31C01100 - Taloustieteen matemaattiset menetelmät - Mathematics for Economists
Aalto University - Fall 2021

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Final Exam, 16.12.2021
Suggested Answers

Instructions:

- The exam has 4 questions. You must answer all parts of all questions
- Please write as clearly as you can
- Explain your reasoning. You don't have to give lengthy explanations for what you do, but I should be able to see how you justify the key steps in your answers

Question 1 [28 points]

Consider the following constrained maximization problem:

$$\max_{x,y} \quad x^2 - ay$$
$$\text{s.t.} \quad 4x^2 + 4y^2 = a^2,$$

where $a > 0$ is a parameter. Note that a solution to this problem exists by Weierstrass's Theorem.

1. [5 points] Show that the non-degenerate constraint qualification (NDCQ) is satisfied at every point of the constraint set.
2. [7 points] Form the Lagrangian function and find all its critical points.
3. [10 points] Find all the solutions of this maximization problem. [*Hint*: You don't need to check second-order conditions.]
4. [6 points] Suppose a is changed to $a + \epsilon$, where $\epsilon > 0$. Use the *envelope theorem* to estimate the corresponding change in the problem's value function.

Solution

Call $h(x, y) := 4x^2 + 4y^2$. We have that $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0$ only at the point $(0, 0)$. Since $(0, 0)$ does not belong to the constraint set, the NDCQ is satisfied. The Lagrangian is

$$L = x^2 - ay - \lambda\left(4x^2 + 4y^2 - a^2\right).$$

The critical points of L are found by solving the following system:

$$2x - 8\lambda x = 0 \quad (1)$$
$$-a - 8\lambda y = 0 \quad (2)$$
$$4x^2 + 4y^2 - a^2 = 0. \quad (3)$$

It follows from (1) that $x = 0$ or $\lambda = \frac{1}{4}$. Suppose $x = 0$. It follows from (3) that $y = \frac{a}{2}$ or $y = -\frac{a}{2}$. If $y = \frac{a}{2}$, then $\lambda = -\frac{1}{4}$ by (2). Similarly, if $y = -\frac{a}{2}$, then $\lambda = \frac{1}{4}$ by (2). Now suppose $\lambda = \frac{1}{4}$. It follows from (2) that $y = -\frac{a}{2}$, which implies $x = 0$ by (3). In sum, there are two critical points:

$$\left(0, \frac{a}{2}, -\frac{1}{4}\right) \quad \text{and} \quad \left(0, -\frac{a}{2}, \frac{1}{4}\right).$$

Since we know that a solution exists, it must be a critical point of the Lagrangian. Evaluating the objective function $f(x, y) := x^2 - ay$ at the two critical points yields

$$f\left(0, \frac{a}{2}\right) = -\frac{a^2}{2} < \frac{a^2}{2} = f\left(0, -\frac{a}{2}\right),$$

from which we conclude that $\left(0, -\frac{a}{2}\right)$ is the unique solution.
Finally, the change in the parameter a is $da = \epsilon$. By the envelope theorem,

$$\begin{aligned} df(x^*, y^*) &= \frac{\partial L}{\partial a}(x^*, y^*, \lambda^*)da \\ &= (-y^* + 2a\lambda^*)\, da \\ &= \left(\frac{a}{2} + \frac{a}{2}\right)\epsilon \\ &= a\epsilon. \end{aligned}$$

Question 2 [22 points]

Consider the following constrained maximization problem:

$$\begin{aligned} \max_{x,y} \quad & -x^2 - y^2 + xy + 7 \\ \text{s.t.} \quad & 4x + 9y \leq 0 \qquad (4) \\ & x \leq 4 \qquad (5) \end{aligned}$$

Note that the problem's objective function is concave and both constraints are linear.

1. [7 points] Form the Lagrangian function and write all the first order conditions.
2. [15 points] Find all the solutions of this maximization problem. [*Hint*: A possible way to proceed is the following. First, show that (5) must be slack. Then, show that (4) must be binding.]

