An introduction to magnetohydrodynamics (MHD)

Omar Maj

Max Planck Institute for Plasma Physics, Garching bei München, Germany.

IPP Summer University,
Greifswald, September 19th, 2016

Thanks to Prof. Marino Bornatici (University of Pavia, Italy), Bruce D. Scott, Roberto Bilato and Emanuele Poli (IPP).

Plan of the lectures

- LECTURE I: MHD equations.
  1. Introduction to basic fluid dynamics.
  2. From two-fluid plasma models to MHD equations.

- LECTURE II: Character of MHD.
  1. Conservation laws.
  2. Basic processes in MHD: magnetic field diffusion, waves.

LECTURE I: MHD equations

1. Basic fluid dynamics:
   Mass, momentum and energy conservation laws.
2. Two-fluid description of plasmas:
   Each particle species is described as an electrically charged fluid.
3. Quasi-neutral/low-frequency limit:
   At low frequency, charge separation in suppressed and plasmas are quasi-neutral.
4. MHD equations:
   A single electrically conducting neutral fluid.

Definition of the basic physical quantities of fluid dynamics

Let W be a arbitrary (measurable) subset of the physical domain occupied by the fluid.

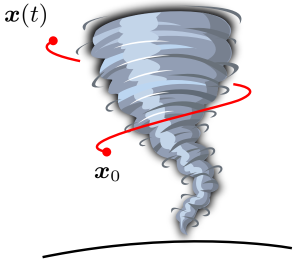
- **MASS DENSITY**: The mass per unit of volume $\rho(t, \boldsymbol{x}) > 0$,
$$\text{mass in } W = \int_W \rho(t, \boldsymbol{x}) dx.$$
- **FLUID VELOCITY**: The velocity $\boldsymbol{u}(t, \boldsymbol{x})$ of the fluid element $\rho(t, \boldsymbol{x})dx$,
$$\text{momentum in } W = \int_W \rho(t, \boldsymbol{x})\boldsymbol{u}(t, \boldsymbol{x}) dx.$$
- **TEMPERATURE**: A scalar $T(t, \boldsymbol{x}) \geq 0$ for the internal energy of a fluid element,
$$\text{internal energy in } W = \int_W \big(\frac{3}{2} n(t, \boldsymbol{x}) k_B T(t, \boldsymbol{x})\big) dx,$$
where $n(t, \boldsymbol{x}) = \rho(t, \boldsymbol{x})/m$ and m is the mass of fluids atoms/molecules.

The flow of a vector field

- Lagrangian trajectories of a fluid element:

$$\begin{cases} \dfrac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{u}\big(t, \boldsymbol{x}(t)\big), \\ \boldsymbol{x}(0) = \boldsymbol{x}_0. \end{cases}$$

  The point $\boldsymbol{x}(t)$ moves at the same velocity of the fluid, thus representing the fluid motion.

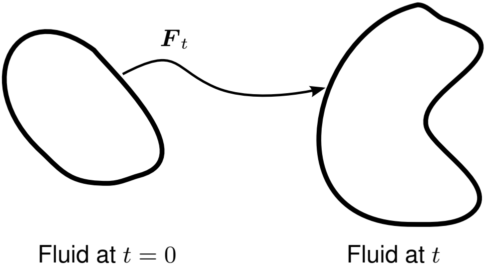


- By (ideally) computing the Lagrangian trajectories for every initial condition, we can construct a one-parameter family of maps

$$\boldsymbol{x}_0 \mapsto \boldsymbol{F}_t(\boldsymbol{x}_0) = \boldsymbol{x}(t), \qquad t \geq 0.$$

  The family of maps $\boldsymbol{F}_t$ is referred to as the *flow of the vector field* $\boldsymbol{u}(t, \boldsymbol{x})$.

The flow of a vector field - cartoon



The flow of a vector field - Basic mathematical properties

- For every given point $\boldsymbol{x}_0$, $\boldsymbol{F}_t(\boldsymbol{x}_0)$ is a Lagrangian trajectory, hence,

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{F}_t(\boldsymbol{x}_0) = \boldsymbol{u}\big(t, \boldsymbol{F}_t(\boldsymbol{x}_0)\big), \\ \boldsymbol{F}_0(\boldsymbol{x}_0) = \boldsymbol{x}_0. \end{cases}$$

- At any given time $t \geq 0$,

$$\boldsymbol{x} = \boldsymbol{F}_t(\boldsymbol{x}_0)$$

  is a coordinate transformation generated by the fluid motion.

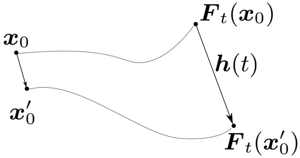
- The Jacobian matrix $\nabla \boldsymbol{F}_t$ and determinant $J_t = \det(\nabla \boldsymbol{F}_t)$ satisfies

$$\begin{cases} \dfrac{d}{dt}\nabla \boldsymbol{F}_t = \nabla \boldsymbol{F}_t \cdot \nabla \boldsymbol{u}(t, \boldsymbol{F}_t), \\ \nabla \boldsymbol{F}_t = I, \end{cases} \qquad \begin{cases} \dfrac{d}{dt} J_t = \big[\nabla \cdot \boldsymbol{u}(t, \boldsymbol{F}_t)\big] J_t, \\ J_0 = 1. \end{cases}$$

Deformation and rotation tensors

- The motion of two neighboring points:

$$\boldsymbol{x}(t) = \boldsymbol{F}_t(\boldsymbol{x}_0),$$
$$\boldsymbol{x}'(t) = \boldsymbol{F}_t(\boldsymbol{x}_0'),$$
$$\boldsymbol{h}(t) = \boldsymbol{x}'(t) - \boldsymbol{x}(t).$$



At the lowest order, the evolution of $\boldsymbol{h}(t)$ is governed by the tensor $\nabla \boldsymbol{u}$, i.e.,

$$\frac{d}{dt}\boldsymbol{h}(t) = \boldsymbol{u}(t, \boldsymbol{x}') - \boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{h}(t) \cdot \nabla \boldsymbol{u}(t, \boldsymbol{x}) + O\big(h(t)^2\big),$$

$$\nabla \boldsymbol{u} = \underbrace{\tfrac{1}{2}\big[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\big]}_{\mathbf{D}=\text{Deformation}} + \underbrace{\tfrac{1}{2}\big[\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^T\big]}_{\mathbf{S}=\text{Rotation}}.$$

- The rotation tensor is related to the **VORTICITY** $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$,

$$\boldsymbol{h} \cdot \mathbf{S} = \tfrac{1}{2}\big[\boldsymbol{h} \cdot \nabla \boldsymbol{u} - \nabla \boldsymbol{u} \cdot \boldsymbol{h}\big] = \tfrac{1}{2}\big(\nabla \times \boldsymbol{u}\big) \times \boldsymbol{h} = \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{h}.$$

- At last,

$$\frac{d}{dt}\boldsymbol{h} = \mathbf{D} \cdot \boldsymbol{h} + \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{h} + O(h^2).$$

Example of pure-deformation flow

- Consider the two-dimensional flow

$$\boldsymbol{u} = \nu \begin{pmatrix} x_1 \\ -x_2 \\ 0 \end{pmatrix}, \quad \nabla \boldsymbol{u} = \begin{pmatrix} \nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

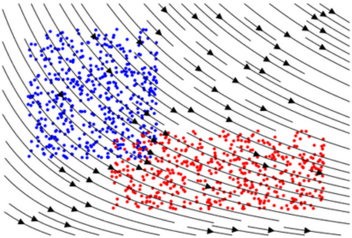
- This is a pure-deformation flow, that is,

$$\mathbf{D} = \nabla \boldsymbol{u}, \qquad \boldsymbol{\omega} = 0.$$

- The flow is

$$\boldsymbol{F}_t(\boldsymbol{x}_0) = \begin{pmatrix} e^{\nu t} x_{0,1} \\ e^{-\nu t} x_{0,2} \\ x_{0,3} \end{pmatrix}.$$

The Lagrangian trajectories are hyperbolas $x_1(t)x_2(t) = x_{0,1}x_{0,2}$.



Example of pure-rotation flow: a simple vortex

- Consider the two-dimensional flow

$$\boldsymbol{u} = \nu \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}, \quad \nabla \boldsymbol{u} = \begin{pmatrix} 0 & -\nu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

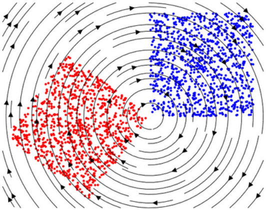
- This is a pure-rotation flow, that is, $\nabla \boldsymbol{u}$ is anti-symmetric,

$$\mathbf{D} = 0, \qquad \boldsymbol{\omega} = (0, 0, -2\nu)^T.$$

- The flow is

$$\boldsymbol{F}_t(\boldsymbol{x}_0) = \begin{pmatrix} x_{0,1}\cos(\nu t) + x_{0,2}\sin(\nu t) \\ -x_{0,1}\sin(\nu t) + x_{0,2}\cos(\nu t) \\ x_{0,3} \end{pmatrix}.$$

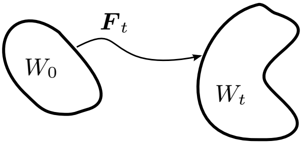
  The Lagrangian trajectories are circles $x_1(t)^2 + x_2(t)^2 = x_{0,1}^2 + x_{0,2}^2$.



