Aalto University
School of Electrical Engineering

ELEC-E8740 — Bootstrap Particle Filtering

Simo Särkkä

Aalto University

December 10, 2021

Contents

Intended Learning Outcomes

After this lecture, you will be able to:

- describe the basic idea of particle filtering,
- explain the three steps in particle filtering: simulation, weighting, resampling,
- identify the differences between Kalman filtering and particle filtering.

Recap: Extended Kalman Filter

- Model approximation:

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n \approx \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{F}_x(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \mathbf{q}_n$$
$$\mathbf{y}_n = \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n \approx \mathbf{g}(\hat{\mathbf{x}}_{n|n-1}) + \mathbf{G}_x(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}) + \mathbf{r}_n$$

- Prediction:

$$\begin{aligned} \hat{\mathbf{x}}_{n|n-1} &= \mathbf{f}(\hat{\mathbf{x}}_{n-1|n-1}), \\ \mathbf{P}_{n|n-1} &= \mathbf{F}_x \mathbf{P}_{n-1|n-1} \mathbf{F}_x^\mathsf{T} + \mathbf{Q}_n, \end{aligned}$$

- Measurement update:

$$\begin{aligned} \mathbf{K}_n &= \mathbf{P}_{n|n-1} \mathbf{G}_x^\mathsf{T} (\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^\mathsf{T} + \mathbf{R}_n)^{-1}, \\ \hat{\mathbf{x}}_{n|n} &= \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{y}_n - \mathbf{g}(\hat{\mathbf{x}}_{n|n-1})), \\ \mathbf{P}_{n|n} &= \mathbf{P}_{n|n-1} - \mathbf{K}_n(\mathbf{G}_x \mathbf{P}_{n|n-1} \mathbf{G}_x^\mathsf{T} + \mathbf{R}_n)\mathbf{K}_n^\mathsf{T}. \end{aligned}$$

Recap: Unscented Kalman Filter

- Uses a nonlinear transformation of deterministic sampling points
- Prediction:
  - Calculate the sigma-points using $\hat{\mathbf{x}}_{n-1|n-1}$ and $\mathbf{P}_{n-1|n-1}$
  - Propagate the sigma-points $\mathbf{x}_n^j = \mathbf{f}(\mathbf{x}_{n-1}^j)$
  - Calculate the mean and covariance $\hat{\mathbf{x}}_{n|n-1}$, $\mathbf{P}_{n|n-1}$
- Measurement update:
  - Calculate the sigma-points using $\hat{\mathbf{x}}_{n|n-1}$ and $\mathbf{P}_{n|n-1}$
  - Propagate the sigma-points $\mathbf{y}_n^j = \mathbf{g}(\mathbf{x}_n^j)$
  - Calculate the mean and covariance $\mathrm{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$, $\mathrm{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$, $\mathrm{Cov}\{\mathbf{x}_n, \mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}$
  - Perform the Kalman filter measurement update:

$$\begin{aligned}
\mathbf{K}_n &= \mathrm{Cov}\{\mathbf{x}_n, \mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} \, \mathrm{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}^{-1}, \\
\hat{\mathbf{x}}_{n|n} &= \hat{\mathbf{x}}_{n|n-1} + \mathbf{K}_n(\mathbf{y}_n - \mathrm{E}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\}), \\
\mathbf{P}_{n|n} &= \mathbf{P}_{n|n-1} - \mathbf{K}_n \, \mathrm{Cov}\{\mathbf{y}_n \mid \mathbf{y}_{1:n-1}\} \mathbf{K}_n^{\mathsf{T}}.
\end{aligned}$$

Discrete-Time Nonlinear State-Space Model

- Discrete-time nonlinear state-space model:

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$
$$\mathbf{y}_n = \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n$$