Solution

Since the objective function is concave and the constraints linear, first order conditions are necessary and sufficient to identify all the solutions to the maximization problem. The Lagrangian is

$$L = -x^2 - y^2 + xy + 7 - \mu_1 (4x + 9y) - \mu_2(x - 4).$$

The first order conditions are:

$$-2x + y - 4\mu_1 - \mu_2 = 0 \qquad (6)$$
$$-2y + x - 9\mu_1 = 0 \qquad (7)$$
$$\mu_1(4x + 9y) = 0 \qquad (8)$$
$$\mu_2(x - 4) = 0 \qquad (9)$$
$$4x + 9y \leq 0 \qquad (10)$$
$$x \leq 4 \qquad (11)$$
$$\mu_1, \mu_2 \geq 0. \qquad (12)$$

Suppose (5) is binding, i.e. $x = 4$. It follows from (10) that $y \leq -\frac{16}{9} < 0$. But (6) implies $y = 8 + 4\mu_1 + \mu_2 > 0$, so leading to a contradiction. Thus (5) must be slack and $\mu_2 = 0$. Now suppose that (4) is slack. By (8), $\mu_1 = 0$. But (6) and (7) imply $x = y = 0$, so contradicting the assumption that (4) is slack. Hence (4) must be binding. Now, (6), (7) and the binding constraint (4) form a system of three linear equations in three unknowns, which solves for $x = y = \mu_1 = 0$. Therefore, we can conclude that the unique solution to this maximization problem is $(0, 0)$.

Question 3 [33 points]

Consider the following matrix:

$$A = \begin{pmatrix} 15 & -4 \\ 9 & 3 \end{pmatrix}$$

1. [6 points] Find all the eigenvalues of A and determine their multiplicity.
2. [8 points] If A has two distinct eigenvalues, find two linearly independent eigenvectors. Alternatively, if A has only one repeated eigenvalue, find an eigenvector and a generalized eigenvector.
3. [5 points] Form the *general* solution of the following system of linear *difference* equations:

   $$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 15 & -4 \\ 9 & 3 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \quad t = 0, 1, 2, \ldots \tag{13}$$

   Note that the system's coefficient matrix is the same matrix A as before.
4. [4 points] Is $(x^*, y^*) = (0, 0)$ the only steady state of system (13)? If so, explain why. If not, find all the steady states of the system.
5. [3 points] Is the steady state $(x^*, y^*) = (0, 0)$ globally asymptotically stable? Explain why or why not.
6. [7 points] Find the *particular* solution of the following linear system of *differential* equations:

   $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 15 & -4 \\ 9 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

   with initial conditions $x(0) = 4$ and $y(0) = 2$. Note that the system's coefficient matrix is the same matrix A as before.

Solution

A has a unique eigenvalue $r = 9$ of multiplicity 2. An eigenvector is any non-zero solution of

$$\begin{pmatrix} 15-9 & -4 \\ 9 & 3-9 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We can choose $\boldsymbol{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. A generalized eigenvector is any solution of

$$\begin{pmatrix} 15-9 & -4 \\ 9 & 3-9 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

We can choose $\boldsymbol{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
The general solution of (13) is

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \left(C_1 9^t + tC_2 9^{t-1}\right) \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 9^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The steady state $(x^*, y^*) = (0, 0)$ is unique because $\det(I - A) = 64 \neq 0$. However, the steady state is not stable because $|r| = 9 \not< 1$.
The general solution of the system of differential equations is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (C_1 + C_2 t)\, e^{9t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{9t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The particular solution is found by solving

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

from which we get $C_1 = -2$ and $C_2 = 8$.

Question 4 [17 points]

Consider the following autonomous first order differential equation:

$$\dot{y} = y^2 - 7ay,$$

where $a \neq 0$ is a parameter.

1. [4 points] This differential equation has two equilibria (or steady states), one of which is $y_1^* = 0$. Find the other equilibrium y_2^*.
2. [6 points] Find all values of $a \neq 0$ such that $y_1^* = 0$ is locally asymptotically stable. [*Hint*: Think what happens when $a > 0$ and when $a < 0$.]
3. [4 points] Consider the case in the previous question where $y_1^* = 0$ is locally asymptotically stable. Is the other equilibrium y_2^* stable or unstable? Explain.
4. [3 points] Now suppose that $a = 0$, so that $y_1^* = 0$ is the unique equilibrium. Is $y_1^* = 0$ locally asymptotically stable in this case? Explain why or why not.

Solution

The two equilibria are $y_1^* = 0$ and $y_2^* = 7a$. Let $f(y) := y^2 - 7ay$. Then $f'(y) = 2y - 7a$. Consequently, $f'(0) = -7a$. Therefore, y_1^* is locally asymptotically stable when $a > 0$ (so that $f'(0) < 0$), and unstable when $a < 0$. When $a > 0$, $f'(7a) = 14a - 7a = 7a > 0$. Hence y_2^* is unstable. Finally, when $a = 0$, $f'(0) = 0$, so the test with derivatives is inconclusive. By sketching a phase portrait, one can verify that $y_1^* = 0$ is unstable in this case.