Reynolds transport theorem

- Let $f(t, \boldsymbol{x})$ be a scalar field and $W_t = \boldsymbol{F}_t(W_0)$ a volume moving with the fluid:

$$\frac{d}{dt} \int_{W_t} f(t, \boldsymbol{x}) dx = ?$$



- We can use the flow $\boldsymbol{F}_t$ as a change of variables,

$$\begin{aligned}
\frac{d}{dt} \int_{W_t} f(t, \boldsymbol{x}) dx &= \frac{d}{dt} \int_{W_0} f\big(t, \boldsymbol{F}_t(\boldsymbol{x}_0)\big) J_t(\boldsymbol{x}_0) dx_0 \\
&= \int_{W_0} \Big[\frac{df}{dt}\big(t, \boldsymbol{F}_t(\boldsymbol{x}_0)\big) J_t(\boldsymbol{x}_0) + f\big(t, \boldsymbol{F}_t\big(\boldsymbol{x}_0)\big) \frac{d}{dt} J_t(\boldsymbol{x}_0) \Big] dx_0 \\
&= \int_{W_0} \Big[\frac{df}{dt}\big(t, \boldsymbol{F}_t(\boldsymbol{x}_0)\big) + f\big(t, \boldsymbol{F}_t(\boldsymbol{x}_0)\big) \nabla \cdot \boldsymbol{u}\big(t, \boldsymbol{F}_t(\boldsymbol{x}_0)\big) \Big] J_t(\boldsymbol{x}_0) dx_0 \\
&= \int_{W_t} \big[\partial_t f + \nabla \cdot (f \boldsymbol{u}) \big] dx,
\end{aligned}$$

where $df/dt = \partial_t f + \boldsymbol{u} \cdot \nabla f$ is the *advective derivative*.

Reynolds transport theorem - cont'd

- Statement of the **Reynolds transport theorem**:

$$\frac{d}{dt}\int_{W_t} f(t,\boldsymbol{x})dx = \int_{W_t}\Big[\underbrace{\frac{df}{dt} + f\nabla\cdot\boldsymbol{u}}_{\text{advective form}}\Big]dx = \int_{W_t}\Big[\underbrace{\partial_t f + \nabla\cdot(f\boldsymbol{u})}_{\text{conservative form}}\Big]dx.$$

- Example: For $f = 1$ one has the evolution of the volume of W_t,

$$\frac{d}{dt}\int_{W_t} dx = \int_{W_t} \nabla\cdot\boldsymbol{u}\, dx.$$

- **Incompressibility**: The volume of every domain W_t is conserved,

$$\text{incompressibility} \quad \Longleftrightarrow \quad \nabla\cdot\boldsymbol{u} = 0.$$

Equations of motion: Mass continuity equation

- The mass contained in a volume W_t which moves with the fluid is constant,

$$\frac{d}{dt} \underbrace{\int_{W_t} \rho(t, \boldsymbol{x}) dx}_{\text{mass in } W_t} = 0.$$

- From the transport theorem (conservative form):

$$\frac{d}{dt} \int_{W_t} \rho(t, \boldsymbol{x}) dx = \int_{W_t} \big[\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u})\big] dx = 0, \quad \text{for every } W_t.$$

  This yields the **MASS CONTINUITY EQUATION**

$$\boxed{\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0.}$$

- For incompressible flows, $\rho = \mathrm{constant}$ is a solution.

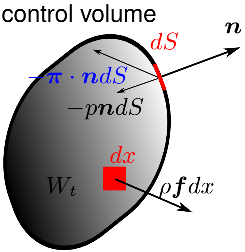
Equations of motion: Momentum balance equation

- Newton's second law:

$$\frac{d}{dt}\underbrace{\int_{W_t}\rho(t,\boldsymbol{x})\boldsymbol{u}(t,\boldsymbol{x})dx}_{\text{momentum in } W_t} = \text{forces acting on } W_t.$$

- Forces on a fluid volume:

$$p = \text{pressure},$$
$$\boldsymbol{\pi} = \text{viscosity} \quad \left[\mathrm{Tr}(\boldsymbol{\pi}) = 0\right],$$
$$\boldsymbol{f} = \text{force per unit of mass}.$$



$$\text{surfaces forces} = -\int_{\partial W_t}\left[p\boldsymbol{n} + \boldsymbol{\pi}\cdot\boldsymbol{n}\right]dS, \quad \text{body forces} = \int_{W_t}\rho(t,x)\boldsymbol{f}(t,x)dx.$$

Equations of motion: Momentum balance equation - cont'd

- By means of the Reynolds theorem, the momentum balance reads

$$\int_{W_t} \big[\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u})\big] dx = -\int_{\partial W_t} \big[p\boldsymbol{n} + \boldsymbol{\pi} \cdot \boldsymbol{n}\big] dS + \int_{W_t} \rho \boldsymbol{f}\, dx.$$

- Gauss divergence theorem allows us to deal with boundary terms

$$\int_{\partial W_t} \big[p\boldsymbol{n} + \boldsymbol{\pi} \cdot \boldsymbol{n}\big] dS = \int_{W_t} \big[\nabla p + \nabla \cdot \boldsymbol{\pi}\big] dx.$$

- Since W_t is an arbitrary volume, the momentum balance becomes

$$\boxed{\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} + \boldsymbol{\pi}) = -\nabla p + \rho \boldsymbol{f}.}$$

Equations of motion: Energy balance equation

- The balance of the total (kinetic plus internal) energy is

$$\frac{d}{dt}\underbrace{\int_{W_t}\Big[\frac{1}{2}\rho u^2+\frac{3}{2}nk_BT\Big]dx}_{\text{energy in } W_t} = (\text{work of forces on } W_t) + (\text{energy exchange in } W_t).$$

- Work done by the forces:

$$\begin{aligned}\text{work of forces on } W_t &= -\int_{\partial W_t}\boldsymbol{u}\cdot\big[p\boldsymbol{n}+\boldsymbol{\pi}\cdot\boldsymbol{n}\big]dS+\int_{W_t}\rho\boldsymbol{u}\cdot\boldsymbol{f}\,dx\\ &= -\int_{W_t}\nabla\cdot\big[p\boldsymbol{u}+\boldsymbol{\pi}\cdot\boldsymbol{u}\big]dx+\int_{W_t}\rho\boldsymbol{u}\cdot\boldsymbol{f}\,dx.\end{aligned}$$

- Energy exchance via heat flux and sources

$$\begin{aligned}\text{energy exchance in } W_t &= -\int_{\partial W_t}\boldsymbol{q}\cdot\boldsymbol{n}dS+\int_{W_t}Qdx\\ &= -\int_{W_t}\nabla\cdot\boldsymbol{q}dx+\int_{W_t}Qdx,\end{aligned}$$

where $\boldsymbol{q}$ is the heat flux and Q the heat source.

Equations of motion: Energy balance equation - cont'd

- Reynolds theorem yields the total energy balance,

$$\partial_t\left(\tfrac{1}{2}\rho u^2 + \tfrac{3}{2}nk_BT\right) + \nabla\cdot\left[\left(\tfrac{1}{2}\rho u^2 + \tfrac{3}{2}nk_BT + p\right)\boldsymbol{u} + \pi\cdot\boldsymbol{u} + \boldsymbol{q}\right] = \rho\boldsymbol{u}\cdot\boldsymbol{f} + Q.$$

- The momentum balance equation implies

$$\partial_t\left(\tfrac{1}{2}\rho u^2\right) + \nabla\cdot\left(\tfrac{1}{2}\rho u^2\boldsymbol{u} + p\boldsymbol{u} + \pi\cdot\boldsymbol{u}\right) = \rho\boldsymbol{u}\cdot\boldsymbol{f} + \boldsymbol{\pi} : \nabla\boldsymbol{u} + p\nabla\cdot\boldsymbol{u},$$

  which can be used to simplify the kinetic energy terms.

- By using the latter identity, one obtains the internal energy balance

$$\boxed{\partial_t\left(\tfrac{3}{2}nk_BT\right) + \nabla\cdot\left(\tfrac{3}{2}nk_BT\boldsymbol{u} + \boldsymbol{q}\right) + p\nabla\cdot\boldsymbol{u} + \boldsymbol{\pi} : \nabla\boldsymbol{u} = Q.}$$

Summary of the equations of motion

General form of the equations of fluid dynamics:

- Mass continuity equation:
$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0.$$
- Momentum balance equation:
$$\partial_t(\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} + \boldsymbol{\pi}) = -\nabla p + \rho \boldsymbol{f}.$$
- Internal energy balance:
$$\partial_t \left(\tfrac{3}{2} n k_B T\right) + \nabla \cdot \left(\tfrac{3}{2} n k_B T \boldsymbol{u} + \boldsymbol{q}\right) + p \nabla \cdot \boldsymbol{u} + \boldsymbol{\pi} : \nabla \boldsymbol{u} = Q.$$

- The equations are not closed! We need to specify

  pressure p, viscosity $\boldsymbol{\pi}$, heat flux $\boldsymbol{q}$, and heat source Q,

  as functions of $(\rho, \boldsymbol{u}, T)$: This is known as *the closure problem*.
- Kinetic theory is needed to obtain good closure relations. In particular,
$$\boxed{p = n k_B T.}$$

Forces acting on an electrically charged fluid

- For a gas of charged particles of mass m_α and electric charge e_α,

$$\text{mass density} = m_\alpha n_\alpha, \quad \text{charge density} = e_\alpha n_\alpha,$$
$$\text{electric current density} = e_\alpha n_\alpha \boldsymbol{u}_\alpha.$$

- Forces acting of an electrically charged fluid are then

$$m_\alpha n_\alpha \boldsymbol{f}_\alpha = e_\alpha n_\alpha \boldsymbol{E} + \frac{(e_\alpha n_\alpha \boldsymbol{u}_\alpha) \times \boldsymbol{B}}{c} + \boldsymbol{R}_\alpha,$$

  where $\boldsymbol{E}$, $\boldsymbol{B}$ are the electromagnetic fields *(c.g.s. units)* and

$$\boldsymbol{R}_\alpha \text{ is the friction force due to collisions,}$$

  cf. the kinetic physics lecture by R. Bilato.

