Mathematics for Economists

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Differential equations

Economics Study Survey by Aalto Economics

Please respond!

Ordinary differential equations

- Difference equations: time is a *discrete* variable, $t = 0, 1, 2, \ldots$
- Differential equations: time is a *continuous* variable, $t \in [0, +\infty)$ or $t \in \mathbb{R}$
- An **ordinary differential equation** is an equation

$$F(t, y(t), y'(t), y''(t), \ldots, y^{(n)}(t)) = 0,$$

  where:

  - t is an independent variable (typically, but not necessarily, time)
  - $y(t)$ is a function of t
  - $y'(t), y''(t), \ldots, y^{(n)}(t)$ are the first, second, ..., nth derivatives of y at t
  - F is a function of $n + 2$ variables.

Ordinary differential equations

- **Example.** All the following are differential equations:

$$y''(t) + 3 = 0$$

$$(y'(t))^2 - t^2 y(t) = 0$$

$$y'(t)(y''(t) + 3) = 0$$

$$y'(t) - t^2 = 0$$

Ordinary differential equations

- **Terminology.** An **ordinary** differential equation describes a relationship between a function of *one variable* and its derivative
- A **partial** differential equation describes a relationship between a function of *several* variables and its partial derivatives
- A differential equation is an ***n*th order** differential equation if it involves derivatives up to and including the nth derivative of $y(t)$. For example, $y''(t) + 3 = 0$ is a second order differential equation
- In this course, we confine ourselves to first and second order ordinary differential equations

Ordinary differential equations

- **Notation.** In the theory of differential equations it is customary to use the **dot notation** for derivatives:

$$y'(t) = \frac{dy}{dt}(t) = \dot{y}$$

$$y''(t) = \frac{d^2y}{dt^2}(t) = \ddot{y}$$

- For example, the equation $y'(t)(y''(t)+3) = 0$ can be written as

$$\dot{y}(\ddot{y}+3) = 0$$

Ordinary differential equations

- Consider the following differential equation

$$\dot{y} = 2t \tag{1}$$

- To **solve** (1) we need to find a function $y(t)$ such that (1) holds for all t. In other words, we need to find a function $y(t)$ whose first derivative w.r.t. t is $2t$ for all t

- $y(t) = t^2$ solves (1). And so do $y(t) = t^2 + 17$, $y(t) = t^2 + \sqrt{2}$, ...

- More generally, any function

$$y(t) = t^2 + C, \tag{2}$$

with $C \in \mathbb{R}$, solves the differential equation (1)

- We say that (2) is the **general solution** of (1)

Ordinary differential equations

- Suppose that $y(t)$ must satisfy $y(0) = 1$ in addition to $\dot{y} = 2t$. That is, the two equations

$$\dot{y} = 2t \tag{3}$$
$$y(0) = 1 \tag{4}$$

must hold simultaneously

- The system (3)-(4) is called an **initial value problem (IVP)**

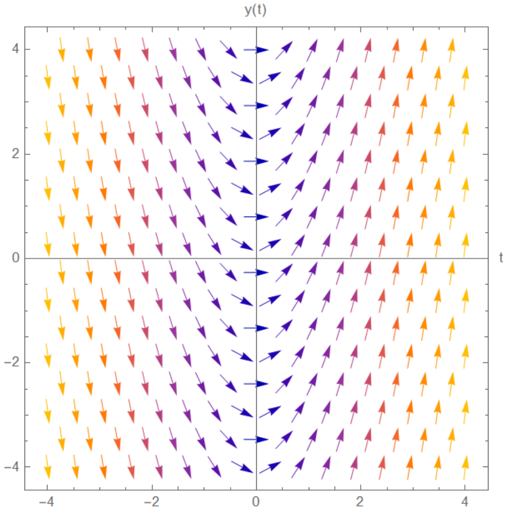
- You can verify that the unique solution of this IVP is

$$y(t) = t^2 + 1 \tag{5}$$

- The solution (5) is called a **particular solution** and is derived from the general solution (2) by choosing the appropriate value of the constant C

