Mathematics for Economists

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Eigenvalues

Eigenvalues and eigenvectors

- Let A be a square matrix of size $n \times n$
- An **eigenvalue** of A is a number r which when subtracted from each of the diagonal entries of A converts A into a *singular* matrix
- In other words, r is an eigenvalue of A if and only if $A - rI$ is singular, where I is the identity matrix
- Recall that a matrix is singular if and only if its determinant is equal to zero. Thus we can say that r is an eigenvalue of A if and only if $\det(A - rI) = 0$

Eigenvalues and eigenvectors

- **Example.** Let

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Then A has two eigenvalues: $r_1 = 2$ and $r_2 = 4$

- Let

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Then B has two eigenvalues: $r_1 = 0$ and $r_2 = 2$

Eigenvalues and eigenvectors

- Suppose r is an eigenvalue of A. By definition, $A - rI$ is singular
- Consider the system of linear equations
$$(A - rI)\boldsymbol{v} = \mathbf{0}, \tag{1}$$
where $\boldsymbol{v}$ is an $n \times 1$ vector and $\mathbf{0}$ is the zero vector
- Since $A - rI$ is singular, there exists a non-zero vector $\boldsymbol{v}$ that solves (1)
- Any non-zero vector that solves (1) is an **eigenvector** of A corresponding to the eigenvalue r
- Note: there are infinitely many non-zero solutions to (1)
- Also note that (1) is equivalent to
$$A\boldsymbol{v} = r\boldsymbol{v}$$

Eigenvalues and eigenvectors

Proposition

Let A be an $n \times n$ matrix and let r be a scalar. The following statements are equivalent:

1. *Subtracting r from each diagonal entry of A transforms A into a singular matrix;*
2. *$A - rI$ is a singular matrix;*
3. $\det(A - rI) = 0$*;*
4. *$(A - rI)\boldsymbol{v} = \mathbf{0}$ for some non-zero vector $\boldsymbol{v}$;*
5. *$A\boldsymbol{v} = r\boldsymbol{v}$ for some non-zero vector $\boldsymbol{v}$.*

Facts about eigenvalues and eigenvectors

- Diagonal elements of a diagonal matrix are eigenvalues
- Eigenvector determines a direction that is left invariant under A
- Matrix is singular if and only if 0 is its eigenvalue
- If a matrix is symmetric, it has n real eigenvalues
- If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A \in \mathbb{R}^{n \times n}$, then $\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{trace}(A)$ and $\lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det(A)$.
  - trace(A) is the sum of diagonal elements of A

Eigenvalues: Example 2

$A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$, because this is not a diagonal matrix we need characteristic equation for finding the eigenvalues

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} -1-\lambda & 3 \\ 2 & 0-\lambda \end{pmatrix} = \\ &= -(1+\lambda)(-\lambda) - 6 = \lambda^2 + \lambda - 6 \\ &= (\lambda + 3)(\lambda - 2) \end{aligned}$$

the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$. The eigenvector corresponding to λ_1 can be obtained from $(A - (-3)I)v = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, e.g., $v = (-3, 2)$ is an eigenvector (like all vectors αv, $\alpha \neq 0$), what is the other eigenvector?

Characteristic polynomial

- How to find eigenvalues and eigenvectors?
- Consider the following matrix

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

- Eigenvalues can be found by calculating the determinant of

$$A - rI = \begin{pmatrix} 2-r & 1 & -1 \\ 0 & 1-r & 1 \\ 2 & 0 & -2-r \end{pmatrix}$$

- The polynomial $\det(A - rI)$ is called the **characteristic polynomial** of A; the eigenvalues of A are the roots of its characteristic polynomial

Characteristic polynomial

- You can verify that $\det(A - rI) = r(2 - r)(1 + r)$
- Therefore, A has three distinct eigenvalues:

$$r_1 = -1, \quad r_2 = 0, \quad r_3 = 2$$

Eigenvectors

- To find an eigenvector associated with $r_1 = -1$ we need to find a non-zero solution to

$$(A - r_1 I)\boldsymbol{v}_1 = 0 \iff \begin{pmatrix} 2-(-1) & 1 & -1 \\ 0 & 1-(-1) & 1 \\ 2 & 0 & -2-(-1) \end{pmatrix} \boldsymbol{v}_1 = 0$$

