)

$$\mathbf{M} = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}$$

Markov processes

Example: households classified according to their neighborhoods as urban (1), suburban (2), rural (3)

- $x^i(k)$ probability that a household is in state i in period k
- Markov matrix

$$\begin{pmatrix} 0.75 & 0.02 & 0.1 \\ 0.2 & 0.9 & 0.2 \\ 0.05 & 0.08 & 0.7 \end{pmatrix}.$$

Markov process as a difference equation

- $x^i(k+1)$= (prob. of transition form state 1 1 to i) $\times$ (prob of state 1)+...+(prob. of transition form state n 1 to i) $\times$ (prob of state n)
- In matrix form $\mathbf{x}(k+1) = \mathbf{M}\mathbf{x}(k)$, i.e.

$$\begin{pmatrix} x^1(k+1) \\ \vdots \\ x^n(k+1) \end{pmatrix} = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} \begin{pmatrix} x^1(k) \\ \vdots \\ x^n(k) \end{pmatrix}.$$

Example