- If $\alpha \in \{\mathrm{i} = \text{ions}, \mathrm{e} = \text{electrons}\}$, and neglecting thermal forces,

$$-\boldsymbol{R}_\mathrm{i} = \boldsymbol{R}_\mathrm{e} = e n_\mathrm{e} \eta \boldsymbol{J}, \qquad \boldsymbol{J} = \sum_{\alpha \in \{\mathrm{i},\mathrm{e}\}} e_\alpha n_\alpha \boldsymbol{u}_\alpha.$$

Two-fluid description of plasmas: basic equations

- Basic equations: for two-species plasmas $\alpha \in \{\mathrm{i}, \mathrm{e}\}$,

$$
\begin{cases}
\partial_t n_\alpha + \nabla \cdot (n_\alpha \boldsymbol{u}_\alpha) = 0, \\
\partial_t (m_\alpha n_\alpha \boldsymbol{u}_\alpha) + \nabla \cdot (m_\alpha n_\alpha \boldsymbol{u}_\alpha \boldsymbol{u}_\alpha + \boldsymbol{\pi}_\alpha) = -\nabla p_\alpha + m_\alpha n_\alpha \boldsymbol{f}_\alpha, \\
\partial_t \big(\tfrac{3}{2} p_\alpha\big) + \nabla \cdot \big(\tfrac{3}{2} p_\alpha \boldsymbol{u}_\alpha + \boldsymbol{q}_\alpha\big) + p_\alpha \nabla \cdot \boldsymbol{u}_\alpha + \boldsymbol{\pi}_\alpha : \nabla \boldsymbol{u}_\alpha = Q_\alpha, \\
\\
\nabla \times \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{J} + \frac{1}{c} \partial_t \boldsymbol{E}, \qquad \big[\boldsymbol{J} = e_\mathrm{i} n_\mathrm{i} \boldsymbol{u}_\mathrm{i} - e n_\mathrm{e} \boldsymbol{u}_\mathrm{e}\big], \\
\partial_t \boldsymbol{B} + c \nabla \times \boldsymbol{E} = 0, \\
\nabla \cdot \boldsymbol{E} = 4\pi \rho_{ch}, \qquad \big[\rho_{ch} = e_\mathrm{i} n_\mathrm{i} - e n_\mathrm{e}\big], \\
\nabla \cdot \boldsymbol{B} = 0,
\end{cases}
$$

where we have used the closure $p_\alpha = n_\alpha k_B T_\alpha$, and

$$
m_\alpha n_\alpha \boldsymbol{f}_\alpha = e_\alpha n_\alpha \Big[\boldsymbol{E} + \frac{\boldsymbol{u}_\alpha \times \boldsymbol{B}}{c}\Big] + \boldsymbol{R}_\alpha, \quad -\boldsymbol{R}_\mathrm{i} = \boldsymbol{R}_\mathrm{e} = e n_\mathrm{e} \eta \boldsymbol{J}.
$$

- Total momentum and energy conservation requires

$$
\sum_\alpha \boldsymbol{R}_\alpha = 0, \qquad \sum_\alpha \big[Q_\alpha + \boldsymbol{u}_\alpha \cdot \boldsymbol{R}_\alpha\big] = 0.
$$

Low-frequency regime and quasi-neutrality

- Typical values of basic quantities:

$$\partial_t(\cdot) \sim \tau^{-1}(\cdot), \qquad \nabla(\cdot) \sim L^{-1}(\cdot), \qquad \boldsymbol{u}_\alpha \sim V, \qquad n_\alpha \sim N,$$

$$k_B T_\alpha \sim \text{energy scale} \sim k_B T, \qquad \boldsymbol{E} \sim \frac{k_B T}{eL}, \qquad \boldsymbol{B} \sim \frac{k_B T}{eL}\frac{c}{V}, \quad \cdots$$

- Assumptions defining the regime of interest:

$\tau = L/V,$	τ is set by the advection time scale (low frequencies),
$m_{\mathrm{i}} V^2 = k_B T,$	kinetic-energy scale is set by temperature scale,
$V/c = \lambda_D/L \ll 1,$	non-relativistic quasi-neutral plasmas,

where the typical Debye length (in c.g.s. units) is

$$\lambda_D = \sqrt{\frac{k_B T}{4\pi e^2 N}}.$$

- With $\epsilon = (\lambda_D/L)^2 \ll 1$, one finds two small terms (relative to the others):

$c^{-1}\partial_t \boldsymbol{E} = O(\epsilon),$	displacement current is negligible,
$\nabla \cdot \boldsymbol{E} = O(\epsilon),$	charge separation is negligible.

Quasi-neutral two-fluid model

- Quasi-neutral limit of the two-fluid model:

$$
\begin{cases}
\partial_t n_\alpha + \nabla \cdot (n_\alpha \boldsymbol{u}_\alpha) = 0, \qquad e_{\mathrm{i}} n_{\mathrm{i}} - e n_{\mathrm{e}} = 0, \\
\partial_t (m_\alpha n_\alpha \boldsymbol{u}_\alpha) + \nabla \cdot (m_\alpha n_\alpha \boldsymbol{u}_\alpha \boldsymbol{u}_\alpha + \boldsymbol{\pi}_\alpha) = -\nabla p_\alpha + m_\alpha n_\alpha \boldsymbol{f}_\alpha, \\
\partial_t \left(\frac{3}{2} p_\alpha\right) + \nabla \cdot \left(\frac{3}{2} p_\alpha \boldsymbol{u}_\alpha + \boldsymbol{q}_\alpha\right) + p_\alpha \nabla \cdot \boldsymbol{u}_\alpha + \boldsymbol{\pi}_\alpha : \nabla \boldsymbol{u}_\alpha = Q_\alpha, \\
\\
\nabla \times \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{J}, \qquad \text{[this defines } \boldsymbol{J} \text{ directly and } \nabla \cdot \boldsymbol{J} = 0\text{]}, \\
\partial_t \boldsymbol{B} + c \nabla \times \boldsymbol{E} = 0, \\
\nabla \cdot \boldsymbol{B} = 0.
\end{cases}
$$

- This model does <u>not</u> imply $\nabla \cdot \boldsymbol{E} = 0$. Gauss law is recovered at first order in ϵ.
- Energy and momentum in the quasi-neutral limit:

$$
\text{energy density} = \sum_\alpha \left(\frac{1}{2} m_\alpha n_\alpha u_\alpha^2 + \frac{p_\alpha}{\gamma - 1}\right) + \underbrace{\frac{E^2}{8\pi}}_{O(\epsilon)} + \frac{B^2}{8\pi},
$$

$$
\text{momentum density} = \sum_\alpha m_\alpha n_\alpha \boldsymbol{u}_\alpha + \underbrace{\frac{\boldsymbol{E} \times \boldsymbol{B}}{4\pi c}}_{O(\epsilon)}.
$$

Equation for the center-of-mass fluid

- Let us consider a single fluid with "center-of-mass parameters":

$$\rho = m_\mathrm{i} n_\mathrm{i} + m_\mathrm{e} n_\mathrm{e}, \qquad \rho \boldsymbol{u} = m_\mathrm{i} n_\mathrm{i} \boldsymbol{u}_\mathrm{i} + m_\mathrm{e} n_\mathrm{e} \boldsymbol{u}_\mathrm{e}, \qquad \rho_{ch} = e_\mathrm{i} n_\mathrm{i} - e n_\mathrm{e} = 0.$$

- By summing over the index α, the two-fluid equations imply

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} + \boldsymbol{\pi}) = -\nabla p + (\boldsymbol{J} \times \boldsymbol{B})/c, \\ \partial_t \big(\frac{3}{2} p\big) + \nabla \cdot \big(\frac{3}{2} p \boldsymbol{u} + \boldsymbol{q}\big) + p \nabla \cdot \boldsymbol{u} + \boldsymbol{\pi} : \nabla \boldsymbol{u} = \boldsymbol{J} \cdot \boldsymbol{E} - \boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B})/c, \end{cases}$$

where the single fluid quantities are

$$p = \sum_\alpha \Big[p_\alpha + \frac{1}{3} m_\alpha n_\alpha (\boldsymbol{u}_\alpha - \boldsymbol{u})^2 \Big],$$

$$\boldsymbol{\pi} = \sum_\alpha \Big[\boldsymbol{\pi}_\alpha + m_\alpha n_\alpha (\boldsymbol{u}_\alpha - \boldsymbol{u})(\boldsymbol{u}_\alpha - \boldsymbol{u}) - \frac{1}{3} m_\alpha n_\alpha (\boldsymbol{u}_\alpha - \boldsymbol{u})^2 \Big],$$

$$\boldsymbol{q} = \sum_\alpha \Big[\boldsymbol{q}_\alpha + p_\alpha (\boldsymbol{u}_\alpha - \boldsymbol{u}) + \boldsymbol{\pi}_\alpha \cdot (\boldsymbol{u}_\alpha - \boldsymbol{u}) + \frac{1}{2} m_\alpha n_\alpha (\boldsymbol{u}_\alpha - \boldsymbol{u})^2 (\boldsymbol{u}_\alpha - \boldsymbol{u}) \Big].$$

- Remark: This is an exact result, no approximations. But the system is not closed.