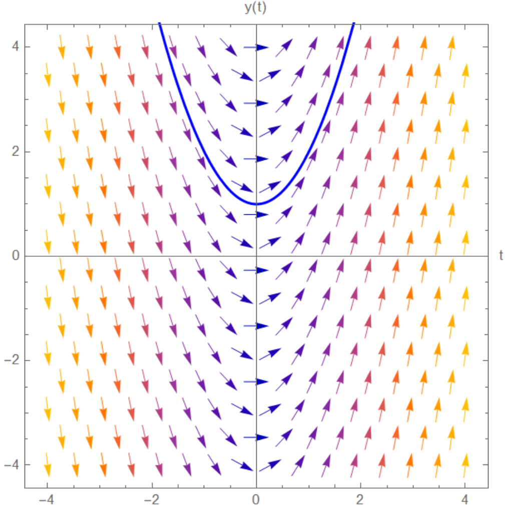
Ordinary differential equations

- Direction field (or integral field) of $\dot{y} = 2t$



Ordinary differential equations

- Solution of the IVP $\dot{y} = 2t$, $y(0) = 1$



Ordinary differential equations

Proposition (Existence and uniqueness of a solution)

Consider the initial value problem

$$\dot{y} = f(t, y), \qquad y(t_0) = y_0.$$

Suppose that f *is continuous at* (t_0, y_0)*.*

- *Then, there exists a* C^1 *function* $y : I \to \mathbb{R}$ *defined on the open interval* $(t_0 - a, t_0 + a)$ *around* t_0 *such that* $y(t_0) = y_0$ *and* $\dot{y}(t) = f(t, y(t))$ *for all* $t \in I$*. That is,* $y(t)$ *is a solution of the initial value problem under consideration.*
- *If in addition the partial derivative of* f *with respect to* y *is continuous at* (t_0, y_0)*, then the solution* $y(t)$ *is unique.*

Ordinary differential equations

- **Example.** Consider the initial value problem

$$\dot{y} = 3y^{\frac{2}{3}}, \qquad y(0) = 0$$

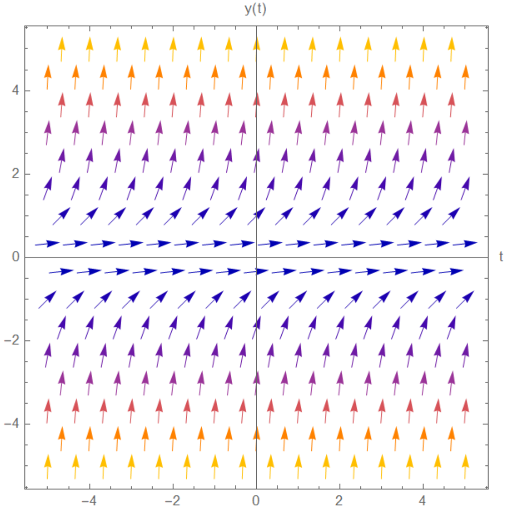
- We have $f(t, y) = 3y^{\frac{2}{3}}$, which is continuous on $\mathbb{R}^2$. This is sufficient to establish that a solution exists

- However, $\frac{\partial f}{\partial y} = \frac{2}{y^{1/3}}$, which is not well-defined at $(t_0, y_0) = (0, 0)$. Hence we cannot apply the second part of the proposition in the previous page about uniqueness. In other words, the solution is not necessarily unique

- In fact, two solutions of this IVP are $y(t) = 0$ and $y(t) = t^3$

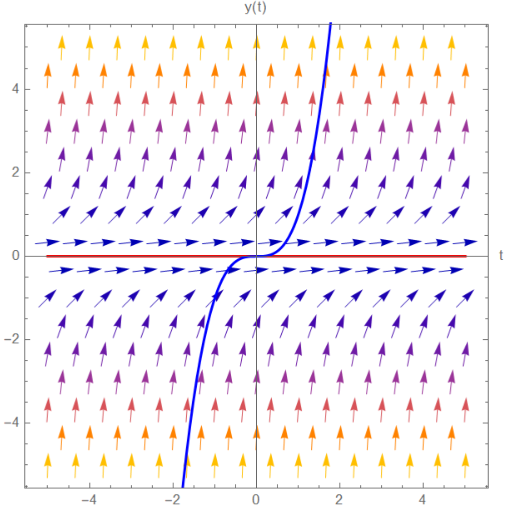
Ordinary differential equations

- Direction field of $\dot{y} = 3y^{\frac{2}{3}}$



Ordinary differential equations

- Two solutions of the IVP $\dot{y} = 3y^{\frac{2}{3}}$, $y(0) = 0$



Ordinary differential equations

- How to solve a differential equation?
- In general, when a solution exists, we cannot always write it down in closed form, i.e. as an *explicit* function $y(t)$
- However, there two important families of differential equations for which explicit solutions can often be found:

  1. Separable equations
  2. **Linear equations**

Linear first order differential equations

- A **linear first order differential equation with constant coefficients** is an equation of the form
$$\dot{y} = ay + b, \tag{6}$$
with $a \neq 0$

