Differential and Integral Calculus 1 - MS-A0111
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Solutions to the exam
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PROBLEM 1 Compute the following limits:

$$\text{(a)} \quad \lim_{x\to 0} \frac{(\cos x)^2 + x^2 - 1}{x^4} \qquad \qquad \text{(b)} \quad \lim_{x\to 0} \frac{e^{3x} - \sin(3x) - 1}{\ln(1-2x)}.$$

SOLUTION.
(a) Let us use the Taylor formulas provided on the exam sheet. For $x \to 0$, we may write

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6),$$

so that

$$\begin{aligned}
(\cos x)^2 &= \big(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\big)^2 \\
&= 1 - x^2 + 2\frac{x^4}{24} + \frac{x^4}{4} + O(x^6) \\
&= 1 - x^2 + \frac{x^4}{3} + O(x^6),
\end{aligned}$$

where in the last two steps we collected as a single $O(x^6)$ all the terms that go to zero at least as fast as x^6. When doing this in the given limit, we get

$$\begin{aligned}
\lim_{x\to 0} \frac{(\cos x)^2 + x^2 - 1}{x^4} &= \lim_{x\to 0} \frac{1 - x^2 + \frac{x^4}{3} + O(x^6) + x^2 - 1}{x^4} \\
&= \lim_{x\to 0} \frac{\frac{x^4}{3} + O(x^6)}{x^4} \\
&= 1/3.
\end{aligned}$$

(b) By plugging $3x$ instead of x in the formulas on the exam sheet, we get

$$e^{3x} = 1 + 3x + \frac{(3x)^2}{2} + O(x^3) \quad \text{and} \quad \sin(3x) = 3x - O(x^3),$$

as $x \to 0$. Similarly, with $-2x$ instead of x we have

$$\ln(1-2x) = -2x + O(x^2),$$

as $x \to 0$. Thus for the given limit we get

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{3x} - \sin(3x) - 1}{\ln(1-2x)} &= \lim_{x\to 0} \frac{1 + 3x + \frac{(3x)^2}{2} + O(x^3) - (3x - O(x^3)) - 1}{-2x + O(x^2)} \\
&= \lim_{x\to 0} \frac{\frac{9}{2}x^2 + O(x^3)}{-2x + O(x^2)} \\
&= \lim_{x\to 0} \frac{\frac{9}{2}x + O(x^2)}{-2 + O(x)}.
\end{aligned}$$

For $x \to 0$, the numerator tends to zero and the denominator to -2, hence the limit is equal to zero.

COMMENTS. There were other ways of computing these limits, for instance by remembering some famous ones or using L'Hopital's rule. In the approach with Taylor polynomials, a recurring mistake was in taking the square in the first limit. Remember that $O(x^n)$ is $O(x^6)$ for any $n \geq 6$, and that the sum of several functions all of which are $O(x^6)$ is itself $O(x^6)$. So when you have something like

$$O(x^6) + O(x^8) + O(x^9),$$

you can simply write it all as $O(x^6)$.

PROBLEM 2 Consider the function $f(x) = (\sin(\cos x))^2 + (\cos(\cos(x)))^2$.

a) Compute the derivative of f using only famous derivatives and differentiation rules, without using trigonometric formulas.

b) What does this tell you about the function f? What is its value?

SOLUTION.
(a) By the Chain Rule (and by the fact that the derivative of a sum of functions is the sum of the respective derivatives), we find

$$\begin{aligned} f'(x) &= -2\sin(x)\cos(\cos(x))\sin(\cos(x)) + 2\sin(x)\cos(\cos(x))\sin(\cos(x)) \\ &= 0. \end{aligned}$$

(b) The fact that the derivative of a continuous function is zero tells us that the function is constant. The value is then the same for any input number, so if we pick for instance $x = \pi/2$, which gives $\cos(\pi/2) = 0$, we find

$$\begin{aligned} f(\pi/2) &= \sin(0)^2 + \cos(0)^2 \\ &= 0^2 + 1^2 \\ &= 1. \end{aligned}$$

PROBLEM 3 Compute the integrals

$$\text{(a)} \quad \int x^2 \sin x \, dx \qquad\qquad \text{(b)} \quad \int_0^{\sqrt{8}} \frac{x^3}{\sqrt{x^2+1}} \, dx.$$

SOLUTION.
(a) We may differentiate by parts twice:

$$\begin{aligned}
\int x^2 \sin x \, dx &= -x^2 \cos x + \int 2x \cos x \, dx \\
&= -x^2 \cos x + \left[2x \sin x - \int 2 \sin x \, dx \right] \\
&= -x^2 \cos x + 2x \sin x + 2 \cos x + C,
\end{aligned}$$

for $C \in \mathbb{R}$.
(b) Notice that in the interval $[0, \sqrt{8}]$ the functions we consider are non-negative. So it makes sense to take only non-negative results of a square root. We perform the substitution $u = \sqrt{x^2+1}$, which yields $du = \frac{2x}{\sqrt{x^2+1}} dx$, and $x^2 = u^2 - 1$, hence

$$\begin{aligned}
\int_0^{\sqrt{8}} \frac{x^3}{\sqrt{x^2+1}} \, dx &= \int_0^{\sqrt{8}} \frac{x^2}{2} \frac{2x}{\sqrt{x^2+1}} \, dx \\
&= \int_1^3 \frac{u^2-1}{2} \, du \\
&= \frac{1}{2} \left(\frac{u^3}{3} - u \right) \Big|_1^3 \\
&= 20/3.
\end{aligned}$$

PROBLEM 4 Find all the solutions to $y' + 2y = 3$.

