Exercise and Homework Round 8

These exercises (except for the last) will be gone through on **Friday, November 18, 12:15–14:00** in the exercise session. The last exercise is a homework which you should return via mycourses by **Friday, November 25 at 12:00**.

Exercise 1. (Discretization of spring model)

(a) Recall the state-space form of the spring model in the Exercise round 6:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \tag{1}$$

where we have put $u(t) = 0$.

(b) Write down the solution in terms of the matrix exponential $\exp(\mathbf{A}\, t)$. Plot the solution by evaluating the solution for $t = 0, \ldots, 1$ on a dense grid. Compute the matrix exponentials using numerical expm function.

(c) Discretize the model with $\Delta t = 0.1$ to the form

$$\mathbf{x}_n = \mathbf{F}\mathbf{x}_{n-1}, \tag{2}$$

and compute $\mathbf{F}$ numerically with expm.

(d) Visualize the solution solution from (a) and the discretized solution from (b), and check that they match at the discretization points.

Exercise 2. (Wiener velocity model)

(a) Recall the 1D Wiener velocity model from the lecture slides.

(b) Write down the discretization of the model in form (with fixed Δt)

$$\mathbf{x}_n = \mathbf{F}\mathbf{x}_{n-1} + \mathbf{q}_n. \tag{3}$$

What does $\mathbf{F}$ look like, what are the mean and covariance of $\mathbf{q}_n$?

(c) Simulate trajectories from the (discretized) model by using a suitable initial mean and covariance.

(d) The mean and covariance matrix for each time step can be computed by the recursions

$$\begin{aligned} \mathbf{m}_n &= \mathbf{F}\mathbf{m}_{n-1}, \\ \mathbf{P}_n &= \mathbf{F}\mathbf{P}_{n-1}\mathbf{F}^\mathsf{T} + \mathbf{Q}_n. \end{aligned} \tag{4}$$

Check that the empirical mean and covariance match the theoretical ones.

Exercise 3. (Euler–Maruyama discretization of robot model)

Consider the following 2D dynamic model of a robot platform:

$$\begin{aligned} \dot{p}^x(t) &= v(t)\cos(\varphi(t)) + w_1(t), \\ \dot{p}^y(t) &= v(t)\sin(\varphi(t)) + w_2(t), \\ \dot{\varphi}(t) &= \omega_{\text{gyro}}(t) + w_3(t), \end{aligned}$$

where p^x, p^y is the position, φ is the orientation angle, v is the speed input, ω_{gyro} is the gyroscope reading, and w_1, w_2, w_3 are independent white noise processes with spectral densities q_1, q_2, q_3.

(a) Form Euler–Maruyama discretization of the model with a discretization step Δt. Simulate (random) trajectories from the model.

(b) Form linearization-based discretization of the model. When does this coincide with the Euler–Maryuama discretization? Simulate (random) trajectories from the model.

Homework 8 (DL Friday, November 25 at 12:00)

Consider the scalar differential equation

$$\dot{x} = a\,x + u, \qquad x(0) = x_0, \tag{5}$$

with $a = -1/2$ and $x_0 = 3$, where $u = u(t)$ is some given input function.

(a) With discretization step Δt, form discretization of the model with zeroth-order-hold (ZOH) approximation in form

$$x_n = f_n\,x_{n-1} + l_n\,u_{n-1}. \tag{6}$$

(b) By assuming that $u(t) = 1$, and $\Delta = 0.1$ simulate trajectory of length 100 steps from the discretized model.

(c) Solve the equation using builtin ODE solver (e.g. Matlab's ode45 or Python's odeint) and check that the solution matches the above at the discretization points.