- The general solution of (6) is
$$y(t) = -\frac{b}{a} + Ce^{at} \tag{7}$$

Linear first order differential equations

- We can derive the solution of (6) also by following another method (*integrating factor*)

- Take the differential equation (6), multiply both sides by e^{-at} (which is called the "integrating factor") and rearrange terms

$$\dot{y}e^{-at} - aye^{-at} = be^{-at} \tag{8}$$

- The left-hand side of (8) is the derivative of ye^{-at} w.r.t. t. Hence we can rewrite (8) as

$$\frac{d}{dt}\left(ye^{-at}\right) = be^{-at}$$

Linear first order differential equations

- Then by the definition of the indefinite integral, we have

$$ye^{-at} = \int be^{-at}dt = -\frac{b}{a}e^{-at} + C$$

- Thus $ye^{-at} = -\frac{b}{a}e^{-at} + C$, and multiplying both sides of this expression by e^{at} we finally get

$$y(t) = -\frac{b}{a} + Ce^{at}$$

Linear first order differential equations

- **Example.** Solve the differential equation

$$\dot{y} + 2y = 8$$

- Rewrite the equation as

$$\dot{y} = -2y + 8$$

- By (7), the general solution is

$$y(t) = 4 + Ce^{-2t}$$

Linear first order differential equations

- Suppose we want to solve the linear equation

$$\dot{y} = ay + b(t), \tag{9}$$

with $a \neq 0$

- We can use the integrating factor e^{-at} to obtain the general solution

$$y(t) = Ce^{at} + e^{at} \int b(t)e^{-at} dt \tag{10}$$

Linear first order differential equations

- **Example.** Solve the differential equation

$$\dot{y} + y = t$$

- Rewrite the equation as

$$\dot{y} = -y + t$$

- By (10), the general solution is

$$\begin{aligned} y(t) &= Ce^{-t} + e^{-t}\int te^{t}dt \\ &= ke^{-t} + t - 1, \end{aligned}$$

where k is a constant

- Notice that one can use *integration by parts* to evaluate $\int te^{t}dt$

Linear second order differential equations

- A **linear second order ordinary differential equation** is an equation of the form

$$a\ddot{y} + b\dot{y} + cy = 0 \tag{11}$$

- Equation (11) is also *homogeneous* because each non-zero term depends directly on the unknown function y or on a derivative of it. Equations like $a\ddot{y} + b\dot{y} + cy = 7$ or $a\ddot{y} + b\dot{y} + cy = 4t$ are not homogeneous

- An expression for a general solution of (11) can be found as follows

- If $a = 0$, then (11) is a first order linear differential equation, and in this case we know that a solution will have the form $y(t) = e^{rt}$ for some parameter r

- The idea is to find conditions under which a function like $y(t) = e^{rt}$ is a solution of (11)

Linear second order differential equations

- If $y(t) = e^{rt}$ is our candidate solution, then we must have

$$y = e^{rt} \tag{12}$$
$$\dot{y} = re^{rt} \tag{13}$$
$$\ddot{y} = r^2 e^{rt} \tag{14}$$

- Inserting (12)-(14) into (11) and rearranging yields

$$e^{rt}\left(ar^2 + br + c\right) = 0$$

- Since e^{rt} is never equal to zero, the latter equation is equivalent to

$$ar^2 + br + c = 0 \tag{15}$$

Linear second order differential equations

- Thus we can conclude that $y = e^{rt}$ is a solution if and only if r satisfies (15)
- Equation (15) is called the **characteristic equation** of the differential equation (11)
- The left-hand side of (15) is a polynomial of degree 2 whose roots can be found through the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The term $b^2 - 4ac$ is called the **discriminant**

Linear second order differential equations

- There are three mutually exclusive cases:

  1. the discriminant is *strictly positive* and the characteristic equation has two distinct real roots

  2. the discriminant is *equal to zero* and the characteristic equation has two identical real roots, i.e. a real root of multiplicity 2

  3. the discriminant is *strictly negative* and the characteristic equation has two distinct complex roots

Linear second order differential equations

- **First case.** $b^2 - 4ac > 0$ and the characteristic equation has two distinct real roots r_1 and r_2
- The general solution of the differential equation (11) is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- Note: C_1 and C_2 are two *distinct* unknown constants