- The above system has infinitely many non-zero solutions. We can choose $\boldsymbol{v}_1 = (1, -1, 2)^T$

- Note: if $\boldsymbol{v}_1$ is an eigenvector associated with r_1, then $\alpha \boldsymbol{v}_1$ is another eigenvector for all scalars $\alpha \neq 0$

- The set of all solutions of the above system (including $\boldsymbol{v} = \mathbf{0}$) is called the **eigenspace** of A with respect to the eigenvalue $r_1 = -1$

Eigenvectors

- You can verify that an eigenvector associated with $r_2 = 0$ is

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

and that an eigenvector associated with $r_3 = 2$ is

$$\boldsymbol{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Eigenvalues: More facts

- Characteristic equation is a polynomial of degree n

$\rightarrow$ there are n eigenvalues some of which may be complex numbers

- in general, finding the roots of nth degree polynomial is hard

Eigenvalues and eigenvectors

- **Exercise.** Consider the system at p. 2 in matrix form

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \quad t = 0, 1, 2, \ldots,$$

- Verify that two eigenvalues of the coefficient matrix A are $r_1 = 2$ and $r_2 = -1$
- Verify that an eigenvector associated with $r_1 = 2$ is

$$\boldsymbol{v}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

and that an eigenvector associated with $r_2 = -1$ is

$$\boldsymbol{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

- Use the above eigenvalues and eigenvectors to solve this system

Diagonalization

- We have learned how to solve a system of difference equations by using the following transformation of the coefficient matrix A:

$$P^{-1}AP = D, \tag{2}$$

where D is a diagonal matrix

- The transformation (2) is called **diagonalization** of the matrix A
- If we can find matrices P, P^{-1}, and D such that (2) holds, we say that A is *diagonalizable*
- Notice that P must be invertible in order for A to be diagonalizable

Diagonalization

Proposition (Diagonalization)

Let A be an $n \times n$ matrix. Let $r_1, r_2, \ldots, r_n$ be eigenvalues of A, and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ the corresponding eigenvectors. Form the matrix

$$P = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_n \end{pmatrix}$$

whose columns are the n eigenvectors of A.

- *If P is invertible, then*

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 & \ldots & 0 \\ 0 & r_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r_n \end{pmatrix}$$

- *Conversely, if $P^{-1}AP$ is a diagonal matrix D, the columns of P must be eigenvectors of A and the diagonal entries of D must be eigenvalues of A*

Diagonalization

- The proposition in the previous page relies on the hypothesis that the matrix P is invertible. This is equivalent to the hypothesis that the n eigenvectors of A are *linearly independent* (that is, no one of them can be written as a linear combination of the others)

Proposition (Linearly independent eigenvectors)

Let $r_1, \dots, r_h$ be h **distinct** *eigenvalues of the $n \times n$ matrix A. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_h$ be the corresponding eigenvectors. Then, $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_h$ are linearly independent.*

Thus, if all the eigenvalues of A are distinct, A has n linearly independent eigenvectors and is diagonalizable

Systems of first-order linear difference equations

- Consider the following system of difference equations

$$\boldsymbol{z}_{t+1} = A\boldsymbol{z}_t, \quad t = 0, 1, 2, \dots$$

where A is an $n \times n$ coefficient matrix

Proposition (Solution of systems of linear difference equations)

Suppose the coefficient matrix A has n **distinct real eigenvalues** *$r_1, r_2, \dots, r_n$ and corresponding eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$. The general solution of the system of difference equations $\boldsymbol{z}_{t+1} = A\boldsymbol{z}_t$ is*

$$\boldsymbol{z}_t = c_1 r_1^t \boldsymbol{v}_1 + c_2 r_2^t \boldsymbol{v}_2 + \cdots + c_n r_n^t \boldsymbol{v}_n, \quad t = 0, 1, 2, \dots \tag{3}$$

where $c_1, c_2, \dots, c_n$ are constants.