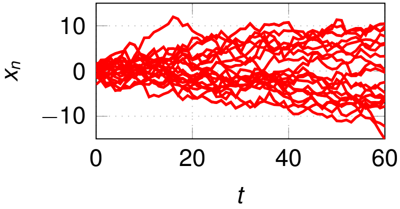
- Process noise: $\mathbf{q}_n \sim p(\mathbf{q}_n)$
- Measurement noise: $\mathbf{r}_n \sim p(\mathbf{r}_n)$
- Initial state: $\mathbf{x}_0 \sim p(\mathbf{x}_0)$

This is a stochastic process, each realization of the state sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is different

Example: Random Walk Process (1/2)

- Dynamic model:

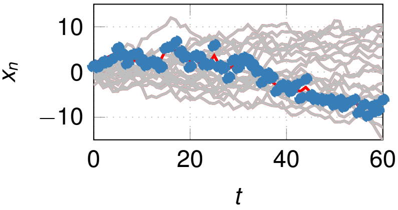
$$
\begin{aligned}
x_n &= x_{n-1} + q_n \\
x_0 &\sim \mathcal{N}(0, 1) \\
q_n &\sim \mathcal{N}(0, 1)
\end{aligned}
$$



Example: Random Walk Process (2/2)

- Only one realization of the process is observed
- Measurement model:

$$
\begin{aligned}
y_n &= x_n + r_n \\
r_n &\sim \mathcal{N}(0, 1)
\end{aligned}
$$



Particle Filtering: Idea

Prediction

- Given: Simulated states $\mathbf{x}_{n-1}^j$ $(j = 1, \ldots, J)$
- Simulate from t_{n-1} to t_n to obtain $\mathbf{x}_n^j$ $(j = 1, \ldots, J)$

Measurement Update

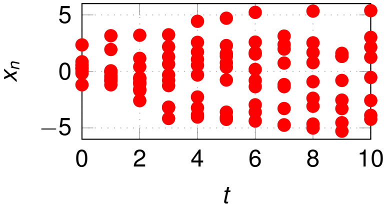
- Evaluate how well $\mathbf{x}_n^j$ explains $\mathbf{y}_n$ $(j = 1, \ldots, J)$
- Assign a weight w_n^j to $\mathbf{x}_n^j$ $(j = 1, \ldots, J)$

Prediction: Simulation

- Intuitive way: Use the dynamic model to simulate one time step

$$\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}) + \mathbf{q}_n$$

- Two step procedure:
  1. Sample $\mathbf{q}_n^j \sim p(\mathbf{q}_n)$,
  2. Calculate $\mathbf{x}_n^j = \mathbf{f}(\mathbf{x}_{n-1}^j) + \mathbf{q}_n^j$.



Measurement Update: Importance Weights

- Weights w_n^j indicate the relevance of each sample
- Importance weights:
  - High weight w_n^j: Explains $\mathbf{y}_n$ well
  - Low weight w_n^j: Explains $\mathbf{y}_n$ poorly
  - Should sum to one:

$$\sum_{j=1}^{J} w_n^j = 1,$$

- Cost function gives low values for good estimates of $\mathbf{x}_n$

Measurement Update: Likelihood (1/2)

- Measurement model:

$$\mathbf{y}_n = \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n$$
$$\mathbf{r}_n \sim p(\mathbf{r}_n)$$

- $\mathbf{r}_n$ is a random variable $\Rightarrow$ $\mathbf{y}_n$ is a random variable too
- $\mathbf{y}_n$ must have a probability density function (pdf)
- Given $\mathbf{x}_n$, the pdf for $\mathbf{y}_n$ is the same as for $\mathbf{r}_n$ but shifted by $\mathbf{g}(\mathbf{x}_n)$
- The pdf for $\mathbf{y}_n$ given $\mathbf{x}_n$ is called the likelihood

$$\mathbf{y}_n \sim p(\mathbf{y}_n \mid \mathbf{x}_n)$$

Measurement Update: Likelihood (2/2)

- The likelihood is a suitable measure for the importance weights w_n^j
- The non-normalized weights are then:

$$\tilde{w}_n^j = p(\mathbf{y}_n \mid \mathbf{x}_n^j).$$

- Normalization:

$$w_n^j = \frac{\tilde{w}_n^j}{\sum_{i=1}^J \tilde{w}_n^i}.$$

Example: Gaussian Likelihood (1/2)