Linear second order differential equations

- **Second case.** $b^2 - 4ac = 0$ and the characteristic equation has a unique root r of multiplicity 2
- The general solution of the differential equation (11) is

$$y(t) = C_1 e^{rt} + C_2\, t\, e^{rt}$$

Linear second order differential equations

- **Third case.** $b^2 - 4ac < 0$ and the characteristic equation has two distinct complex roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where i is the imaginary unit
- The general solution of the differential equation (11) is

$$y(t) = e^{\alpha t}\left(C_1 \cos \beta t + C_2 \sin \beta t\right)$$

- Note: The two complex roots of the characteristic equation are always *conjugates* of each other

Linear second order differential equations

- **Example.** Let $\ddot{y} - 7y = 0$
- The characteristic equation is $r^2 - 7 = 0$
- There are two real roots $r_1 = \sqrt{7}$ and $r_2 = -\sqrt{7}$
- The general solution is
$$y(t) = C_1 e^{\sqrt{7}t} + C_2 e^{-\sqrt{7}t}$$

Linear second order differential equations

- **Example.** Let $\ddot{y} - 6\dot{y} + 9y = 0$
- The characteristic equation is $r^2 - 6r + 9 = (r-3)^2$
- The two identical roots are $r_1 = r_2 = 3$
- The general solution is

$$y(t) = C_1 e^{3t} + C_2 t e^{3t} = e^{3t}(C_1 + C_2 t)$$

Linear second order differential equations

- In an initial value problem with a second order differential equation, we need to specify *two initial conditions*:

$$y(t_0) = y_0, \qquad \dot{y}(t_0) = y_1$$

- Since the general solution of the differential equation depends on two independent parameters C_1 and C_2, we need two initial conditions to pin down a particular solution of the IVP under consideration

- **Example.** Consider the following initial value problem

$$\ddot{y} - \dot{y} - 2y = 0, \quad y(0) = 3, \quad \dot{y}(0) = 0$$

- You can verify that the general solution of the differential equation is

$$y(t) = C_1 e^{-t} + C_2 e^{2t}$$

- **Example (cont'd).** To find the particular solution, we need to solve the system

$$
\begin{aligned}
y(0) = 3 &\iff C_1 e^0 + C_2 e^0 = 3 \\
\dot{y}(0) = 0 &\iff -C_1 e^0 + 2C_2 e^0 = 0
\end{aligned}
$$

- We easily get $C_1 = 2$ and $C_2 = 1$

- Thus the solution of the IVP is

$$y(t) = 2e^{-t} + e^{2t}$$

Equilibria and stability

- In economics and other disciplines, it is often important to understand the **stability** of solutions of differential equations
- Consider a first order differential equation that can be written as

$$\dot{y} = f(y) \tag{16}$$

- In words, the independent variable t does not explicitly appear on the right-hand side of (16)
- The equation in (16) is called **autonomous**

Equilibria and stability

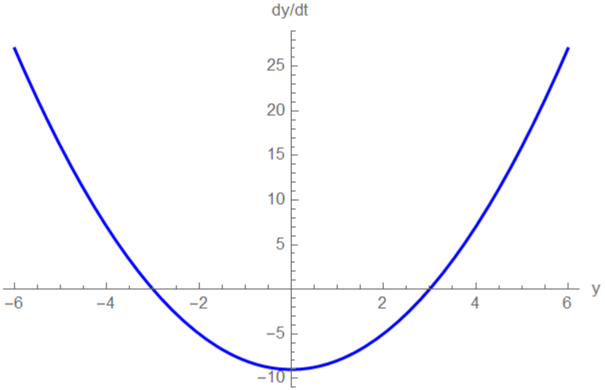
- If there exists a value y^* such that $f(y^*) = 0$, we say that y^* is an **equilibrium** or a **stationary state** or a **steady state** for the equation in (16)
- Given an equilibrium y^*, the constant function $y(t) = y^*$ for all t is a solution of (16). Intuitively, an equilibrium is a solution that does not change over time
- The question we are going to address is this. Suppose $y(t)$ is a solution of (16) with initial condition $y(t_0) = y_0$. Will this solution converge to the steady state y^* as t goes to infinity? Differently put, will the "system" described by (16) ever reach the equilibrium y^* if the system itself starts from (t_0, y_0)?

Equilibria and stability

- Consider the autonomous equation $\dot{y} = y^2 - 9$
- There are two steady states: $y_1^* = 3$ and $y_2^* = -3$
- A useful tool in the analysis of stability is the **phase diagram** or **phase portrait**

Equilibria and stability

- Phase portrait of $\dot{y} = y^2 - 9$



Equilibria and stability

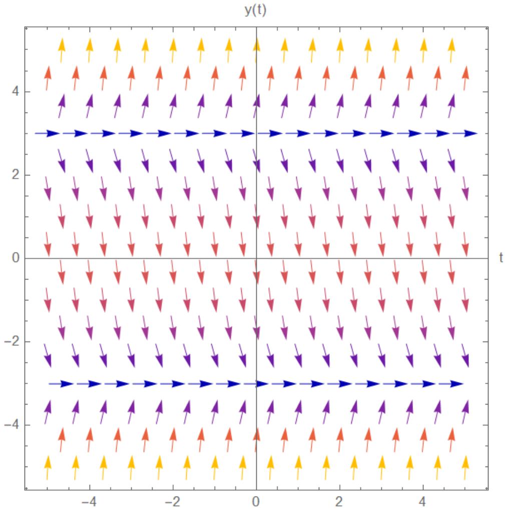
- Suppose $y(t)$ is a solution of the given equation
- For any t, the pair $(y(t), \dot{y}(t))$ is a point on the curve in the phase diagram
- If $(y(t), \dot{y}(t))$ lies *above* the horizontal axis, then $\dot{y}(t) = f(y(t)) > 0$. That is, $y(t)$ is increasing with respect to t. In the diagram, we move from $(y(t), \dot{y}(t))$ to the right and along the curve
- On the other hand, if $(y(t), \dot{y}(t))$ lies *below* the horizontal axis, then $\dot{y}(t) = f(y(t)) < 0$. That is, $y(t)$ is decreasing with respect to t. In the diagram, we move from $(y(t), \dot{y}(t))$ to the left and along the curve

Equilibria and stability

- Consider the two equilibria $y_1^* = 3$ and $y_2^* = -3$
- If a solution $y(t)$ of $\dot{y} = y^2 - 9$ starts close to $y_2^* = -3$, but not at y_2^*, then $y(t)$ will approach y_2^* as time t goes to infinity. We say that the equilibrium $y_2^* = -3$ is **locally asymptotically stable**
- If a solution $y(t)$ of $\dot{y} = y^2 - 9$ starts close to $y_1^* = 3$, but not at y_1^*, then $y(t)$ will move away from y_1^* as time t goes to infinity. We say that the equilibrium $y_1^* = 3$ is **unstable**

Equilibria and stability

- Direction field of $\dot{y} = y^2 - 9$

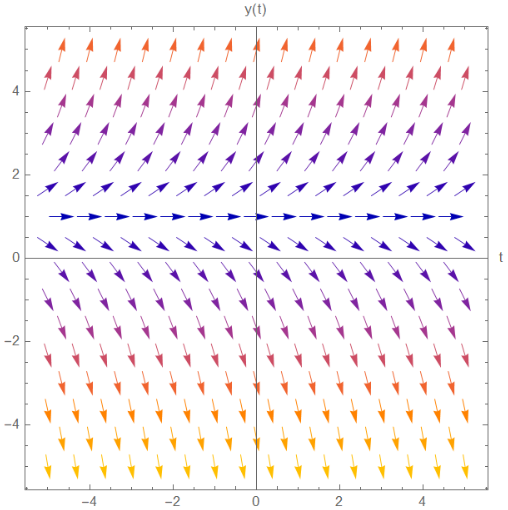


Equilibria and stability

- Now consider the autonomous equation $\dot{y} = y - 1$
- The unique equilibrium is $y^* = 1$
- You can verify that y^* is unstable

Equilibria and stability

- Direction field of $\dot{y} = y - 1$

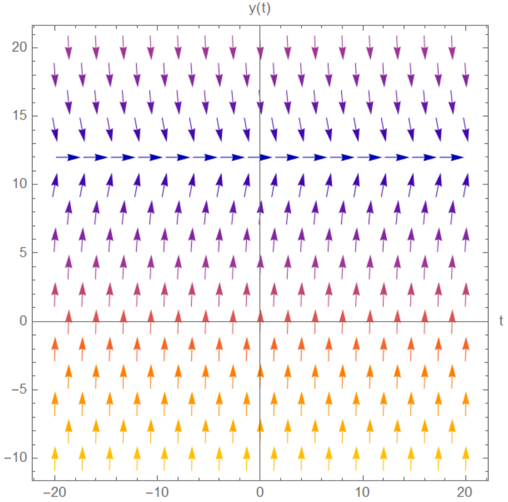


Equilibria and stability

- Consider yet another autonomous equation, $\dot{y} = 24 - 2y$
- The unique equilibrium is $y^* = 12$
- You can verify that y^* is stable
- More specifically, y^* is **globally asymptotically stable** because a solution $y(t)$ with initial condition $y(t_0) = y_0$ will always converge to y^* for any start point (t_0, y_0)

