Model Solutions 10

1. (a) The efficient amount of care maximizes the joint surplus of the consumer and the producer. The surpluses are:

$$CS(c) = V_{\text{gadget}} - p_{\text{gadget}} - c = 100 - p - c$$
$$PS(c) = p_{\text{gadget}} - C_{\text{gadget}} - C_{\text{gadget}} \times Pr(\text{broken gadget}) = p - 64 - 64 \times \frac{1}{c}$$
$$TS(c) = CS(c) + PS(c) = 36 - c - 64 \times \frac{1}{c}$$

Let's differentiate $TS(c)$ wrt. c to get the optimal amount of care:

$$\frac{\partial TS(c)}{\partial c} = -1 + \frac{64}{c^2} = 0$$
$$\implies c^* = 8$$

The efficient amount of care is €8.

(b) Since the consumer is fully insured against breaking the gadget, there is a problem of moral hazard and it is optimal for the consumer to expend zero euros worth of care. Thus, the probability of the gadget breaking down is 1/2. Let's solve for the firm's break-even price:

$$PS(c) = p - 64 - \frac{64}{2} = 0$$
$$\implies p^* = 96$$

The break-even price is €96.

(c) As in part 1a, we are maximizing the sum of consumer and producer surplus. The surpluses are:

$$CS(x, c) = 100 - p - c - x \times \frac{1}{c}$$
$$PS(c) = p - 64 - 64 \times \frac{1}{c}$$
$$TS(x, c) = 36 - c - 64 \times \frac{1}{c} - x \times \frac{1}{c}$$

The expression for $TS(x, c)$ can be simplified to a function that depends only on c, since whatever the hassle cost x, the consumer will choose c optimally so that the cost of care equals the expected hassle cost: $c = \frac{x}{c}$. Then, we can optimize:

$$TS(c) = 36 - 2c - \frac{64}{c}$$
$$\frac{\partial TS(c)}{\partial c} = -2 + \frac{64}{c^2} = 0$$
$$\implies c^* = \sqrt{32}$$

Since c $= \frac{x}{c}$ and $c^* = \sqrt{32}$, the welfare-maximizing hassle is $x = 32$. The lowest break-even price for gadgets is:

$$PS(c) = p - 64 - \frac{64}{\sqrt{32}} = 0$$
$$\implies p^* \approx €75.31$$

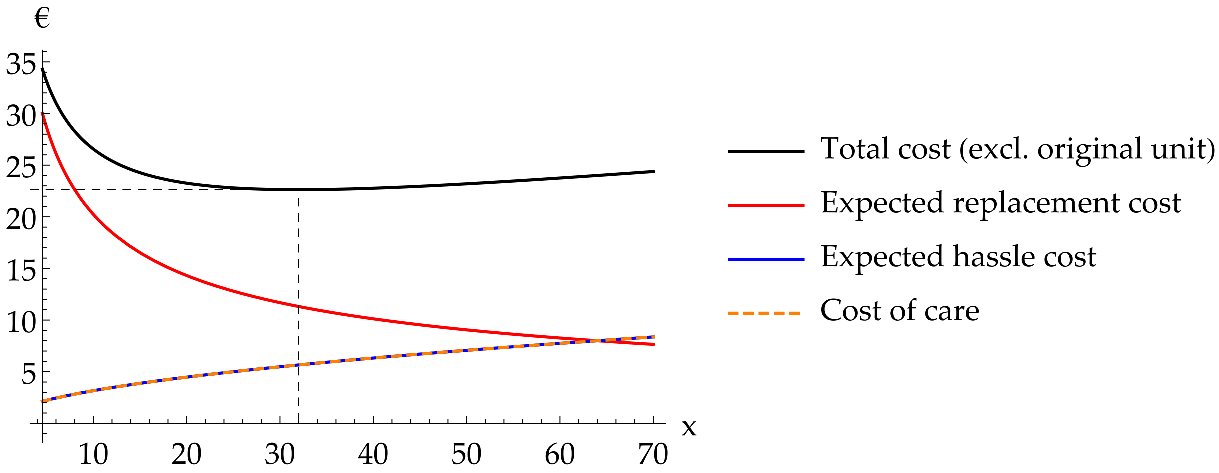

Figure 1: Firm's replacement costs and consumer's hassle and care costs as a function of hassle cost