Systems of first-order linear difference equations

- The expression (3) is the **general solution** of the system $\mathbf{z}_{t+1} = A\mathbf{z}_t$ in the sense that one can use (3) to solve the system for any given vector $\mathbf{z}_0$ of initial conditions

- Given an initial vector $\mathbf{z}_0$, we must have by (3) that

$$\begin{aligned}\mathbf{z}_0 &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \\ &= \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = P \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}\end{aligned}$$

  from which we obtain

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = P^{-1}\mathbf{z}_0$$

Systems of first-order linear difference equations

- **Example.** Consider the following system:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \quad t = 0, 1, 2, \ldots$$

- The two eigenvalues of the coefficient matrix are $r_1 = 3$ and $r_2 = 2$, with eigenvectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- The general solution is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = c_1 3^t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 2^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t = 0, 1, 2, \ldots$$

Systems of first-order linear difference equations

- **Example (cont'd).** Suppose the initial condition is $x_0 = y_0 = 1$
- We can determine the coefficients c_1 and c_2 by solving

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- You can verify that $c_1 = -2$ and $c_2 = 3$
- Therefore, with the initial condition $x_0 = y_0 = 1$, the solution to the system of difference equations is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = (-2) \cdot 3^t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 3 \cdot 2^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t = 0, 1, 2, \ldots$$

Systems of first-order linear difference equations

- The solution of a system of difference equations can also be found by using the *powers* of the coefficient matrix
- Given the system $\boldsymbol{z}_{t+1} = A\boldsymbol{z}_t$ with initial condition $\boldsymbol{z}_0$, we have

$$
\begin{aligned}
\boldsymbol{z}_1 &= A\boldsymbol{z}_0 \\
\boldsymbol{z}_2 &= A\boldsymbol{z}_1 = A(A\boldsymbol{z}_0) = A^2\boldsymbol{z}_0 \\
\boldsymbol{z}_3 &= A\boldsymbol{z}_2 = A(A^2\boldsymbol{z}_0) = A^3\boldsymbol{z}_0,
\end{aligned}
$$

  and so on
- The solution is $\boldsymbol{z}_t = A^t\boldsymbol{z}_0$
- But how to compute A^t?

Systems of first-order linear difference equations

- Let's use again the diagonalization $A = PDP^{-1}$ (which is equivalent to $P^{-1}AP = D$):

$$
\begin{aligned}
A &= PDP^{-1} \\
A^2 &= (PDP^{-1})(PDP^{-1}) = PD^2P^{-1} \\
A^3 &= (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1},
\end{aligned}
$$

  and so on

- In general,

$$A^t = PD^tP^{-1} \tag{4}$$

Systems of first-order linear difference equations

- In (4), we can use the following fact. For a diagonal matrix D, we have that

$$D^t = \begin{pmatrix} r_1^t & 0 & \dots & 0 \\ 0 & r_2^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_n^t \end{pmatrix}$$

Systems of first-order linear difference equations

- In sum, if A is diagonalizable as $A = PDP^{-1}$, the solution to the system of difference equations is:

$$
\begin{aligned}
\boldsymbol{z}_t &= A^t \boldsymbol{z}_0 \\
&= PD^t P^{-1} \boldsymbol{z}_0 \\
&= P \begin{pmatrix} r_1^t & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_n^t \end{pmatrix} P^{-1} \boldsymbol{z}_0 .
\end{aligned}
$$

- A useful property of A^t is this: If r is an eigenvalue of A, then r^t is an eigenvalue of A^t

Example

- Consider again the system

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \quad t = 0, 1, 2, \ldots$$

  with initial condition $x_0 = y_0 = 1$

- Here we have that

$$P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Example

- The solution is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & 2^t \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-2) \cdot 3^t + 3 \cdot 2^t \\ 4 \cdot 3^t + (-3) \cdot 2^t \end{pmatrix}$$

$$= (-2) \cdot 3^t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 3 \cdot 2^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Information of eigenvalues

- The nature of the linear function defined by A can be sketched by using eigenvalues
  - if an eigenvalue of A is positive, then A scales the vectors in the direction of the corresponding eigenvector
  - if an eigenvalue is negative, A reverses the direction of the corresponding eigenvector and scales
  - example: how does $A = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ behave?
  - if there are no real eigenvectors, there is no directions in which A behaves as described above

  $\rightarrow$ the matrix turns all the directions (and possibly scales them)
  - e.g., $A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ reverses vectors by angle θ (what are the eigenvalues?)