Equilibria and stability

- Direction field of $\dot{y} = 24 - 2y$



Equilibria and stability

- Building on the graphical analysis with phase diagrams, we can state the following result
- Let $\dot{y} = f(y)$ be an autonomous differential equation:
  - If $f(y^*) = 0$ and $f'(y^*) < 0$, then y^* is a locally asymptotically stable equilibrium;
  - If $f(y^*) = 0$ and $f'(y^*) > 0$, then y^* is an unstable equilibrium.
- If $f(y^*) = 0$ and $f'(y^*) = 0$, then y^* can be either stable or unstable. For example, $\dot{y} = y^3$ has a unique equilibrium $y^* = 0$, which is unstable. On the other hand, $\dot{y} = -y^3$ has a unique equilibrium $y^* = 0$, which is globally asymptotically stable

Equilibria and stability

- We can also determine the stability of a second order linear differential equation

$$a\ddot{y} + b\dot{y} + cy = 0, \tag{17}$$

  with $a \neq 0$

- Notice that $y(t) = 0$ is always a solution of (17). In other words, $y^* = 0$ is a steady state of (17)

- The equilibrium $y^* = 0$ is globally asymptotically stable if and only if:
  - $a, b, c > 0$ or, equivalently,
  - both roots of the characteristic equation $ar^2 + br + c = 0$ have negative real part.

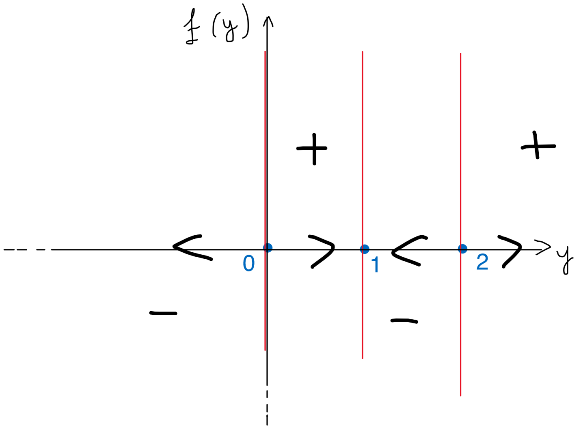
Ordinary differential equations

- **Exercise.** Find the solution of each of the following differential equations for the initial conditions $y(0) = 1, \dot{y}(0) = 0$

1. $6\ddot{y} - \dot{y} - y = 0$
2. $\ddot{y} + 2\dot{y} + 2y = 0$
3. $4\ddot{y} - 4\dot{y} + y = 0$

Equilibria and stability

- **Exercise** $\dot{y} = y(y-1)(y-2)$
- The three equilibria are $y_1^* = 0$, $y_2^* = 1$, and $y_3^* = 2$
- $y_2^* = 1$ is locally asymptotically stable whereas both $y_1^* = 0$ and $y_3^* = 2$ are unstable



Systems of ordinary differential equations

- A **first order** system of two **ordinary differential equations** has the form

$$\dot{x} = F(x, y, t) \tag{18}$$
$$\dot{y} = G(x, y, t) \tag{19}$$

- A **solution** of (18)-(19) is a pair of functions $x^*(t)$ and $y^*(t)$ such that, for every t, both (18) and (19) are satisfied

- If both F and G do not depend explicitly on t, then the system is called *autonomous* or *time-independent*

Systems of ordinary differential equations

- **Example.** Consider the system

$$\dot{x} = 2x + e^t y - e^t$$
$$\dot{y} = 4e^{-t} x + y$$

- You can verify that the *general solution* is

$$x(t) = C_1 + C_2 e^{4t} - \frac{1}{3} e^t$$
$$y(t) = -2C_1 e^{-t} + 2C_2 e^{3t} + \frac{4}{3}$$

- Note: The general solution of a system of n first order equations in n unknowns contains n independent parameters

Systems of ordinary differential equations

- **Example (cont'd).** Suppose we also have the two initial conditions:

$$x(0) = 0$$
$$y(0) = 0$$

- To find the solution of this initial value problem we have to solve

$$0 = C_1 + C_2 e^0 - \frac{1}{3} e^0$$
$$0 = -2C_1 e^0 + 2C_2 e^0 + \frac{4}{3},$$

from which we get $C_1 = \frac{1}{2}$ and $C_2 = -\frac{1}{6}$

- Thus the particular solution is

$$x(t) = \frac{1}{2} - \frac{1}{6} e^{4t} - \frac{1}{3} e^{t}$$
$$y(t) = -e^{-t} - \frac{1}{3} e^{3t} + \frac{4}{3}$$