2. (a) The efficient level of effort maximizes the output of Raymond's work minus possible costs to Raymond. With low effort, it is clearly best that Raymond works for the other company and makes €100k, since with low effort, the probability of sales is zero. With high effort, working for Öky-Alus, the expected value of Raymond's work is:

$$\text{EV}_{\text{high}} = 0.8 \times 1000 - 40 = €760\text{k}$$

Since this is more than the €100k that Raymond currently makes with low effort, high effort is economically efficient.

(b) Since Raymond is risk-neutral, he compares expected payoffs. The pay package needs to satisfy two criteria. Firstly, it must incentivize Raymond to exert high effort at work. Secondly, It must give a higher expected compensation to Raymond than the outside option of €100k. Let's first solve for the sales bonus that would make high effort optimal for Raymond:

$$\begin{aligned} \text{EV}_{\text{high}} &\geq \text{EV}_{\text{low}} \\ x + 0.8b - 40 &\geq x \\ b &\geq 50 \end{aligned}$$

The bonus needs to be at least €50k to incentivize high effort. Let's then solve for the smallest base wage that would make Raymond work for Öky-Alus:

$$\begin{aligned} x + 0.8 \times 50 - 40 &\geq 100 \\ x &\geq 100 \end{aligned}$$

The base wage needs to be at least €100k if bonus is €50k. There are many other combinations of base wage and bonus that would maximize the profits of Öky-Alus and incentivize Raymond to work for Öky-Alus, but $x = 100$ and $b = 50$ is the solution with the highest base wage. Clearly, expected profits are also above zero.

(c) Let's start by expressing Raymond's utility when he gets a bonus (v_2) and when he doesn't (v_1):

$$\begin{aligned} v_1 &= u(x + w_0) = (x + 116)^{2/3} \\ v_2 &= u(x + b + w_0) = (x + b + 116)^{2/3} \end{aligned}$$

Raymond is now risk averse, but the pay package still needs to satisfy the same two criteria as in part 2b. Let's use the expressions from above and formulate the conditions. 1.) The bonus needs to be high enough to incentivize high effort:

$$\begin{aligned} \text{EV}_{\text{high}} &\geq \text{EV}_{\text{low}} \\ 0.8v_2 + 0.2v_1 - 40 &\geq v_1 \\ v_2 &\geq v_1 + 50 \end{aligned}$$

2.) The overall payoff (with high effort) needs to be higher than at the other firm:

$$\begin{aligned} 0.8v_2 + 0.2v_1 - 40 &\geq (100 + 116)^{2/3} \\ 0.8v_2 + 0.2v_1 - 40 &\geq 36 \end{aligned}$$

Let's plug the v_2 solved from the first condition into the second condition and solve for the optimal base wage:

$$\begin{aligned} 0.8(v_1 + 50) + 0.2v_1 - 40 &\geq 36 \\ v_1 &\geq 36 \\ (x + 116)^{2/3} &\geq 36 \\ x &\geq 100 \end{aligned}$$

Thus, the optimal base wage is 100. Let's then plug this into the first first condition and solve for the optimal bonus:

$$\begin{aligned} v_2 &\geq v_1 + 50 \\ (100 + b + 116)^{2/3} &\geq 86 \\ 100 + b + 116 &\geq 797.53 \\ b &\geq 581.53 \end{aligned}$$

The optimal base wage is €100k and the optimal bonus €581.53k. Let's verify that expected profits are above zero:

$$E[\Pi(x = 100, b = 581.53)] = 0.8(1000 - 581.53) - 100 = 234.78$$

Expected profits are €234.78k, so this indeed is the profit-maximizing pay package.