SOLUTION. There are at least four ways of solving this problem, based on what was shown in the course: recall the formula for the general solution and take it for granted; use the method of the integrating factor; use the variation of parameter; observe that this is a separable equation. Denote $\mu(x) = 2x$, an antiderivative of the function $p(x) = 2$. And denote $q(x) = 3$. Here we just use the general formula and separate the variables.

a) *Take the general formula for granted.* We saw in two different ways that the general solution for such a DE is

$$y(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x)\, dx,$$

which in this case gives

$$\begin{aligned} y(x) &= e^{-\mu(x)} \int e^{\mu(x)} q(x)\, dx \\ &= e^{-2x} \int e^{2x} \cdot 3\, dx \\ &= e^{-2x} \frac{3}{2}(e^{2x} + C) \qquad (*) \\ &= \frac{3}{2} + Ce^{-2x}, \end{aligned}$$

where C can be any real number.

b) *Separating the variables.* You can also observe that this is a separable equation: you may write $\frac{dy}{dx} + 2y = 3$ and separate

$$\frac{1}{3 - 2y}\, dy = dx.$$

Taking the integral on both sides yields

$$-\ln|3 - 2y| = x + C,$$

so that $|3 - 2y| = e^{-x+C}$, which means that $3 - 2y = \pm e^{-x+C} = \pm e^{-x} e^{C} = C' e^{-x}$, where we simply rename $C' = \pm e^{C}$, which can be any nonzero real number, but in case $C' = 0$ we still get a solution. So to conclude

$$y = \frac{1}{2}(3 + C' e^{-x}).$$

COMMENTS. The most popular solutions were by far separating the variables or recalling the general formula. The most frequent mistakes:

- forgetting the absolute value, or the $\pm$
- recalling the wrong formula, for instance with wrong operations, like a sum instead of a product: $y(x) = e^{-\mu(x)} + \int e^{\mu(x)} q(x)\, dx$, or also $y(x) = e^{-\mu(x)} + \int (e^{\mu(x)} + q(x))\, dx$
- in step $(*)$ in method a), many forgot the integrating constant C. This gives just one single solution for our DE, although there is an infinite amount of solutions.
- In the step following $(*)$ in method a), many forgot to multiply the constant C by e^{-2x}, which resulted in solutions of the form $\frac{3}{2} + C$.

NB: In general, you can always check whether you got some right solutions for an equation: you plug them inside it, and you see if they satisfy it!

PROBLEM 5 Consider the function $f(x) = \frac{\sin(2x)}{16}$.

a) Compute the third Taylor polynomial P_3 for f about $a = \pi/2$.

b) If you approximate $f(\frac{\pi}{2} + 1)$ with $P_3(\frac{\pi}{2} + 1)$, is the error smaller than $\frac{1}{20}$? Explain why.

SOLUTION.
(a) The third Taylor polynomial is

$$
\begin{aligned}
P_3(x) &= \sum_{i=0}^{3} \frac{f^{(k)}(\pi/2)}{k!}(x - \pi/2)^k \\
&= \frac{f^{(0)}(\pi/2)}{0!}(x - \pi/2)^0 + \frac{f^{(1)}(\pi/2)}{1!}(x - \pi/2)^1 + \frac{f^{(2)}(\pi/2)}{2!}(x - \pi/2)^2 \\
&\qquad + \frac{f^{(3)}(\pi/2)}{3!}(x - \pi/2)^3.
\end{aligned}
$$

The derivatives of the given function are

$$
f^{(0)} = f \qquad f'(x) = \frac{\cos(2x)}{8} \qquad f''(x) = -\frac{\sin(2x)}{4} \qquad f'''(x) = -\frac{\cos(2x)}{2}.
$$