Systems of ordinary differential equations

- In this course, we'll learn how to solve first order **linear systems** with constant coefficients

- More specifically, we'll focus on systems that can be written as

$$
\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + \cdots + a_{1n}x_n + b_1 \\
\dot{x}_2 &= a_{21}x_1 + \cdots + a_{2n}x_n + b_2 \\
\vdots \quad & \quad\;\; \vdots \qquad\qquad\quad\;\; \vdots \\
\dot{x}_n &= a_{n1}x_1 + \cdots + a_{nn}x_n + b_n
\end{aligned}
$$

  or, in matrix notation,

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}$$

- When $\boldsymbol{b} = \mathbf{0}$, the system is *homogeneous*

Systems of ordinary differential equations

- Let's consider a homogeneous system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$
- If the coefficient matrix A is diagonal, then the system is *uncoupled* and consists of n independent equations:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 \\ \dot{x}_2 &= a_{22}x_2 \\ &\vdots \\ \dot{x}_n &= a_{nn}x_n \end{aligned}$$

- We can solve each equation in isolation, and we already know how to do it. The general solution of the system is

$$x_1(t) = C_1 e^{a_{11}t}, x_2(t) = C_2 e^{a_{22}t}, \ldots, x_n(t) = C_n e^{a_{nn}t}$$

Systems of ordinary differential equations

- If the coefficient matrix A is not diagonal (yet diagonalizable), then we can adopt the same strategy we used with systems of *difference* equations. That is, we can make a change of variables by diagonalizing the coefficient matrix A, find the solution of the resulting uncoupled system, and then transform the solution back to the original variables

Proposition

Suppose the $n \times n$ coefficient matrix A has n distinct real eigenvalues $r_1, \ldots, r_n$, with corresponding eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. Then, the general solution of the linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ is

$$\boldsymbol{x}(t) = C_1 e^{r_1 t}\boldsymbol{v}_1 + C_2 e^{r_2 t}\boldsymbol{v}_2 + \cdots + C_n e^{r_n t}\boldsymbol{v}_n.$$

Systems of ordinary differential equations

- **Example.** Consider the following initial value problem:

$$\dot{x}_1 = 5x_1 - \frac{1}{2}x_2$$
$$\dot{x}_2 = -2x_1 + 5x_2$$
$$x_1(0) = 12, \; x_2(0) = 4.$$

The coefficient matrix is

$$A = \begin{pmatrix} 5 & -\frac{1}{2} \\ -2 & 5 \end{pmatrix}$$

- The characteristic polynomial of A is

$$(5 - r)(5 - r) - 1 = (r - 4)(r - 6)$$

- Hence the two eigenvalues are $r_1 = 4$ and $r_2 = 6$

Systems of ordinary differential equations

- **Example (cont'd).** You can verify that two eigenvectors corresponding to r_1 and r_2 are

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

  respectively

- The general solution of the system is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Systems of ordinary differential equations

- **Example (cont'd).** To find the particular solution of the IVP, we need to solve

$$\begin{pmatrix} 12 \\ 4 \end{pmatrix} = C_1 e^0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^0 \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

  from which we get $C_1 = 7$ and $C_2 = 5$

- Thus the unique solution of this IVP is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 7e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 5e^{6t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Systems of ordinary differential equations

- We can still apply the proposition at p. 11 even when some of the eigenvalues are repeated, *provided that* each eigenvalue of multiplicity $h > 1$ has h linearly independent eigenvectors

- **Example.** Consider the uncoupled system

$$\dot{x}_1 = 3x_1$$
$$\dot{x}_2 = 3x_2$$

- The coefficient matrix is the diagonal matrix

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Systems of ordinary differential equations

- **Example (cont'd).** A has one eigenvalue $r = 3$ of multiplicity 2. However, $r = 3$ has two linearly independent eigenvectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Thus we can write the general solution of the system as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 e^{3t} \\ C_2 e^{3t} \end{pmatrix}$$

Systems of ordinary differential equations

- Consider the linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}$
- Any vector $\boldsymbol{x}^*$ such that $A\boldsymbol{x}^* + \boldsymbol{b} = \boldsymbol{0}$ is an *equilibrium* or *steady state* of the system
- Given a steady state $\boldsymbol{x}^*$, the constant function $\boldsymbol{x}(t) = \boldsymbol{x}^*$ is clearly a solution of $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}$
- The steady state $\boldsymbol{x}^* = -A^{-1}\boldsymbol{b}$ is unique if and only if A is invertible