The generalized Ohm's law

- The momentum balance in advective form times e_α/m_α reads

$$e_\alpha n_\alpha \frac{d\boldsymbol{u}_\alpha}{dt} + \frac{e_\alpha}{m_\alpha}\nabla\cdot\boldsymbol{\pi}_\alpha + \frac{e_\alpha}{m_\alpha}\nabla p_\alpha = \frac{\omega_{p,\alpha}^2}{4\pi}\boldsymbol{E} + \frac{e_\alpha^2 n_\alpha}{m_\alpha c}\boldsymbol{u}_\alpha\times\boldsymbol{B} + \frac{e_\alpha}{m_\alpha}\boldsymbol{R}_\alpha,$$

  where the plasma frequencies are (c.g.s. units)

$$\omega_\alpha^2 = 4\pi e_\alpha^2 n_\alpha/m_\alpha, \qquad \omega_p^2 = \omega_{p,\mathrm{i}}^2 + \omega_{p,\mathrm{e}}^2.$$

- The sum over $\alpha\in\{\mathrm{i},\mathrm{e}\}$, *using quasi-neutrality* $\boldsymbol{u}_\mathrm{i}\approx\boldsymbol{u}$, yields

$$\underbrace{en_\mathrm{e}\frac{d\boldsymbol{u}_\mathrm{i}}{dt} + \frac{e_\mathrm{i}}{m_\mathrm{i}}\nabla\cdot\boldsymbol{\pi}_\mathrm{i} + \frac{e_\mathrm{i}}{m_\mathrm{i}}\nabla p_\mathrm{i}}_{\text{ion inertia, viscosity, and pressure}} - \underbrace{en_\mathrm{e}\frac{d\boldsymbol{u}_\mathrm{e}}{dt} - \frac{e}{m_\mathrm{e}}\nabla\cdot\boldsymbol{\pi}_\mathrm{e} - \frac{e}{m_\mathrm{e}}\nabla p_\mathrm{e}}_{\text{electron inertia, viscosity, and pressure}}$$

$$= \frac{\omega_p^2}{4\pi}\Big[\boldsymbol{E} + \frac{\boldsymbol{u}\times\boldsymbol{B}}{c} - \eta\boldsymbol{J}\Big] - \underbrace{\frac{\boldsymbol{J}\times\boldsymbol{B}}{m_\mathrm{e}c}}_{\text{Hall term}}.$$

- Since the plasma frequency corresponds to the dominating term,

$$\boxed{\boldsymbol{E} + \frac{\boldsymbol{u}\times\boldsymbol{B}}{c} - \eta\boldsymbol{J} = 0, \qquad \text{Ohm's law in MHD.}}$$

MHD equations

- Single-fluid equations, together with the Ampèrè and Faraday laws, as well as the leading term in the generalized Ohm's law:

$$
\begin{cases}
\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\
\partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} + \boldsymbol{\pi}) = -\nabla p + (\boldsymbol{J} \times \boldsymbol{B})/c, \\
\partial_t \big(\frac{3}{2} p\big) + \nabla \cdot \big(\frac{3}{2} p \boldsymbol{u} + \boldsymbol{q}\big) + p \nabla \cdot \boldsymbol{u} + \boldsymbol{\pi} : \nabla \boldsymbol{u} = \eta \boldsymbol{J}^2, \\
\\
\nabla \times \boldsymbol{B} = \dfrac{4\pi}{c} \boldsymbol{J}, \qquad \big[\nabla \cdot \boldsymbol{J} = 0, \quad \text{quasi-neutrality}\big], \\
\partial_t \boldsymbol{B} + c \nabla \times \boldsymbol{E} = 0, \qquad \big[\nabla \cdot \boldsymbol{B} = 0\big], \\
\boldsymbol{E} + \dfrac{\boldsymbol{u} \times \boldsymbol{B}}{c} - \eta \boldsymbol{J} = 0.
\end{cases}
$$

- We shall apply this to the case of ideal fluids for which $\boldsymbol{\pi} = 0$, $\boldsymbol{q} = 0$.
- Reconstructing single fluid variables (validity conditions):

$$
\boldsymbol{u}_{\mathrm{i}} \approx \boldsymbol{u}, \qquad \boldsymbol{u}_{\mathrm{e}} \approx \boldsymbol{u} - \boldsymbol{J}/(e n_{\mathrm{e}}), \qquad \text{subject to } |\boldsymbol{J}|/(e n_{\mathrm{e}}) \ll |\boldsymbol{u}|.
$$

Magnetic field induction equation

- Electrodynamics sector of the model:

$$\boldsymbol{E} + \frac{\boldsymbol{u} \times \boldsymbol{B}}{c} = \eta \boldsymbol{J}, \qquad \partial_t \boldsymbol{B} + c\nabla \times \boldsymbol{E} = 0, \qquad \nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J}.$$

- Solve for $\boldsymbol{J}$ and then $\boldsymbol{E}$:

$$\boldsymbol{J} = \frac{c}{4\pi}\nabla \times \boldsymbol{B}, \qquad \boldsymbol{E} = \frac{c\eta}{4\pi}\nabla \times \boldsymbol{B} - \frac{\boldsymbol{u} \times \boldsymbol{B}}{c}.$$

- The whole electrodynamics sector reduces to the MHD induction equation:

$$\boxed{\partial_t \boldsymbol{B} = \nabla \times \big(\boldsymbol{u} \times \boldsymbol{B}\big) - \nabla \times \big(\frac{c^2\eta}{4\pi}\nabla \times \boldsymbol{B}\big),}$$

  which is an equation of advection-diffusion type.

Magnetic Reynolds number

- MHD induction equation:

$$\partial_t \boldsymbol{B} = \nabla \times \big(\boldsymbol{u} \times \boldsymbol{B}\big) - \nabla \times \big(\frac{c^2\eta}{4\pi} \nabla \times \boldsymbol{B}\big).$$

- Estimating the order of magnitude:

$$\nabla \times \big(\boldsymbol{u} \times \boldsymbol{B}\big) \sim \frac{VB}{L}, \qquad \nabla \times \big(\frac{c^2\eta}{4\pi} \nabla \times \boldsymbol{B}\big) \sim \frac{c^2\eta B}{4\pi L^2},$$

where V, B, and L^{-1} are typical scales for $\boldsymbol{u}$, $\boldsymbol{B}$, and ∇.

- **Magnetic Reynolds number**: advection-to-diffusion ratio

$$\boxed{\mathrm{Rm} = 4\pi \frac{VL}{c^2\eta}.}$$

- Ideal MHD:

$$\mathrm{Rm} \gg 1, \qquad \partial_t \boldsymbol{B} = \nabla \times \big(\boldsymbol{u} \times \boldsymbol{B}\big).$$

- Magnetic field diffusion (linear problem, decoupled, less interesting as $\eta \propto T^{-3/2}$):

$$\mathrm{Rm} \ll 1, \qquad \partial_t \boldsymbol{B} = -\nabla \times \big(\frac{c^2\eta}{4\pi} \nabla \times \boldsymbol{B}\big).$$

Magnetic pressure and field line tension

- The magnetic field acts on the flow via the force

$$
\begin{aligned}
\frac{\boldsymbol{J}\times\boldsymbol{B}}{c} &= \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \frac{1}{4\pi}(\boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla\boldsymbol{B}\cdot\boldsymbol{B}) \\
&= \frac{1}{4\pi}\boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla\Big(\frac{B^2}{8\pi}\Big).
\end{aligned}
$$

- The momentum equation becomes

$$
\rho(\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) = -\nabla\Big(p + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic pressure}}\Big) + \underbrace{\frac{1}{4\pi}\boldsymbol{B}\cdot\nabla\boldsymbol{B}}_{\text{field-line tension}}.
$$

- The magnetic field energy contribute to the pressure. Plasma beta:

$$
\beta = \frac{8\pi p}{B^2}, \quad \begin{cases} \beta \ll 1, & \text{(tokamaks, coronal funnels)}, \\ \beta \approx 1, & \text{(solar wind, coronal loops)}, \\ \beta \gg 1, & \text{(?)}. \end{cases}
$$

- The magnetic field-line tension reacts to field-line bending.

Equation of state in the ideal case

- Continuity and pressure equations with $\eta = 0$:

$$\frac{d\rho}{dt} + \rho \nabla \cdot \boldsymbol{u} = 0, \qquad \frac{dp}{dt} + \gamma p \nabla \cdot \boldsymbol{u} = 0.$$

- The pressure equation is equivalent to,

$$\frac{dp}{dt} - \frac{\gamma p}{\rho} \frac{d\rho}{dt} = \rho^{\gamma} \frac{d}{dt}\left(p\rho^{-\gamma}\right) = 0$$

hence, we can take

$$\boxed{p(\rho) = C\rho^{\gamma}, \qquad \text{(equation of state).}}$$

- Isentropic flows: The equation of state implies

$$\rho^{-1}\nabla p = C\gamma\rho^{\gamma-2}\nabla\rho = \nabla h(\rho),$$

where $h(\rho) = C\gamma\rho^{\gamma-1}/(\gamma-1)$ is the **ENTHALPY** of the fluid.

Summary of MHD

Resistive MHD: $\eta > 0$

$$\begin{cases} (\partial_t + \boldsymbol{u}\cdot\nabla)\rho + \rho\nabla\cdot\boldsymbol{u} = 0, \\ \rho(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \dfrac{\boldsymbol{J}\times\boldsymbol{B}}{c}, \qquad \boldsymbol{J} = \dfrac{c}{4\pi}\nabla\times\boldsymbol{B}, \\ (\partial_t + \boldsymbol{u}\cdot\nabla)p + \gamma p\nabla\cdot\boldsymbol{u} = \eta\boldsymbol{J}^2, \qquad \gamma = 5/3, \\ \partial_t\boldsymbol{B} = \nabla\times\big(\boldsymbol{u}\times\boldsymbol{B}\big) - \nabla\times\big(\dfrac{c^2\eta}{4\pi}\nabla\times\boldsymbol{B}\big). \end{cases}$$

Ideal MHD: $\eta = 0$

$$\begin{cases} (\partial_t + \boldsymbol{u}\cdot\nabla)\rho + \rho\nabla\cdot\boldsymbol{u} = 0, \\ \rho(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \dfrac{\boldsymbol{J}\times\boldsymbol{B}}{c}, \qquad \boldsymbol{J} = \dfrac{c}{4\pi}\nabla\times\boldsymbol{B}, \\ (\partial_t + \boldsymbol{u}\cdot\nabla)p + \gamma p\nabla\cdot\boldsymbol{u} = 0, \qquad (p(\rho) = C\rho^\gamma), \qquad \gamma = 5/3, \\ \partial_t\boldsymbol{B} = \nabla\times\big(\boldsymbol{u}\times\boldsymbol{B}\big). \end{cases}$$

LECTURE II: Character of MHD

1. Global resistive conservation laws:
   Mass, momentum and energy conservation laws in ideal and resistive MHD.
2. Global ideal conservation laws:
   Magnetic helicity and cross helicity.
3. Local ideal conservation laws:
   Frozen-in law and flux conservation.
4. Basic processes:
   Frozen-in fields, resistive diffusion of magnetic field, MHD waves.

Conservation of total mass, momentum and energy

- A (sufficiently regular) solution of **RESISTIVE** MHD on a domain $\Omega \subseteq \mathbb{R}^3$ with boundary conditions $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, and $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ satisfies:

$$\frac{d}{dt}\int_\Omega \rho dx = 0, \qquad \text{(mass conservation)},$$

$$\frac{d}{dt}\int_\Omega \rho \boldsymbol{u} dx = -\int_{\partial\Omega} \left(p + \frac{B^2}{8\pi}\right)\boldsymbol{n} dS, \qquad \text{(momentum conservation)},$$

$$\frac{d}{dt}\int_\Omega w dx = -\frac{c}{4\pi}\int_{\partial\Omega} \boldsymbol{B} \cdot \left[\boldsymbol{n} \times (\eta \boldsymbol{J})\right] dS, \qquad \text{(energy conservation)}.$$

- In the quasi-neutral limit the energy amounts to

$$w = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1} + \frac{B^2}{8\pi}, \quad (\gamma = 5/3).$$

- The conservation laws remain valid in the ideal case $\eta = 0$.
- Remark: boundary terms are usually zero if Ω is properly chosen.

Magnetic helicity and cross helicity in ideal MHD

- Definition of **MAGNETIC AND CROSS HELICITIES**:

$$H_m = \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B} dx, \quad \nabla \times \boldsymbol{B} = \boldsymbol{A}, \qquad \text{(magnetic helicity)},$$

$$H_c = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{B} dx, \qquad \text{(cross helicity)}.$$

- If $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, then magnetic helicity is invariant under gauge transformations:

$$\boldsymbol{A} \mapsto \boldsymbol{A}' = \boldsymbol{A} + \nabla\varphi \quad \Rightarrow \quad H_m \mapsto H_m' = H_m + \int_{\partial\Omega} \varphi \boldsymbol{B} \cdot \boldsymbol{n} dS = H_m.$$

- A (sufficiently regular) solution of <u>**IDEAL**</u> MHD on a domain $\Omega \subseteq \mathbb{R}^3$ with boundary conditions $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ satisfies:

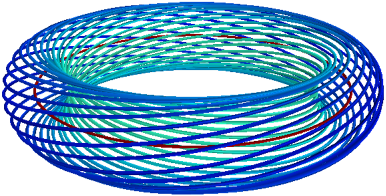
$$\cdots \qquad \text{(mass, momentum and energy conservation)},$$

$$\frac{d}{dt} \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B} dx = 0, \qquad \text{(magnetic-helicity conservation)},$$

$$\frac{d}{dt} \int_\Omega \boldsymbol{u} \cdot \boldsymbol{B} dx = 0, \qquad \text{(cross-helicity conservation)}.$$

- Remark: for cross helicity, one needs $\rho^{-1}\nabla p(\rho) = \nabla h(\rho)$ (isentropic flow).

Frozen-in law: definition of magnetic field lines



- Basic definition of field lines

$$\boldsymbol{b} = \boldsymbol{B}/B,$$

$$\frac{d}{ds}\boldsymbol{x}_t(s) = \boldsymbol{b}\big(t, \boldsymbol{x}_t(s)\big), \quad t = \text{fixed},$$

- Re-parametrization: given a function $f(t, \boldsymbol{x}) > 0$,

$$\sigma(s) = \int^s \frac{ds'}{f\big(t, \boldsymbol{x}_t(s')\big)}, \quad \frac{d\sigma}{ds} = \frac{1}{f\big(t, \boldsymbol{x}_t\big)},$$

$$\frac{d\boldsymbol{x}_t}{d\sigma} = \frac{ds}{d\sigma}\frac{d\boldsymbol{x}_t}{ds} = f(t, \boldsymbol{x}_t)\boldsymbol{b}(t, \boldsymbol{x}_t),$$

$$\frac{d}{d\sigma}\boldsymbol{x}_t(\sigma) = \boldsymbol{\mu}\big(t, \boldsymbol{x}_t(\sigma)\big), \quad \boldsymbol{\mu}(t, \boldsymbol{x}) = f(t, \boldsymbol{x})\boldsymbol{b}(t, \boldsymbol{x}).$$

This shows that we can re-scale the vector field by a strictly positive function.

Frozen-in law: general frozen-in condition for vector fields

- At time $t = 0$, let us consider a field line,

$$\boldsymbol{x} = \boldsymbol{x}_0(\sigma), \qquad \frac{d\boldsymbol{x}_0(\sigma)}{d\sigma} = \boldsymbol{\mu}\big(0, \boldsymbol{x}_0(\sigma)\big),$$

  and let it be frozen with the fluid

$$\boldsymbol{x}_t(\sigma) = \boldsymbol{F}_t\big(\boldsymbol{x}_0(\sigma)\big).$$

- Since $\boldsymbol{F}_t$ is one-to-one, $\boldsymbol{x}_t(\sigma)$ is a curve and its tangent is

$$\frac{d\boldsymbol{x}_t}{d\sigma} = \frac{d\boldsymbol{x}_0}{d\sigma} \cdot \nabla \boldsymbol{F}_t(\boldsymbol{x}_0) = \boldsymbol{\mu}(0, \boldsymbol{x_0}) \cdot \nabla \boldsymbol{F}_t(\boldsymbol{x}_0).$$

- If the new curve $\boldsymbol{x}_t(\sigma)$ is a field line of $\boldsymbol{\mu}(t, \boldsymbol{x})$, we must have

$$\boldsymbol{\mu}(0, \boldsymbol{x_0}) \cdot \nabla \boldsymbol{F}_t(\boldsymbol{x}_0) = \boldsymbol{\mu}(t, \boldsymbol{x}_t).$$

- Applying the operator ∂_t we obtain a condition on the field $\boldsymbol{\mu}$:

$$\partial_t \big[\boldsymbol{\mu}(t, \boldsymbol{x}_t)\big] = \partial_t \boldsymbol{\mu} + \boldsymbol{u} \cdot \nabla \boldsymbol{\mu},$$

$$\partial_t \big[\boldsymbol{\mu}(0, \boldsymbol{x_0}) \cdot \nabla \boldsymbol{F}_t(\boldsymbol{x}_0)\big] = \boldsymbol{\mu}(0, \boldsymbol{x}_0) \cdot \nabla \boldsymbol{F}_t \cdot \nabla \boldsymbol{u} = \boldsymbol{\mu}(t, \boldsymbol{x}_t) \cdot \nabla \boldsymbol{u}(t, \boldsymbol{x}),$$

$$\boxed{\partial_t \boldsymbol{\mu} + \boldsymbol{u} \cdot \nabla \boldsymbol{\mu} - \boldsymbol{\mu} \cdot \nabla \boldsymbol{u} = 0, \qquad \text{frozen-in condition for } \boldsymbol{\mu}.}$$

Frozen-in law: magnetic field lines in ideal MHD

- MHD induction equation:

$$\partial_t \boldsymbol{B} = \nabla \times \big(\boldsymbol{u} \times \boldsymbol{B}\big) = \boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \nabla \cdot \boldsymbol{u},$$
$$\partial_t \boldsymbol{B} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \boldsymbol{u} + \boldsymbol{B} \nabla \cdot \boldsymbol{u} = 0.$$

- Simple case: for incompressible flows $\nabla \cdot \boldsymbol{u} = 0$,

$$\partial_t \boldsymbol{B} + \boldsymbol{u} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \boldsymbol{u} = 0, \quad \Longrightarrow \quad \text{frozen-in field lines.}$$

- For compressible flows, let us set $\boldsymbol{\mu} = \boldsymbol{B}/\rho$, assuming $\rho(t, \boldsymbol{x}) \geq \rho_0 > 0$,

$$\begin{aligned}
\partial_t \boldsymbol{\mu} + \boldsymbol{u} \cdot \nabla \boldsymbol{\mu} &= \rho^{-1}\big(\partial_t \boldsymbol{B} + \boldsymbol{u} \cdot \nabla \boldsymbol{B}\big) - \rho^{-2}\big(\partial_t \rho + \boldsymbol{u} \cdot \nabla \rho\big) \\
&= \rho^{-1}\big(\boldsymbol{B} \cdot \nabla \boldsymbol{u} - \boldsymbol{B} \nabla \cdot \boldsymbol{u}\big) + \rho^{-1} \boldsymbol{B} \nabla \cdot \boldsymbol{u} \\
&= \boldsymbol{\mu} \cdot \nabla \boldsymbol{u} \quad \Longrightarrow \quad \text{frozen-in field lines.}
\end{aligned}$$

The “frozen-in law” of ideal MHD:

In ideal MHD, magnetic field lines move with the plasma flow.

Frozen-in law: vortex lines

- Vortex lines are the field lines of the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$.
- From the momentum balance equation in advective form,

$$\rho\big[\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\big] = -\nabla p + \rho \boldsymbol{f},$$

  and using the vector identity

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nabla(u^2/2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}),$$

  one obtains

$$\partial_t \nabla \times \boldsymbol{u} - \nabla \times \big(\boldsymbol{u} \times (\nabla \times \boldsymbol{u})\big) = -\nabla \times (\rho^{-1} \nabla p) + \nabla \times \boldsymbol{f}.$$

- For isentropic flows, $\rho^{-1}\nabla p = \nabla h$, and conservative forces $\nabla \times \boldsymbol{f} = 0$,

$$\partial_t \boldsymbol{\omega} - \nabla \times \big(\boldsymbol{u} \times \boldsymbol{\omega}\big) = 0,$$

  which is formally identical to the MHD induction equation.
- For isentropic flows with conservative forces, vortex lines are frozen-in.

Flux conservation law: statement of the result

- Consider a surface which moves frozen in the plasma flow:

$$S_0 = \text{surface with boundary } C_0 = \partial S_0, \text{ a simple curve.}$$
$$S_t = \boldsymbol{F}_t(S_0) \text{ with boundary } C_t = \partial S_t = \boldsymbol{F}_t(C_0), \text{ a simple curve.}$$

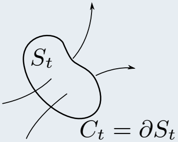
- The flux of the magnetic field through S_t is

$$\Phi(t) = \int_{S_t} \boldsymbol{B} \cdot \boldsymbol{n} dS = \int_{C_t} \boldsymbol{A} \cdot \boldsymbol{t} ds,$$
$$\boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

The flux conservation law in ideal MHD:

The magnetic field flux through a surface moving with the fluid is conserved,

$$\frac{d\Phi(t)}{dt} = 0.$$



Flux conservation law: Proof

- Step 1. An equation for the vector potential $\boldsymbol{A}$:

$$\partial_t \boldsymbol{B} - \nabla \times \big(\boldsymbol{u} \times \boldsymbol{B}\big) = 0, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A},$$
$$\nabla \times \big[\partial_t \boldsymbol{A} - \boldsymbol{u} \times (\nabla \times \boldsymbol{A})\big] = 0, \quad \Longrightarrow \quad \partial_t \boldsymbol{A} - \boldsymbol{u} \times (\nabla \times \boldsymbol{A}) = \nabla w,$$

for some scalar w (and for simply connected domains!). The identity

$$\boldsymbol{u} \times (\nabla \times \boldsymbol{A}) = \nabla \boldsymbol{A} \cdot \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{A},$$

gives at last

$$\partial_t \boldsymbol{A} + \boldsymbol{u} \cdot \nabla \boldsymbol{A} - \nabla \boldsymbol{A} \cdot \boldsymbol{u} = \nabla w.$$

- Step 2. Parametrization of the curve C_t:

$$C_0 \text{ given by } \boldsymbol{x} = \boldsymbol{x}_0(\sigma) \text{ for } 0 \leq \sigma \leq 1,$$
$$C_t \text{ given by } \boldsymbol{x} = \boldsymbol{x}_t(\sigma) \text{ with } \boldsymbol{x}_t(\sigma) = \boldsymbol{F}_t\big(\boldsymbol{x}_0(\sigma)\big).$$

The the arc-element is

$$\boldsymbol{t} ds = \frac{\partial \boldsymbol{x}_t(\sigma)}{\partial \sigma} d\sigma.$$

Flux conservation law: Proof - cont'd

- Step 3. Direct calculation:

$$\frac{d}{dt}\int_{C_t} \boldsymbol{A}\cdot\boldsymbol{t}ds = \frac{d}{dt}\int_0^1 \boldsymbol{A}\big(t,\boldsymbol{x}_t(\sigma)\big)\cdot\frac{\partial\boldsymbol{x}_t(\sigma)}{\partial\sigma}d\sigma$$
$$= \int_0^1 \Big[\big(\partial_t\boldsymbol{A} + \partial_t\boldsymbol{x}_t\cdot\boldsymbol{A}\big)\cdot\frac{\partial\boldsymbol{x}_t(\sigma)}{\partial\sigma} + \boldsymbol{A}\cdot\frac{\partial}{\partial t}\frac{\partial\boldsymbol{x}_t(\sigma)}{\partial\sigma}\Big]d\sigma.$$

With the identities

$$\partial_t\boldsymbol{x}_t = \boldsymbol{u}(t,\boldsymbol{x}_t), \qquad \frac{\partial}{\partial t}\frac{\partial\boldsymbol{x}_t(\sigma)}{\partial\sigma} = \frac{\partial}{\partial\sigma}\Big[\boldsymbol{u}\big(t,\boldsymbol{x}_t(\sigma)\big)\Big] = \frac{\partial\boldsymbol{x}_t(\sigma)}{\partial\sigma}\cdot\nabla\boldsymbol{u},$$

one finds,

$$\frac{d\Phi(t)}{dt} = \int_0^1 \Big[\partial_t\boldsymbol{A} + \boldsymbol{u}\cdot\nabla\boldsymbol{A} + \nabla\boldsymbol{u}\cdot\boldsymbol{A}\Big]\cdot\frac{\partial\boldsymbol{x}_t(\sigma)}{\partial\sigma}d\sigma$$
$$= \int_0^1 \Big[\partial_t\boldsymbol{A} + \boldsymbol{u}\cdot\nabla\boldsymbol{A} + \nabla(\boldsymbol{u}\cdot\boldsymbol{A}) - \nabla\boldsymbol{A}\cdot\boldsymbol{u}\Big]\cdot\frac{\partial\boldsymbol{x}_t(\sigma)}{\partial\sigma}d\sigma$$
$$= \int_0^1 \big[\nabla(w+\boldsymbol{u}\cdot\boldsymbol{A})\big]\frac{\partial\boldsymbol{x}_t(\sigma)}{\partial\sigma}d\sigma = \int_{C_t}\big[\nabla(w+\boldsymbol{u}\cdot\boldsymbol{A})\big]\cdot\boldsymbol{t}ds,$$

and the circulation of a gradient is zero.

Flux conservation law: Analogous conservation law for the vorticity

- Kelvin's circulation theorem:

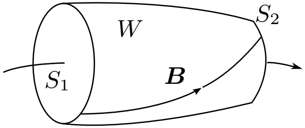
$$\frac{d}{dt} \int_{C_t} \boldsymbol{u} \cdot \boldsymbol{t} ds = 0,$$

  for isentropic fluid subject to conservative forces.
- The magnetic field versus the vorticity field:

electrodynamics	plasma	
$\boldsymbol{A}$ (magnetic potential)	$\boldsymbol{u}$ (velocity field)	circulation
$\boldsymbol{B} = \nabla \times \boldsymbol{A}$ (magnetic field)	$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ (vorticity field)	flux

Flux conservation law: Flux tubes (and vortex tubes)

- Flux tubes:



$$0 = \int_W \nabla \cdot \boldsymbol{B} dx = -\int_{S_1} \boldsymbol{B} \cdot \boldsymbol{n} dS + \int_{S_2} \boldsymbol{B} \cdot \boldsymbol{n} dS,$$

$$\Phi_{\text{tube}} = \int_{S_1} \boldsymbol{B} \cdot \boldsymbol{n} dS.$$

To each flux tube, we can associate a constant flux Φ_{tube}.

- Analogous for isentropic flows: Helmholtz's theorem,

$$\int_{C_1} \boldsymbol{u} \cdot \boldsymbol{t} ds = \int_{C_2} \boldsymbol{u} \cdot \boldsymbol{t} ds$$

for every two closed path C_1 and C_2 encircling a vortex tube.

Flux conservation law: Flux surfaces (and vortex sheets)

- A flux surface is a surface S that is tangent to the magnetic field:

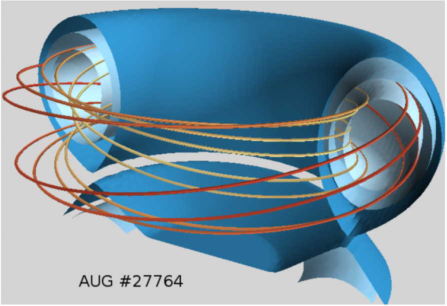
$$\boldsymbol{B} \cdot \boldsymbol{n} = 0, \quad \text{on } S\text{, where } \boldsymbol{n} \text{ is the unit normal to } S.$$

- If S_0 is a flux surface, then so is $S_t = \boldsymbol{F}_t(S_0)$: In fact, for every area A_t on S_t,

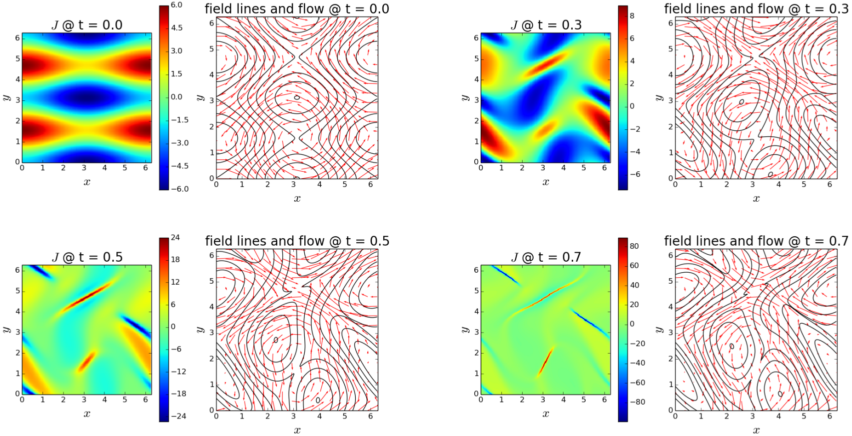
$$\int_{A_t} \boldsymbol{B} \cdot \boldsymbol{n} dS = \int_{A_0} \boldsymbol{B} \cdot \boldsymbol{n} dS = 0,$$

  in view of the flux conservation.

- Flux surfaces in ASDEX upgrade:



An example of frozen-in field: the Orszag-Tang vortex



Magnetic field diffusion at low Rm: Diffusion equation

- MHD induction equation with resistivity:

$$\begin{aligned} \partial_t \boldsymbol{B} &= \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \frac{c^2 \eta}{4\pi} \nabla \times (\nabla \times \boldsymbol{B}), && \eta = \text{constant}, \\ &= \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \kappa_m \Delta \boldsymbol{B}, && \kappa_m = \frac{c^2 \eta}{4\pi}. \end{aligned}$$

- Simplest case: $\mathrm{Rm} \ll 1$,

$$\partial_t \boldsymbol{B}(t, \boldsymbol{x}) = \kappa_m \Delta \boldsymbol{B}(t, \boldsymbol{x}), \quad \boldsymbol{B}(0, \boldsymbol{x}) = \boldsymbol{B}_0(\boldsymbol{x}), \quad \text{(heat equation)}.$$

- Standard Fourier analysis on the full space $\mathbb{R}^3$:

$$\boldsymbol{B}(t, \boldsymbol{x}) = \frac{1}{(2\pi)^3} \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \widehat{\boldsymbol{B}}(t, \boldsymbol{k}) dk, \quad \widehat{\boldsymbol{B}}(t, \boldsymbol{k}) = \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \boldsymbol{B}(t, \boldsymbol{x}) dx.$$

In Fourier representation, the heat equation becomes

$$\partial_t \widehat{\boldsymbol{B}}(t, \boldsymbol{k}) = -\kappa_m k^2 \widehat{\boldsymbol{B}}(t, \boldsymbol{k}), \qquad \widehat{\boldsymbol{B}}(0, \boldsymbol{k}) = \widehat{\boldsymbol{B}}_0(\boldsymbol{k}) = \text{ initial condition},$$
$$\widehat{\boldsymbol{B}}(t, \boldsymbol{k}) = \widehat{\boldsymbol{B}}_0(\boldsymbol{k}) e^{-t\kappa_m k^2}.$$

Magnetic field diffusion at low Rm: Solution via the heat kernel

- Solution via Fourier transform: $\widehat{\boldsymbol{B}}(t,\boldsymbol{k}) = \widehat{\boldsymbol{B}}_0(\boldsymbol{k})e^{-t\kappa_m k^2}$,

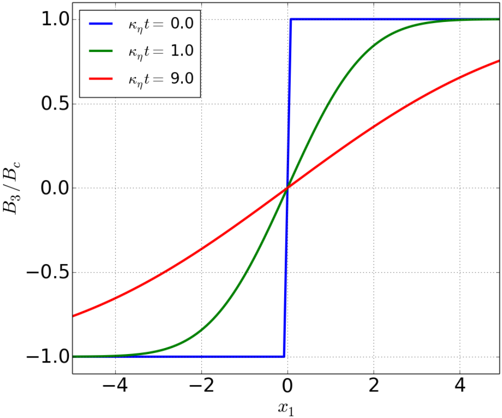
$$\begin{aligned}\boldsymbol{B}(t,\boldsymbol{x}) &= \frac{1}{(2\pi)^3}\int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\widehat{\boldsymbol{B}}_0(\boldsymbol{k})e^{-t\kappa_m k^2}dk \\ &= \frac{1}{(2\pi)^3}\int e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')-t\kappa_m k^2}\boldsymbol{B}_0(\boldsymbol{x}')dx'dk \\ &= \int K(t,\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{B}_0(\boldsymbol{x}')dx', \quad K(t,\boldsymbol{x}) = \frac{e^{-|\boldsymbol{x}|^2/(4t\kappa_m)}}{(4\pi t\kappa_m)^{3/2}}.\end{aligned}$$

  Typical diffusive spreading: $|\Delta\boldsymbol{x}| = \sqrt{4t\kappa_m}$.

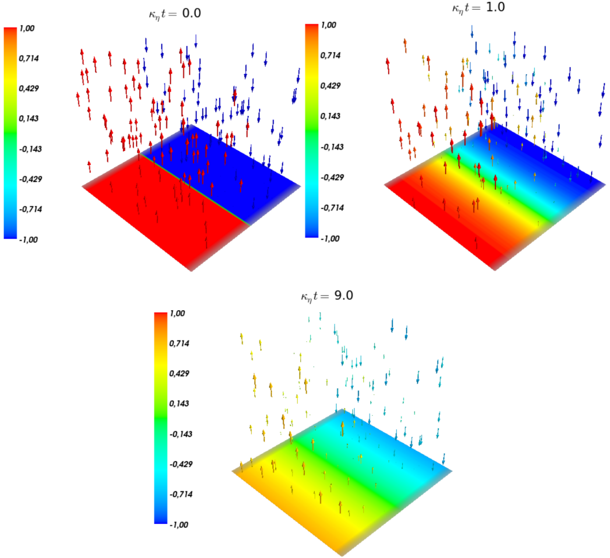
- Analytical example: with $\boldsymbol{x} = (x,y,z)$,

$$\boldsymbol{B}_0(\boldsymbol{x}) = \begin{cases} -\boldsymbol{B}_0, & x<0, \\ 0, & x=0, \\ \boldsymbol{B}_0, & x>0, \end{cases} \qquad \begin{aligned}&\boldsymbol{B}_0(t,\boldsymbol{x}) = \boldsymbol{B}_0\,\mathrm{erf}\left(x/\sqrt{4\kappa_m t}\right), \\ &\boldsymbol{B}_0 = B_c\boldsymbol{e}_3.\end{aligned}$$

Magnetic field diffusion at low Rm: Example



Magnetic field diffusion at low Rm: Example - cont'd



MHD waves: Linearized MHD equations and dispersion tensor

- Linearization of MHD equations over a uniform steady-state background

$$\rho = \rho_0 + \delta\rho, \qquad p(\rho) = p(\rho_0) + \frac{dp(\rho_0)}{d\rho}\delta\rho + O(\delta\rho^2),$$

$$\boldsymbol{u} = \delta\boldsymbol{u},$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}, \qquad c_S^2 = \frac{dp(\rho_0)}{d\rho} = \text{(sound speed)},$$

$$\begin{cases} \partial_t\delta\rho = -\rho_0\nabla\cdot\delta\boldsymbol{u}, \\ \rho_0\partial_t\delta\boldsymbol{u} = -c_S^2\nabla\delta\rho - \dfrac{1}{4\pi}\Big(\nabla\delta\boldsymbol{B}\cdot\boldsymbol{B}_0 - \boldsymbol{B}_0\cdot\nabla\delta\boldsymbol{B}\Big), \\ \partial_t\delta\boldsymbol{B} = \boldsymbol{B}_0\cdot\nabla\delta\boldsymbol{u} - \boldsymbol{B}_0\nabla\cdot\delta\boldsymbol{u}. \end{cases}$$

- Plane-wave solution: $(\delta\rho, \delta\boldsymbol{u}, \delta\boldsymbol{B}) = (\delta\hat{\rho}, \delta\hat{\boldsymbol{u}}, \delta\hat{\boldsymbol{B}})e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$

$$\underbrace{\left[\omega^2 - c_A^2k_\parallel^2 - (c_S^2 + c_A^2)\boldsymbol{k}\boldsymbol{k} + c_A^2k_\parallel(\boldsymbol{b}\boldsymbol{k} + \boldsymbol{k}\boldsymbol{b})\right]}_{\text{dispersion tensor } \boldsymbol{D}(\omega,\boldsymbol{k})}\cdot\delta\hat{\boldsymbol{u}} = 0, \iff \boldsymbol{D}(\omega,\boldsymbol{k})\cdot\delta\hat{\boldsymbol{u}} = 0,$$

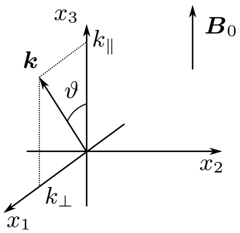
$$c_A = \frac{B_0}{\sqrt{4\pi\rho_0}} = \text{Alfvén speed}.$$

The general form of the dispersion relation is $\det\boldsymbol{D}(\omega,\boldsymbol{k}) = 0$.