Complex eigenvalues

- Imaginary unit i, $i^2 = -1$
- Complex number $a + ib$
- Some algebra
  - $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$ for $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$
  - $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 - a_2 b_1)$
  - De Moivre's formula: $z^k = r^k(\cos(k\theta) + i\sin(k\theta))$, where $r = \sqrt{a^2 + b^2}$ and θ satisfies $\cos(\theta) = a/r$
- If an eigenvalue is a complex number, then the corresponding eigenvector is a complex vector
- Assume that $\mathbf{A}$ is 2×2 that has complex eigenvalues $z = \alpha \pm i\beta$
  - eigenvectors $\mathbf{u} \pm i\mathbf{v}$ solution of the difference equation $\mathbf{z}_{k+1} = \mathbf{A}\mathbf{z}_k$ is

$$\mathbf{z}_k = 2r^k[(c_1 \cos k\theta - c_2 \sin k\theta)u - (c_2 \cos k\theta + c_1 \sin k\theta)v]$$

Example

$\mathbf{z}_{k+1} = \mathbf{A}\mathbf{z}_k$ with

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -9 & 1 \end{pmatrix}.$$

Characteristic polynomial: $\lambda^2 - 2\lambda + 10 = 0$. Solutions are

$$\lambda_{1,2} = (2 \pm \sqrt{4 - 4 \cdot 10})/2 = (2 \pm \sqrt{-36}))/2 = (2 \pm 6\sqrt{-1})/2 = 1 \pm i3.$$

Eigenvectors:

$$(\mathbf{A} - \lambda_1 \mathbf{I}) = \begin{pmatrix} -3i & 1 \\ -9 & -3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., $-3iv_1 + v_2 = 0$, e.g. $\mathbf{v} = (1, 3i)$ is an eigenvector and another is $(1, -3i)$. In this case $r = \sqrt{1^2 + 3^2} = \sqrt{10}$, $\theta = \arccos(1/\sqrt{10}) \approx 1.249$. Solution of the difference equation $\mathbf{z} = (x, y)$

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \sqrt{10}^k \left[(c_1 \cos k\theta - c_2 \sin k\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - (c_2 \cos k\theta + c_1 \sin k\theta) \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right].$$

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Eigenvalues

Eigenvalues and eigenvectors

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- In other words, r is an eigenvalue of A if and only if $A - rI$ is singular, where I is the identity matrix
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Eigenvalues and eigenvectors

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Eigenvalues and eigenvectors

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Facts about eigenvalues and eigenvectors

- Diagonal elements of a diagonal matrix are eigenvalues
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Characteristic polynomial

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- Consider the following matrix

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Characteristic polynomial

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Eigenvectors

- To find an eigenvector associated with $r_1 = -1$ we need to find a non-zero solution to

$$(A - r_1 I)\boldsymbol{v}_1 = 0 \iff \begin{pmatrix} 2-(-1) & 1 & -1 \\ 0 & 1-(-1) & 1 \\ 2 & 0 & -2-(-1) \end{pmatrix} \boldsymbol{v}_1 = 0$$

- The above system has infinitely many non-zero solutions. We can choose $\boldsymbol{v}_1 = (1, -1, 2)^T$

- Note: if $\boldsymbol{v}_1$ is an eigenvector associated with r_1, then $\alpha \boldsymbol{v}_1$ is another eigenvector for all scalars $\alpha \neq 0$

- The set of all solutions of the above system (including $\boldsymbol{v} = \mathbf{0}$) is called the **eigenspace** of A with respect to the eigenvalue $r_1 = -1$

Eigenvectors

- You can verify that an eigenvector associated with $r_2 = 0$ is

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Eigenvalues and eigenvectors

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Diagonalization

- We have learned how to solve a system of difference equations by using the following transformation of the coefficient matrix A:

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  where D is a diagonal matrix

- The transformation (2) is called **diagonalization** of the matrix A
- If we can find matrices P, P^{-1}, and D such that (2) holds, we say that A is *diagonalizable*
- Notice that P must be invertible in order for A to be diagonalizable

Diagonalization

Proposition (Diagonalization)

Let A be an $n \times n$ matrix. Let $r_1, r_2, \ldots, r_n$ be eigenvalues of A, and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ the corresponding eigenvectors. Form the matrix

$$P = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \ldots & \boldsymbol{v}_n \end{pmatrix}$$

whose columns are the n eigenvectors of A.

- *If P is invertible, then*

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 & \ldots & 0 \\ 0 & r_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r_n \end{pmatrix}$$

- *Conversely, if $P^{-1}AP$ is a diagonal matrix D, the columns of P must be eigenvectors of A and the diagonal entries of D must be eigenvalues of A*

Diagonalization

- The proposition in the previous page relies on the hypothesis that the matrix P is invertible. This is equivalent to the hypothesis that the n eigenvectors of A are *linearly independent* (that is, no one of them can be written as a linear combination of the others)

Proposition (Linearly independent eigenvectors)

Let $r_1, \ldots, r_h$ be h **distinct** *eigenvalues of the $n \times n$ matrix A. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_h$ be the corresponding eigenvectors. Then, $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_h$ are linearly independent.*

Thus, if all the eigenvalues of A are distinct, A has n linearly independent eigenvectors and is diagonalizable

Systems of first-order linear difference equations

- Consider the following system of difference equations

$$\boldsymbol{z}_{t+1} = A\boldsymbol{z}_t, \quad t = 0, 1, 2, \ldots$$

where A is an $n \times n$ coefficient matrix

Proposition (Solution of systems of linear difference equations)

Suppose the coefficient matrix A has n **distinct real eigenvalues** *$r_1, r_2, \ldots, r_n$ and corresponding eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. The general solution of the system of difference equations $\boldsymbol{z}_{t+1} = A\boldsymbol{z}_t$ is*

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where $c_1, c_2, \ldots, c_n$ are constants.

Systems of first-order linear difference equations

- The expression (3) is the **general solution** of the system $\boldsymbol{z}_{t+1} = A\boldsymbol{z}_t$ in the sense that one can use (3) to solve the system for any given vector $\boldsymbol{z}_0$ of initial conditions

- Given an initial vector $\boldsymbol{z}_0$, we must have by (3) that

$$\begin{aligned}\boldsymbol{z}_0 &= c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n \\ &= \begin{pmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_n\end{pmatrix}\begin{pmatrix}c_1 \\ c_2 \\ \vdots \\ c_n\end{pmatrix} = P\begin{pmatrix}c_1 \\ c_2 \\ \vdots \\ c_n\end{pmatrix}\end{aligned}$$

  from which we obtain

$$\begin{pmatrix}c_1 \\ c_2 \\ \vdots \\ c_n\end{pmatrix} = P^{-1}\boldsymbol{z}_0$$

Systems of first-order linear difference equations

- **Example.** Consider the following system:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \quad t = 0, 1, 2, \dots$$

- The two eigenvalues of the coefficient matrix are $r_1 = 3$ and $r_2 = 2$, with eigenvectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- The general solution is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = c_1 3^t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 2^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t = 0, 1, 2, \dots$$

Systems of first-order linear difference equations

- **Example (cont'd).** Suppose the initial condition is $x_0 = y_0 = 1$
- We can determine the coefficients c_1 and c_2 by solving

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- You can verify that $c_1 = -2$ and $c_2 = 3$
- Therefore, with the initial condition $x_0 = y_0 = 1$, the solution to the system of difference equations is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = (-2) \cdot 3^t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 3 \cdot 2^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t = 0, 1, 2, \ldots$$

Systems of first-order linear difference equations

- The solution of a system of difference equations can also be found by using the *powers* of the coefficient matrix

- Given the system $\boldsymbol{z}_{t+1} = A\boldsymbol{z}_t$ with initial condition $\boldsymbol{z}_0$, we have

$$\begin{aligned} \boldsymbol{z}_1 &= A\boldsymbol{z}_0 \\ \boldsymbol{z}_2 &= A\boldsymbol{z}_1 = A(A\boldsymbol{z}_0) = A^2\boldsymbol{z}_0 \\ \boldsymbol{z}_3 &= A\boldsymbol{z}_2 = A(A^2\boldsymbol{z}_0) = A^3\boldsymbol{z}_0, \end{aligned}$$

  and so on

- The solution is $\boldsymbol{z}_t = A^t\boldsymbol{z}_0$

- But how to compute A^t?