Proposition

Suppose the $n \times n$ coefficient matrix A has n distinct real eigenvalues $r_1, \ldots, r_n$, with corresponding eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. Let $\boldsymbol{x}^*$ be a steady state of the linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}$. Then, the general solution of $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}$ is

$$\boldsymbol{x}(t) = C_1 e^{r_1 t}\boldsymbol{v}_1 + C_2 e^{r_2 t}\boldsymbol{v}_2 + \cdots + C_n e^{r_n t}\boldsymbol{v}_n + \boldsymbol{x}^*.$$

Systems of ordinary differential equations

- **Example.** Consider the system

$$\dot{x} = 4x + 7y + 31$$
$$\dot{y} = x - 2y + 4$$

- The system's coefficient matrix is $A = \begin{pmatrix} 4 & 7 \\ 1 & -2 \end{pmatrix}$ and is invertible

- The unique steady state (x^*, y^*) can be found either by direct computation:

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = -A^{-1} \begin{pmatrix} 31 \\ 4 \end{pmatrix},$$

  or by solving

$$0 = 4x^* + 7y^* + 31$$
$$0 = x^* - 2y^* + 4$$

Systems of ordinary differential equations

- **Example (cont'd).** You can verify that $(x^*, y^*) = (-6, -1)$
- The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{5t} \begin{pmatrix} 7 \\ 1 \end{pmatrix} + \begin{pmatrix} -6 \\ -1 \end{pmatrix}$$

Systems of ordinary differential equations

- When the system's coefficient matrix is non-diagonalizable, we can form the general solution by using *generalized* eigenvectors

Proposition

Suppose the 2×2 *matrix* A *has equal eigenvalues* $r_1 = r_2 = r$ *and only one independent eigenvector* $\boldsymbol{v}$*. Let* $\boldsymbol{w}$ *be a generalized eigenvector for* A*. Then, the general solution of the linear system of differential equations* $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ *is*

$$\boldsymbol{x}(t) = (C_1 + C_2 t)\, e^{rt} \boldsymbol{v} + C_2 e^{rt} \boldsymbol{w}.$$

Systems of ordinary differential equations

- **Example.** Consider the system

$$\dot{x} = 4x + y$$
$$\dot{y} = -x + 2y$$

- The system's coefficient matrix is

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$$

and it has only one eigenvalue $r = 3$

- An eigenvector for A is $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and a generalized eigenvector is $\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (C_1 + C_2 t)\, e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Systems of ordinary differential equations

- First order linear systems can be used to *reduce the order* of a given differential equation

- Consider the second order equation

$$\ddot{y} + 2\dot{y} - 8y = 0$$

- Define two new variables (functions) $x_1 = y$ and $x_2 = \dot{y}$, and form the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_2 + 8x_1 \end{aligned}$$

- In words, we've just transformed a second order equation into an equivalent first order system of two equations. A solution of the system gives us also a solution of the initial differential equation

Systems of ordinary differential equations

- The system's coefficient matrix is

$$A = \begin{pmatrix} 0 & 1 \\ 8 & -2 \end{pmatrix}$$

- You can verify that A has two distinct eigenvalues $r_1 = -4$ and $r_2 = 2$, and the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- The general solution of the system is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^{-4t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- Thus the solution of the second order differential equation is

$$y(t) = x_1(t) = C_1 e^{-4t} + C_2 e^{2t}$$

Equilibria and stability

- As we did for first order equations, we want to examine the **stability** of systems of differential equations
- Let's consider the linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}$
- Let $\boldsymbol{x}^*$ be a steady state. We say that $\boldsymbol{x}^*$ is *globally asymptotically stable* if every solution $\boldsymbol{x}(t)$ of $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}$ converges to $\boldsymbol{x}^*$ as $t \to \infty$. Otherwise, we say that $\boldsymbol{x}^*$ is unstable

Proposition (Stability of linear systems)

Consider the linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}$ *and suppose* $\det A \neq 0$.

1. *If every real eigenvalue of A is negative and every complex eigenvalue of A has negative real part, then the steady state $\boldsymbol{x}^*$ is globally asymptotically stable.*
2. *If A has a positive real eigenvalue or a complex eigenvalue with positive real part, then $\boldsymbol{x}^*$ is an unstable equilibrium.*