- Measurement model:

$$\mathbf{y}_n = \mathbf{g}(\mathbf{x}_n) + \mathbf{r}_n$$

- The measurement noise is often (assumed) Gaussian:

$$p(\mathbf{r}_n) = \mathcal{N}(\mathbf{r}_n; 0, \mathbf{R}_n)$$

- Then, the likelihood is Gaussian too:

$$p(\mathbf{y}_n \mid \mathbf{x}_n) = \mathcal{N}(\mathbf{y}_n; \mathbf{g}(\mathbf{x}_n), \mathbf{R}_n).$$

Example: Gaussian Likelihood (2/2)

- Example: Scalar case with

$$
\begin{aligned}
y_n &= \mathbf{g}(x_n) + r_n \\
r_n &\sim \mathcal{N}(0, \sigma_r^2)
\end{aligned}
$$

Point Estimates

- Moments of the state can be calculated using weighted sums of the weighted samples
- Mean:

$$\hat{\mathbf{x}}_{n|n} = \sum_{j=1}^{J} w_n^j \mathbf{x}_n^j$$

- Covariance:

$$\mathbf{P}_{n|n} = \sum_{j=1}^{J} w_n^j (\mathbf{x}_n^j - \hat{\mathbf{x}}_{n|n})(\mathbf{x}_n^j - \hat{\mathbf{x}}_{n|n})^\mathsf{T}.$$

Summary: Sequential Sampling and Weighing

- Initialization: Sample J particles:

$$\mathbf{x}_0^j \sim p(\mathbf{x}_0)$$

- Prediction: Sample $\mathbf{q}_n^j$ and propagate particles:

$$\mathbf{q}_n^j \sim p(\mathbf{q}_n)$$
$$\mathbf{x}_n^j = \mathbf{f}(\mathbf{x}_n^j) + \mathbf{q}_n^j$$

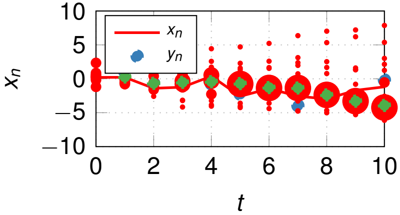
- Measurement update: Calculate and normalize the particle weights:

$$\tilde{w}_n^j = p(\mathbf{y}_n \mid \mathbf{x}_n^j)$$
$$w_n^j = \frac{w_n^j}{\sum_{i=1}^{J} \tilde{w}_n^i}$$

Example: Random Walk Process

- State-space model:

$$x_n = x_{n-1} + q_n$$
$$y_n = x_n + r_n$$



Resampling (1/2)

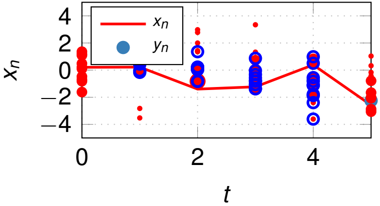
- Problem: The particles diverge after a few samples
- Resampling:
  - Remove samples with low weights
  - Replicate samples with high weights
  - Samples should be represented proportional to their weight:

$$\lfloor w_n^j J \rceil$$

- Equivalent interpretation

$$\Pr\{\tilde{\mathbf{x}}_n^i = \mathbf{x}_n^j\} = w_n^j,$$

Resampling (2/2)



Bootstrap Particle Filter

Algorithm 1 Bootstrap Particle Filter (Gaussian Noises)

Initialize: $\mathbf{x}_0^j \sim \mathcal{N}(\mathbf{m}_0, \mathbf{P}_0)$ $(j = 1, \ldots, J)$
for $n = 1, 2, \ldots$ **do**
  for $j = 1, 2, \ldots, J$ **do**
    Sample: $\mathbf{q}_n^j \sim \mathcal{N}(0, \mathbf{Q})$
    Propagate the state: $\mathbf{x}_n^j = \mathbf{f}(\mathbf{x}_{n-1}^j) + \mathbf{q}_n^j$
    Calculate the weights: $\tilde{w}_n^j = \mathcal{N}(\mathbf{y}_n; \mathbf{g}(\mathbf{x}_n^j), \mathbf{R}_n)$
  end for
  Normalize the importance weights $(j = 1, \ldots, J)$