Systems of first-order linear difference equations

- Let's use again the diagonalization $A = PDP^{-1}$ (which is equivalent to $P^{-1}AP = D$):

$$
\begin{aligned}
A &= PDP^{-1} \\
A^2 &= (PDP^{-1})(PDP^{-1}) = PD^2P^{-1} \\
A^3 &= (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1},
\end{aligned}
$$

  and so on

- In general,

$$A^t = PD^tP^{-1} \tag{4}$$

Systems of first-order linear difference equations

- In (4), we can use the following fact. For a diagonal matrix D, we have that

$$D^t = \begin{pmatrix} r_1^t & 0 & \dots & 0 \\ 0 & r_2^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_n^t \end{pmatrix}$$

Systems of first-order linear difference equations

- In sum, if A is diagonalizable as $A = PDP^{-1}$, the solution to the system of difference equations is:

$$\begin{aligned} \boldsymbol{z}_t &= A^t \boldsymbol{z}_0 \\ &= PD^t P^{-1} \boldsymbol{z}_0 \\ &= P \begin{pmatrix} r_1^t & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & r_n^t \end{pmatrix} P^{-1} \boldsymbol{z}_0. \end{aligned}$$

- A useful property of A^t is this: If r is an eigenvalue of A, then r^t is an eigenvalue of A^t

Example

- Consider again the system

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \quad t = 0, 1, 2, \ldots$$

with initial condition $x_0 = y_0 = 1$

- Here we have that

$$P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Example

- The solution is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 3^t & 0 \\ 0 & 2^t \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} (-2) \cdot 3^t + 3 \cdot 2^t \\ 4 \cdot 3^t + (-3) \cdot 2^t \end{pmatrix}$$
$$= (-2) \cdot 3^t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 3 \cdot 2^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Information of eigenvalues

- The nature of the linear function defined by A can be sketched by using eigenvalues
  - if an eigenvalue of A is positive, then A scales the vectors in the direction of the corresponding eigenvector
  - if an eigenvalue is negative, A reverses the direction of the corresponding eigenvector and scales
  - example: how does $A = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ behave?
  - if there are no real eigenvectors, there is no directions in which A behaves as described above

  $\rightarrow$ the matrix turns all the directions (and possibly scales them)
  - e.g., $A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ reverses vectors by angle θ (what are the eigenvalues?)

Complex eigenvalues

- Imaginary unit i, $i^2 = -1$
- Complex number $a + ib$
- Some algebra
  - $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$ for $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$
  - $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 - a_2 b_1)$
  - De Moivre's formula: $z^k = r^k(\cos(k\theta) + i\sin(k\theta))$, where $r = \sqrt{a^2 + b^2}$ and θ satisfies $\cos(\theta) = a/r$
- If an eigenvalue is a complex number, then the corresponding eigenvector is a complex vector
- Assume that **A** is 2×2 that has complex eigenvalues $z = \alpha \pm i\beta$
  - eigenvectors $\mathbf{u} \pm i\mathbf{v}$ solution of the difference equation $\mathbf{z}_{k+1} = \mathbf{A}\mathbf{z}_k$ is

$$\mathbf{z}_k = 2r^k[(c_1 \cos k\theta - c_2 \sin k\theta)u - (c_2 \cos k\theta + c_1 \sin k\theta)v]$$

Example

$\mathbf{z}_{k+1} = \mathbf{A}\mathbf{z}_k$ with

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -9 & 1 \end{pmatrix}.$$

Characteristic polynomial: $\lambda^2 - 2\lambda + 10 = 0$. Solutions are

$$\lambda_{1,2} = (2 \pm \sqrt{4 - 4 \cdot 10})/2 = (2 \pm \sqrt{-36}))/2 = (2 \pm 6\sqrt{-1})/2 = 1 \pm i3.$$

Eigenvectors:

$$(\mathbf{A} - \lambda_1 \mathbf{I}) = \begin{pmatrix} -3i & 1 \\ -9 & -3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., $-3iv_1 + v_2 = 0$, e.g. $\mathbf{v} = (1, 3i)$ is an eigenvector and another is $(1, -3i)$. In this case $r = \sqrt{1^2 + 3^2} = \sqrt{10}$, $\theta = \arccos(1/\sqrt{10}) \approx 1.249$. Solution of the difference equation $\mathbf{z} = (x, y)$

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \sqrt{10}^k \left[(c_1 \cos k\theta - c_2 \sin k\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - (c_2 \cos k\theta + c_1 \sin k\theta) \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right].$$