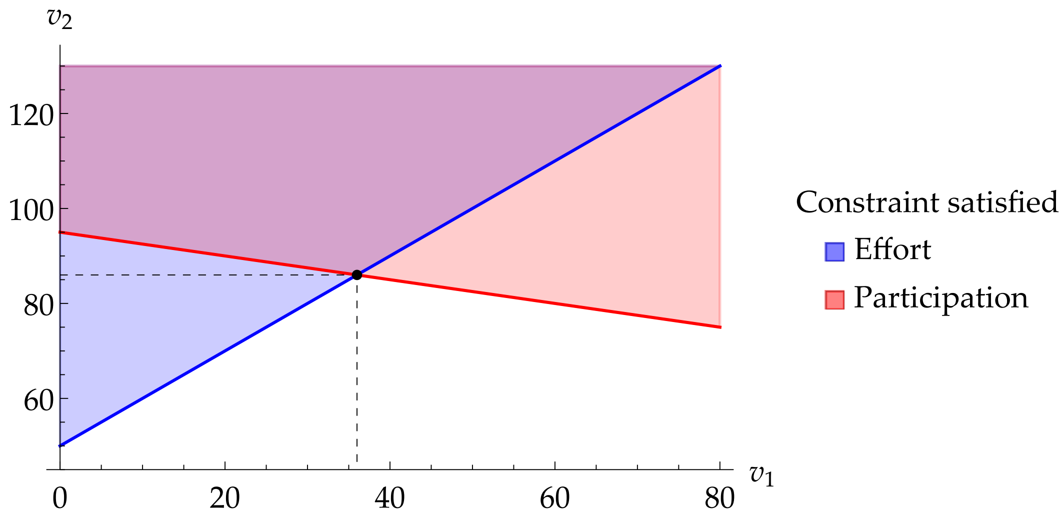

Figure 2: Participation (work for Öky-Alus) and effort (exert high effort) constraints of Raymond, in terms of the transformed variables v_1 and v_2.

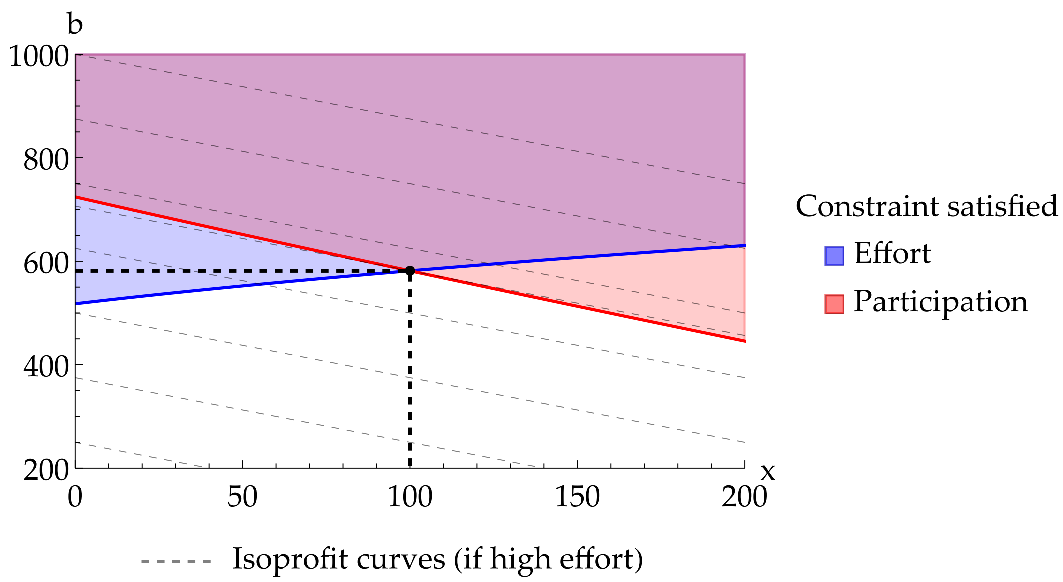

Figure 3: Participation and effort constraints in terms of base wage x and bonus b.

Additional comment. There is no need to check whether any other point that satisfies both constraints could give higher expected profits to the employer. The employer could offer a contract that is to the left on the participation constraint in Figure 3, while still eliciting high effort. However, this would involve exposing the worker to more risk (due to lower base wage, higher bonus) while giving the same expected utility—for which the risk averse worker has to be compensated with a risk premium. At the optimal point the employer is assuming as much risk as possible while both constraints are still satisfied for the worker.

3. (a) In a second-price procurement auction, the lowest bidder wins and gets the second-lowest bidder's price for completing the project. The dominant strategy is to bid your valuation. Thus, Asfaltti Oy should bid €3 billion for the project.

   To see why, consider first a case where Asfaltti Oy would bid below €3 billion. This would not increase its probability of winning if Raxa Group's bid is above €3 billion. And if Raxa Group's bid is below €3 billion and Asfaltti Oy won, it would make a loss. Bidding above €3 billion is not optimal either, since then Asfaltti Oy would lose some of the auctions that would have been profitable for it. Conditional on Asfaltti Oy winning the auction, bidding above €3 billion would also not result in more profit from the project, since the procurement price is defined by Raxa Group's losing bid.

   (b) Now the situation is trickier, since bidding your valuation is generally not the optimal strategy in a first-price auction. Asfaltti Oy must balance the probability of winning (increases with a lower bid) and the profit from the project, conditional on winning (increases with a higher bid). Asfaltti Oy knows that Raxa's bids are uniformly distributed between €1.25 billion and €5 billion. The expected profit of Asfaltti Oy is:

$$\begin{aligned} E[\pi_A(b)] &= \underbrace{\Pr(b \leq b_R)}_{\text{Prob. that bid under Raxa's}} \times \underbrace{(b-3)}_{\text{Profit, if win}} \\ &= (1 - F_R(b))(b-3) \\ &= (1 - \frac{b-1.25}{5-1.25})(b-3) \\ &= \frac{(5-b)(b-3)}{3.75} \end{aligned}$$

   The optimal bid is then:

$$\begin{aligned} \frac{\partial E[\pi_A(b)]}{\partial b} &= \frac{8-2b}{3.75} = 0 \\ &\implies b = 4 \end{aligned}$$

   The optimal bid for Asfaltti Oy is €4 billion.

   (c) In a second-price auction, finding out Raxa's exact cost would not benefit Asfaltti Oy, since knowing the costs would not alter Asfaltti's optimal bid. Asfaltti will win the auction and get the project at Raxa's bid if Raxa's bid is above €3 billion and lose the auction if Raxa's bid is below €3 billion.

In a first-price auction, the situation is different. With the information, Asfaltti will be able to win all auctions where winning is profitable (ie. Raxa's bid is above €3 billion) by bidding slightly below Raxas bid ($b = b_R - \epsilon$):

$$E[\pi_A(b = b_R - \epsilon)] = \underbrace{\Pr(b_R \geq 3)}_{\text{Prob. that Raxa's bid over €3B}} \times \underbrace{E[\pi_A(b = b_R - \epsilon)|b_R \geq 3]}_{\text{Expected profit with optimal bid}}$$

The probability that Raxa's bid is above €3B is $1 - F_R(3) = 1 - \frac{3-1.25}{5-1.25} = 53.33\%$. Since bids above €3B by Raxa are uniformly distributed, Raxa's expected bid, conditional on the bid being above €3B, is €4B. Thus, $E[\pi_A(b = b_R - \epsilon)] = 0.5333 \times (4 - 3) \approx 0.53$ billion euros.

To determine how much the information is worth, let's compare these profits to the expected profits without full information about Raxa's costs:

$$E[\pi_A(b)] = \frac{(5-b)(b-3)}{3.75}$$
$$\implies E[\pi_A(4)] = \frac{(5-4)(4-3)}{3.75} = \frac{1}{3.75} \approx 0.267$$

The value of the information is $0.533 - 0.267 \approx 0.27$ billion euros.

4. (a) Since the valuations are uniformly distributed, each valuation between 0 and 200 euros is equally likely for Hanne (buyer) and each valuation between 0 and 100 euros is equally likely for Jonne (seller).

   For trade to be efficient, buyer valuation needs to be at least as high as seller valuation. When buyer valuation is above 100, trade is always efficient. This happens 50% of the time. When buyer valuation is below 100, trade is efficient half the time. Thus, with $50\% \times 1 + 50\% \times 0.5 = 75\%$ probability, trade would be efficient.

   (b) When buyer makes the TIOLI offer, the expected profit function and optimal price is:

$$\pi_b(p) = (b-p)\Pr(s \leq p) = (b-p)p$$
$$\frac{\partial \pi_b(p)}{\partial p} = b - 2p = 0 \implies$$
$$p_b(b) = \frac{b}{2}$$

   Since buyer valuation b is uniformly distributed between 0 and 200 euros, the price $p_b(b)$ is uniformly distributed between 0 and 100 euros. And since seller valuation is also uniformly distributed between 0 and 100 euros, trade happens with 50% probability.

(c) When seller makes the TIOLI offer, the expected profit function and optimal price is:

$$\begin{aligned}\pi_s(p) &= (p-s)\Pr(b \geq p) = (p-s)(1-\frac{p-0}{200-0}) \\ &= (p-s)(\frac{200-p}{200}) \\ \frac{\partial \pi_s(p)}{\partial p} &= \frac{200-2p+s}{200} = 0 \implies \\ p_s(s) &= \frac{200+s}{2}\end{aligned}$$

Seller valuation is between 0 and 100 euros. When seller valuation is 0, the optimal price is $\frac{200+0}{2} = 100$, and trade occurs 50% (probability that buyer value is above 100) of the time. When seller valuation is 100, the optimal price is $\frac{200+100}{2} = 150$, and trade occurs 25% of the time. Since the price is uniformly distributed between 100 and 150 euros, trade occurs $\frac{50\%+25\%}{2} = 37.5\%$ of the time, when seller makes the TIOLI offer.

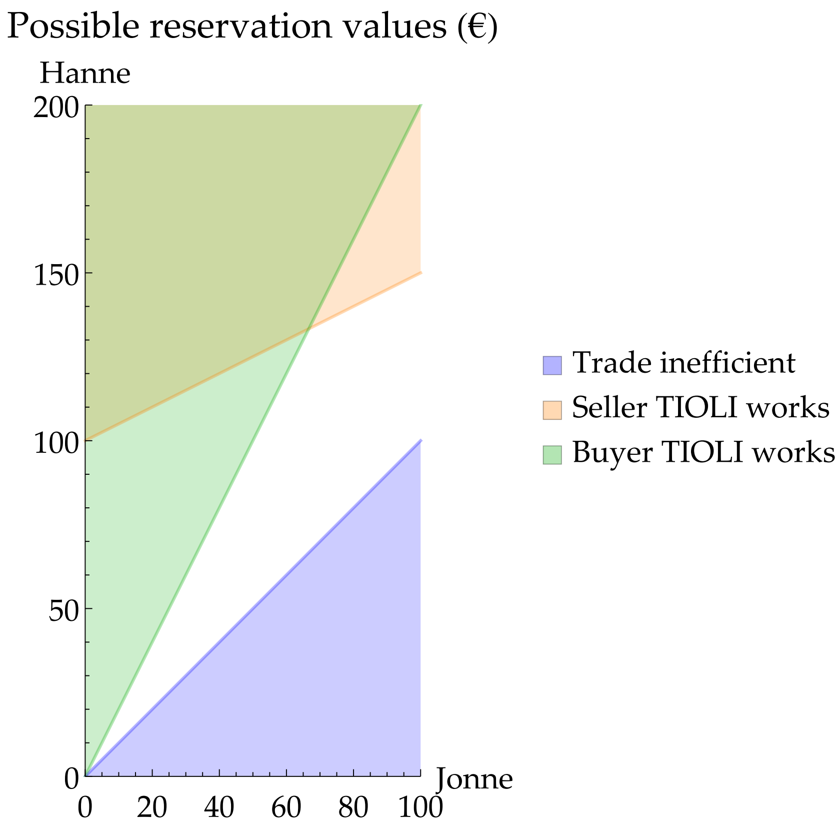

Figure 4: The green and orange lines show the optimal buyer and seller TIOLI prices as functions of buyer and seller valuations. The blue area shows where seller valuation is above buyer valuation, and thus trade is inefficient. In the white area, trade doesn't occur even though it would be efficient.