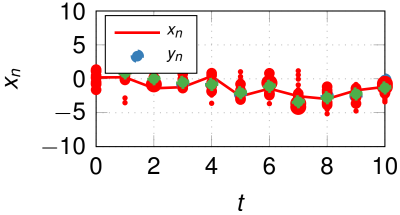
$$w_n^j = \frac{\tilde{w}_n^j}{\sum_{i=1}^J \tilde{w}_n^i}$$

  Calculate the mean $\hat{\mathbf{x}}_{n|n}$ and covariance $\mathbf{P}_{n|n}$
  Resample such that $\Pr\{\tilde{\mathbf{x}}_n^i = \mathbf{x}_n^j\} = w_n^j$
end for

Example: Random Walk

- State-space model:

$$x_n = x_{n-1} + q_n$$
$$y_n = x_n + r_n$$



Example: Object Tracking (1/3)

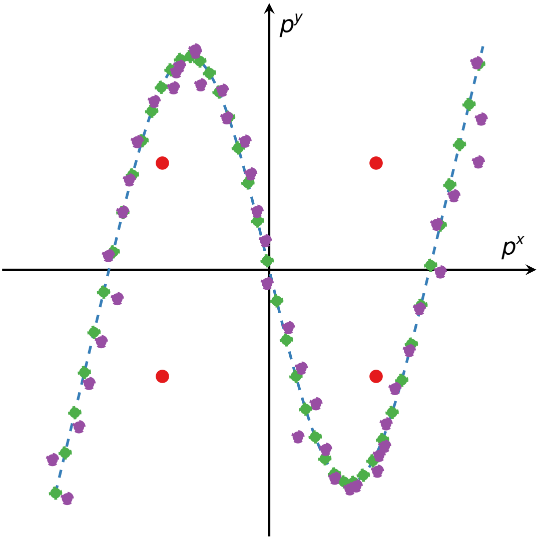
- Quasi-constant turn model:

$$\begin{bmatrix} \dot{p}^x(t) \\ \dot{p}^y(t) \\ \dot{v}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} v(t)\cos(\varphi(t)) \\ v(t)\sin(\varphi(t)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{w}(t)$$

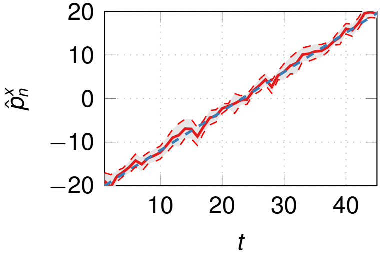
- Range (distance) measurements:

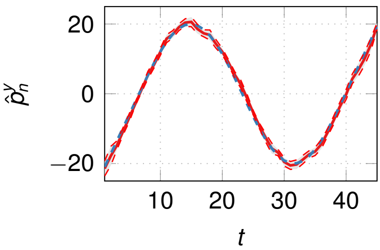
$$\mathbf{y}_n = \begin{bmatrix} |\mathbf{p}_n - \mathbf{p}_1^s| \\ |\mathbf{p}_n - \mathbf{p}_2^s| \\ \vdots \\ |\mathbf{p}_n - \mathbf{p}_K^s| \end{bmatrix} + \mathbf{r}_n$$

Example: Object Tracking (2/3)



Example: Object Tracking (3/3)





Summary

- The particle filter uses a set of random samples to estimate the state
- During prediction, the samples are simulated from t_{n-1} to t_n
- The bootstrap particle filter uses the dynamic model to simulate the samples
- The measurement update evaluates the likelihood to assign an importance weight to each sample
- Resampling is used to mitigate particle degeneracy
- Particle filtering is a universal approach equally applicable to linear and nonlinear system
- It can be shown that particle filters are asymptotically ($J \to \infty$) optimal in many cases