MHD waves: The linear shear Alfvén wave

- Stix reference frame:

$$\boldsymbol{B}_0 = (0, 0, B_0),$$
$$\boldsymbol{k} = (k_\perp, 0, k_\parallel).$$



- Matrix form of the dispersion tensor:

$$\begin{pmatrix} \omega^2 - c_S^2 k_\perp^2 - c_A^2 k^2 & 0 & -c_S^2 k_\perp k_\parallel \\ 0 & \omega^2 - c_A^2 k_\parallel^2 & 0 \\ -c_S^2 k_\perp k_\parallel & 0 & \omega^2 - c_S^2 k_\parallel^2 \end{pmatrix} \begin{pmatrix} \delta\hat{u}_1 \\ \delta\hat{u}_2 \\ \delta\hat{u}_3 \end{pmatrix} = 0.$$

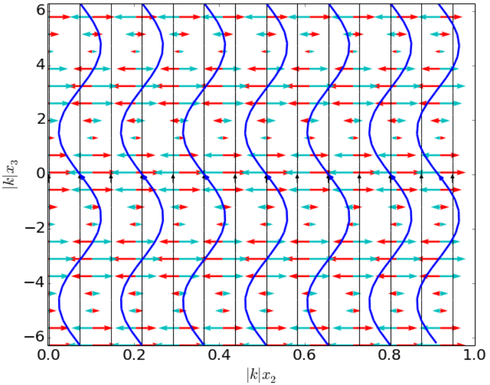
- The diagonal entry corresponds to the *shear Alfvén wave*,

$$\omega^2 = c_A^2 k_\parallel^2, \quad \text{with} \quad \delta\hat{\boldsymbol{u}} \cdot \boldsymbol{k} = 0, \quad \delta\hat{\boldsymbol{u}} \cdot \boldsymbol{B}_0 = 0,$$

and from the linearized induction equation $\omega\delta\hat{\boldsymbol{B}} = \boldsymbol{B}_0 \boldsymbol{k} \cdot \delta\hat{\boldsymbol{u}} - \boldsymbol{B}_0 \cdot \boldsymbol{k}\delta\hat{\boldsymbol{u}}$,

$$\delta\hat{\boldsymbol{B}} = -\frac{\boldsymbol{B}_0 \cdot \boldsymbol{k}}{\omega}\delta\hat{\boldsymbol{u}}.$$

MHD waves: Representation of the Alfvén wave



velocity field perturb. = red arrows   magnetic field perturb. = blue arrows

equilibrium field lines = black   perturbed field lines = blue

MHD waves: Magnetosonic waves

- The remaining 2×2 block corresponds to the compressional MHD waves:

$$\begin{pmatrix} \omega^2 - c_S^2 k_\perp^2 - c_A^2 k^2 & -c_S^2 k_\perp k_\parallel \\ -c_S^2 k_\perp k_\parallel & \omega^2 - c_S^2 k_\parallel^2 \end{pmatrix} \begin{pmatrix} \delta\hat{v}_1 \\ \delta\hat{v}_3 \end{pmatrix} = 0,$$

$$\omega^4 - (c_A^2 + c_S^2) k^2 \omega^2 + c_A^2 c_S^2 k^2 k_\parallel^2 = 0,$$

$$\omega^2 = \frac{1}{2}\Big[(c_A^2 + c_S^2) \pm \sqrt{(c_A^2 + c_S^2)^2 - 4 c_A^2 c_S^2 \cos\vartheta}\Big] k^2.$$

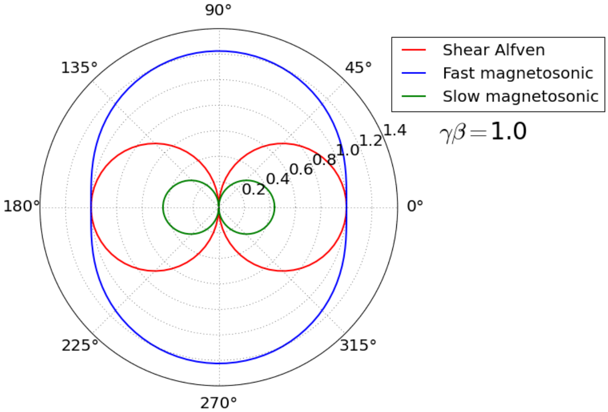
  Those are the fast $(+)$ and the slow $(-)$ magnetosonic waves.

- The names “fast” and “slow” are related to their phase velocity, which depends on the propagation angle ϑ with respect to the background magnetic field,

$$\frac{v_{\mathrm{ph}}}{c_A} = \frac{\omega}{c_A k} = f(\vartheta).$$

MHD waves: Phase velocities

- Phase velocity as a function of the angle ϑ:



- Note the strong anisotropy of the wave dispersion.

MHD waves: Nonlinear shear Alfvén waves

- Ideal incompressible MHD ($\rho = \text{constant}$):

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \dfrac{1}{4\pi\rho}\boldsymbol{B}\cdot\nabla\boldsymbol{B} = -\nabla P, \quad P = \rho^{-1}\Big(p + \dfrac{B^2}{8\pi}\Big), \\ \partial_t \boldsymbol{B} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} - \boldsymbol{B}\cdot\nabla\boldsymbol{u} = 0, \\ \nabla\cdot\boldsymbol{u} = 0. \end{cases}$$

- Elsässer variables:

$$\boldsymbol{z}_\pm = \boldsymbol{u} \pm \frac{\boldsymbol{B}}{\sqrt{4\pi\rho}}, \qquad \Longrightarrow \qquad \begin{cases} \partial_t \boldsymbol{z}_\pm + \boldsymbol{z}_\mp\cdot\nabla\boldsymbol{z}_\pm = -\nabla P, \\ \nabla\cdot\boldsymbol{z}_\pm = 0. \end{cases}$$

- In presence of a (uniform) guide field $\boldsymbol{B}_0$,

$$\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}, \qquad \boldsymbol{z}_\pm = \pm\boldsymbol{c}_A + \delta\boldsymbol{z}_\pm, \qquad \boldsymbol{c}_A = \frac{\boldsymbol{B}_0}{\sqrt{4\pi\rho}} = \text{Alfvén velocity},$$

$$\begin{cases} \partial_t \delta\boldsymbol{z}_\pm \mp \boldsymbol{c}_A\cdot\nabla\delta\boldsymbol{z}_\pm + \delta\boldsymbol{z}_\mp\cdot\nabla\delta\boldsymbol{z}_\pm = -\nabla P, \\ \nabla\cdot\delta\boldsymbol{z}_\pm = 0. \end{cases}$$

This is <u>not</u> a linearization. The only nonlinear term is $\delta\boldsymbol{z}_\mp\cdot\nabla\delta\boldsymbol{z}_\pm$.

MHD waves: Nonlinear shear Alfvén waves - cont'd

- Large-amplitude shear Alfvén waves:

$$\underbrace{\begin{cases} \delta \boldsymbol{z}_- = 0, \\ \partial_t \boldsymbol{z}_+ - \boldsymbol{c}_A \cdot \nabla \boldsymbol{z}_+ = 0, \\ \nabla \cdot \boldsymbol{z}_+ = 0, \end{cases}}_{\text{(“regressive” Alfvén wave)}} \qquad \underbrace{\begin{cases} \delta \boldsymbol{z}_+ = 0, \\ \partial_t \boldsymbol{z}_- + \boldsymbol{c}_A \cdot \nabla \boldsymbol{z}_- = 0, \\ \nabla \cdot \boldsymbol{z}_- = 0. \end{cases}}_{\text{(“progressive” Alfvén wave)}}$$

- Dispersion relation:

$$\delta \boldsymbol{z}_\pm = \zeta_\pm e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})},$$

$$\omega \mp \boldsymbol{c}_A \cdot \boldsymbol{k} = 0,$$

$$\omega^2 = c_A^2 k_\parallel^2, \quad k_\parallel = \boldsymbol{B_0} \cdot \boldsymbol{k}/B_0 = k \cos\vartheta.$$

- Polarization:

$$\delta \boldsymbol{z}_- = 0, \quad \Rightarrow \quad \delta \boldsymbol{v} = \delta \boldsymbol{B}/\sqrt{4\pi\rho}, \quad \text{when } \omega = -c_A k_\parallel,$$

$$\delta \boldsymbol{z}_+ = 0, \quad \Rightarrow \quad \delta \boldsymbol{v} = -\delta \boldsymbol{B}/\sqrt{4\pi\rho}, \quad \text{when } \omega = c_A k_\parallel.$$

Selected references

- Fluid mechanics:
  - A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Dynamics*, Springer (1993).

- Basic Magnetohydrodynamics:
  - D. D. Schnack, *Lectures on Magnetohydrodynamics*, Springer (2009).
  - D. Biskamp, *Nonlinear Magnetohydrodynamics*, Cambridge University Press (1993).

- Magnetohydrodynamics in fusion and astrophysics:
  - H. Zohm, *Magnetohydrodynamic Stability of Tokamaks*, Wiley (2014).
  - J. P. GoedBloed and S. Poedts, *Principles of Magnetohydrodynamics: With Applications to Laboratory and Astrophysical Plasmas*, Cambridge University Press (2004).
  - J. P. GoedBloed, R. Keppens and S. Poedts, *Advanced Magnetohydrodynamics: With Applications to Laboratory and Astrophysical Plasmas*, Cambridge University Press (2004).
  - E. Priest, *Magnetohydrodynamics of the Sun*, Cambridge University Press (2014).

- Magnetohydrodynamics as a Hamiltonian system:
  - P. J. Morrison, Rev. Mod. Phys. 70, 467-521 (1998).