Taking their values at $\pi/2$, as written in the formulas on the exam sheet, one gets

$$
f(\pi/2) = 0 \qquad f'(\pi/2) = -\frac{1}{8} \qquad f''(x) = 0 \qquad f'''(x) = \frac{1}{2},
$$

so that the Taylor polynomial is

$$
P_3(x) = -\frac{1}{8}(x - \pi/2) + \frac{1}{2 \cdot 3!}(x - \pi/2)^3.
$$

(b) One could solve this part by recalling the error formula, which says that for any given x, the error is equal to

$$
f(x) - P_3(x) = \frac{f^{(4)}(s)}{4!}(x - \pi/2)^4,
$$

where s is some number between $\pi/2$ and x. Since the x we are interested in is $x = \frac{\pi}{2} + 1$, in this case the formula says that the error is equal to

$$
\frac{f^{(4)}(s)}{4!} \cdot (\pi/2 + 1 - \pi/2)^4 = \frac{f^{(4)}(s)}{4!} = \frac{f^{(4)}(s)}{24},
$$

where s is some number between $\frac{\pi}{2}$ and $\frac{\pi}{2}+1$. The fourth derivative of f is $f^{(4)}(x) = \sin(x)$, which means that for any possible value of s, we have $-1 \leq f^{(4)}(s) \leq 1$, but then we also have

$$-\frac{1}{20} \leq -\frac{1}{24} \leq \frac{f^{(4)}(s)}{24} \leq \frac{1}{24} \leq \frac{1}{20}.$$

So the answer is yes.

Alternatively, without using the error formula, one could simply remember that the error for value x is defined as $f(x) - P_3(x)$, and write this explicitly for $x = \frac{\pi}{2} + 1$:

$$\begin{aligned} f(\frac{\pi}{2}+1) - P_3(\frac{\pi}{2}+1) &= \frac{\sin(\pi+2)}{16} + \frac{1}{8} - \frac{1}{2 \cdot 3!} \\ &= (1/4)\frac{3\sin(\pi+2)+6-4}{12} \\ &= \frac{3\sin(\pi+2)+2}{48}. \end{aligned}$$

So one could show directly that the error is smaller than $1/20$ by showing that it is smaller than $1/24$, which is the same as showing that $-2 \leq 3\sin(\pi+2)+2 \leq 2$. This is true because $\pi+2$ is between π and 2π, where the value of the sine is between -1 and 0.

PROBLEM 6 Consider the function $f(x) = \frac{e^x - 2}{1 - e^x}$.

a) For what values of x is the function defined? Compute the limits

$$\lim_{x \to -\infty} f(x), \qquad \lim_{x \to +\infty} f(x),$$

and at the points a where f is not defined, compute

$$\lim_{x \to a^-} f(x), \qquad \lim_{x \to a^+} f(x).$$

b) Compute the first derivative of f and study its sign: where is it positive, negative, zero? Where does f increase/decrease?

c) Compute the second derivative of f and study its sign: where is it positive, negative, zero? Where is f convex (happy)/concave (sad)?

d) Use the information above to sketch the graph of f.

SOLUTION.
(a) The function is defined when the denominator is different from zero, that is, for $x \neq 0$. The limits at infinity are

$$\lim_{x \to -\infty} \frac{e^x - 2}{1 - e^x} = \frac{-2}{1} = -2,$$

since $\lim_{x \to -\infty} e^x = 0$, and

$$\lim_{x \to -\infty} \frac{e^x - 2}{1 - e^x} = \lim_{x \to \infty} \frac{e^x(1 - \frac{2}{e^x})}{e^x(\frac{1}{e^x} - 1)} = \lim_{x \to \infty} \frac{1 - \frac{2}{e^x}}{\frac{1}{e^x} - 1} = \frac{1}{-1} = -1,$$

since $\lim_{x \to +\infty} e^x = +\infty$. For the limits at 0, observe that for $x > 0$ the value of $1 - e^x$ is slightly less than zero, so that it has negative sign, and for $x < 0$ the value of $1 - e^x$ is slightly greater than zero. But for x very close to zero, regardless of the side, $e^x - 2$ is close to -1, so its sign is negative. Hence we get

$$\lim_{x \to 0^-} \frac{e^x - 2}{1 - e^x} = -\infty \qquad \text{and} \qquad \lim_{x \to 0^+} \frac{e^x - 2}{1 - e^x} = +\infty.$$

(b) The derivative is

$$\begin{aligned} f'(x) &= \frac{e^x(1 - e^x) - (e^x - 2)(-e^x)}{(1 - e^x)^2} \\ &= -\frac{e^x}{(1 - e^x)^2}. \end{aligned}$$

Both numerator and denominator are positive, wherever f is defined. So the sign is always negative because of the minus sign in front. This means that f is always decreasing.
(c) The second derivative is

$$\begin{aligned} f''(x) &= -\frac{e^x(1-e^x)^2 - e^x 2(1-e^x)(-e^x)}{(1-e^x)^4} \\ &= -\frac{e^x(1-2e^x+e^{2x}+2e^x-2e^{2x})}{1-e^x)^4} \\ &= -\frac{e^x(1-e^x)(1+e^x)}{(1-e^x)^4} \\ &= -\frac{e^x(1+e^x)}{(1-e^x)^3} \\ &= \frac{e^x(1+e^x)}{(e^x-1)^3}. \end{aligned}$$

The numerator is always positive, and the denominator changes sign when $e^x - 1$ does, which happens around $x = 0$. For $x > 0$, one has $f''(x) > 0$, and instead $f''(x) < 0$ for $x < 0$. This means that for $x > 0$ the graph is happy and for $x < 0$ the graph is sad.
(d) Based on all of the above, the following is a sketch of the function's graph: