IPP Summer University

Some Concepts of the Kinetic Theory

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Aim of Kinetic Theory: Connecting Macro- to Micro-Description

"The **extremely small** size of the basic constituents of matter is such that we cannot immediately obtain from it an image of the world at a **macroscopic** level. There are hierarchies of structures, and *new concepts* arise at *each* level. Even if the real world is made up of atoms (or even smaller particles), it is too difficult to describe what occurs in that world in terms of those basic constituents. What we can do is to **establish a bridge** between the various levels in order to form a **coherent picture**."
(C. Cercignani)

"WE'D BETTER LOOK AT THOSE PLANS AGAIN."

Part I: A few but essential physical pictures presented in a casual fashion

OUTLINE

- **Coulomb Collisions**: particle–particle interaction;
- **Irreversibility**: diffusion and Fokker-Planck equations;

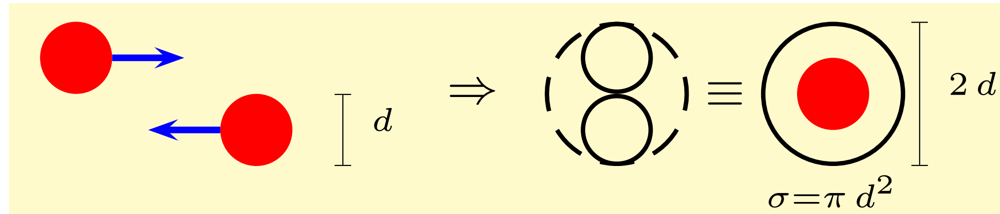
Coulomb Collisions

particle–particle interaction

Collisions in a Normal Gas: Head-on Collisions

- In a gas of **neutral** particles (of atomic diameter d) **collisions** are mainly of the "head-on" type: The interaction potential can be approximated with

$$U_n(r) = \begin{cases} 0 & \text{if} : r > d \\ \infty & \text{if} : r < 0 \end{cases}$$



The *effective collisional area* (collisional cross section) is simply $\sigma = \pi d^2$.

- The *average distance between collisions* (**mean free path**) is the traveled distance divided by the number of collisions:

$$\lambda_{\mathrm{mfp}} = \frac{\overbrace{\langle v\rangle\, t}^{\text{traveled distance}}}{\underbrace{\sigma\ \langle v_{\mathrm{rel}}\rangle\, t}_{\text{interaction vol.}}\quad n} = \frac{1}{\sqrt{2}\,\sigma\, n}, \qquad \text{with} \quad \langle v_{\mathrm{rel}}\rangle = \sqrt{2}\langle v\rangle$$

- The collision frequency is thus:

$$\nu_c = \frac{\langle v\rangle}{\lambda_{\mathrm{mfp}}} = \sqrt{2}\,\sigma\, n\, \langle \mathbf{v}\rangle \propto \sigma\, n\, \sqrt{\mathbf{T}}$$

$$\boxed{\text{In a gas the collisional frequency } \textbf{increases} \text{ with the square root of temperature}}$$

Collisions in an Ionized Gas

- The **Coulomb force** decreases **slowly** with distance:
  - *many small* deflections already present at large distances (*grazing collisions*)
  - **grazing** collisions overwhelm the **head-on** collisions.
- However, beyond the **Debye sphere** (of radius λ_D) the Coulomb force is balanced by the presence of other particles (**Debye shielding**)

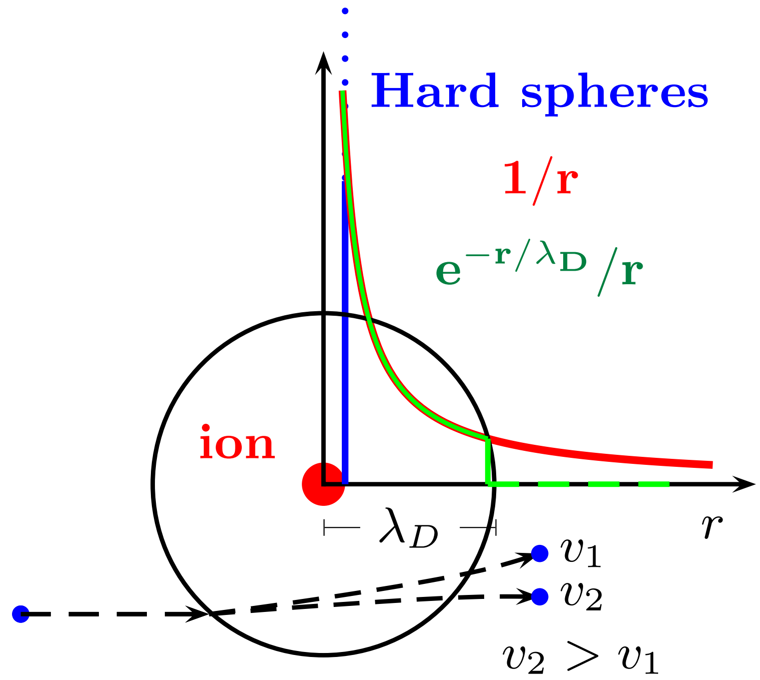
$$\lambda_D := \frac{v_t}{\sqrt{2}\,\omega_p} = \sqrt{\frac{T}{4\pi n\, e^2}}$$

$$= 7.43 \cdot 10^{-5} \sqrt{\frac{T_{[\mathrm{keV}]}}{n_{[10^{19}\mathrm{m}^{-3}]}}}\ \mathrm{m}$$

with

$$v_t := \sqrt{\frac{2T}{m}} = \text{thermal velocity}$$

$$\omega_p := \sqrt{\frac{4\pi\, e^2 n}{m}} = \text{electron plasma (angular) frequency}$$



Collisions in an Ionized Gas: Definition of the Collision Frequency

The collisional time $\tau_c = \nu_c^{-1}$ is the time required by *random* Coulomb interactions to change the velocity of a test particle of an amount Δv equal to the thermal velocity

$$\frac{\langle \Delta v^2 \rangle}{v_t^2} = 1$$

We need to consider $\langle \Delta v^2 \rangle$, since $\langle \Delta v \rangle = 0$ due to the randomness of the motion.

In the following slides,

- We evaluate ν_c by expressing $\langle \Delta v^2 \rangle$ in terms of τ_c.
- In analogy with the estimate done for the normal gas, we follow test particle α colliding against an ensemble of field particles β, which are not necessarily of the same type of α.

Collisions in an Ionized Gas: Test particle α colliding against field particles β

In a collisional time $\tau_c^{\alpha\to\beta}$ a test particle α undergoes on average $N_c = \tau_c^{\alpha\to\beta}/t_{\rm sc}^{\alpha\to\beta}$ ($t_{\rm sc}^{\alpha\to\beta}$ the time of the elementary interactions) small velocity changes δv which are

- *random*: $\langle \delta v \rangle = 0$
- *uncorrelated*: $\langle \delta v_i \delta v_{j\neq i} \rangle = 0$

Therefore, in a collisional time we have

$$\langle \Delta v_\alpha^2 \rangle = \langle \left(\sum_{i=1}^{N_c} \delta v_i \right)^2 \rangle = \sum_{i=1}^{N_c} \langle \delta v_i^2 \rangle = N_c \langle \delta v^2 \rangle$$

- $t_{\rm sc}^{\alpha\to\beta}$ is proportional to the time necessary to the test particles to cross the Debye sphere, $t_{\rm sc}^{\alpha\to\beta} \propto \lambda_D / \langle v_{\rm rel} \rangle$.
- δv is valuated by assuming that $t_{\rm sc}^{\alpha\to\beta}$ is short enough that the particle experiences a constant acceleration

$$\delta v \approx t_{\rm sc}^{\alpha\to\beta} \, \frac{Z_\alpha \, \delta E_\beta}{m_\alpha}$$

with δE_β the electric field due to the field particles.

Collisions in an Ionized Gas: Test particle α colliding against field particles β (con't)

$$\delta v \approx t_{\rm sc}^{\alpha\to\beta}\ \frac{Z_\alpha\ \delta E_\beta}{m_\alpha}$$

- δE_β are *random*, $\langle \delta E_\beta \rangle = 0$, and *decorrelated*, $\langle \delta E_{\beta_i} \delta E_{\beta_{j\neq i}} \rangle = 0$
- Nature of Coulomb collisions in a plasma
  - *Grazing*: $\delta E_\beta = Z_\beta / \lambda_D^2$ (at the Debye sphere),
  - *Many*: $N_D \propto n_\beta \lambda_D^3$ (the number of field particles in the Debye sphere).

Using these reasonable assumptions, we have

$$\langle \delta v^2 \rangle = \left(t_{\rm sc}^{\alpha\to\beta} \frac{Z_\alpha}{m_\alpha} \right)^2\ \langle \sum_{\beta_i=1}^{N_D} \delta E_{\beta_i}^2 \rangle = \left(t_{\rm sc}^{\alpha\to\beta} \frac{Z_\alpha}{m_\alpha} \right)^2\ N_D\ \langle \delta E_\beta^2 \rangle$$

The collisional frequency becomes

$$\boxed{\nu_c^{\alpha\to\beta} \propto \frac{1}{t_{\rm sc}^{\alpha\to\beta}}\ \frac{\langle \delta v^2 \rangle}{v_{{\rm t},\alpha}^2} = \frac{Z_\alpha^2\ Z_\beta^2\ n_\beta}{m_\alpha^2\ \langle v_{\rm rel} \rangle\ v_{t,\alpha}^2}}$$

(from previous slide $t_{\rm sc}^{\alpha\to\beta} \propto \lambda_D / \langle v_{\rm rel} \rangle$)

Collisions in an Ionized Gas: Collision Frequency

$$\nu_c^{\alpha\rightarrow\beta} \propto \frac{Z_\alpha^2\ Z_\beta^2\ n_\beta}{m_\alpha^2\ \langle v_{\rm rel}\rangle\ v_{t,\alpha}^2}$$

Since the velocity of the test and field particles are uncorrelated,

$$\langle v_{\rm rel}\rangle = \sqrt{v_{t,\alpha}^2 + v_{t,\beta}^2} = \sqrt{\frac{2T_\alpha}{m_\alpha} + \frac{2T_\beta}{m_\beta}} = \sqrt{2T}\sqrt{\frac{m_\alpha + m_\beta}{m_\alpha m_\beta}}$$

where the last equality holds if we assume $T_\alpha = T_\beta = T$, it follows

$$\boxed{\nu_c^{\alpha\rightarrow\beta} \propto Z_\alpha^2\ Z_\beta^2\ \sqrt{\frac{m_\beta}{m_\alpha(m_\alpha + m_\beta)}}\ \frac{n_\beta}{\mathbf{T^{3/2}}}}$$

In a plasma the collisional frequency **decreases** with temperature

Collisions in an Ionized Gas: Energy Exchange Rate

Temperature equilibration is described by

$$\frac{\mathrm{d}T_\alpha}{\mathrm{d}t} = \sum_\beta \nu_E^{\alpha\to\beta} \left(T_\beta - T_\alpha\right)$$

- With collisions particles exchange not only momentum but also energy.
- Since the scattering is elastic, energy conservation imposes a symmetry on the *energy exchange rate*:

$$n_\alpha \, \nu_E^{\alpha\to\beta} = n_\beta \, \nu_E^{\beta\to\alpha} \qquad (1)$$

- $\nu_E^{\alpha\to\beta}$ is proportional to the collisional frequency $\nu_c^{\alpha\to\beta}$, and to guarantee the symmetry (1) we symmetrize the dependence on the masses in $\tau_c^{\alpha\to\beta}$

$$\nu_E^{\alpha\to\beta} \propto \frac{Z_\alpha^2 \, Z_\beta^2 \, n_\beta}{m_\alpha m_\beta \, (\langle v_{\mathrm{rel}}\rangle)^3} \propto \frac{m_\alpha}{m_\beta + m_\alpha} \, \nu_c^{\alpha\to\beta}$$

$$\propto Z_\alpha^2 \, Z_\beta^2 \sqrt{\frac{m_\alpha \, m_\beta}{(m_\alpha + m_\beta)^3}} \, \frac{n_\beta}{T^{3/2}}$$

Note: This is just a heuristic derivation, which however gives acceptable dependencies and orderings.

Collisions in an Ionized Gas: Ordering

Summarizing:

$$\nu_c^{\alpha\to\beta} \approx \sqrt{\frac{m_\beta}{m_\alpha\left(m_\alpha+m_\beta\right)}}\;\frac{n_\beta}{T^{3/2}}$$

$$\nu_E^{\alpha\to\beta} = \nu_E^{\beta\to\alpha} \propto \frac{m_\alpha}{m_\beta+m_\alpha}\,\nu_c^{\alpha\to\beta}$$

recalling that $m_p/m_e \approx 1836$ and $\sqrt{m_p/m_e} \approx 43$ and normalizing with respect to $\nu_c^{e\to e}$, in a proton-e plasma the ordering is

$$\boxed{\nu_E^{e\to e} \propto \nu_c^{e\to e}, \quad \nu_E^{e\to p} \propto \frac{\mathbf{m_e}}{\mathbf{m_p}}\,\nu_c^{e\to e}, \quad \nu_E^{p\to p} \propto \sqrt{\frac{\mathbf{m_e}}{\mathbf{m_p}}}\,\nu_c^{e\to e}, \quad \nu_E^{p\to e} \propto \frac{\mathbf{m_e}}{\mathbf{m_p}}\,\nu_c^{e\to e}}$$

Collisions in an Ionized Gas: Relaxation Times (proton-e gas)

$$\nu_{\rm coll}^{e\to e} : \nu_{\rm coll}^{e\to i} : \nu_E^{e\to e} : \nu_{\rm coll}^{i\to i} : \nu_E^{i\to i} : \nu_{\rm coll}^{i\to e} : \nu_E^{i\to e} : \nu_E^{e\to i}$$

$$1 : 1 : 1 : \sqrt{\frac{m_e}{m_i}} : \sqrt{\frac{m_e}{m_i}} : \frac{m_e}{m_i} : \frac{m_e}{m_i} : \frac{m_e}{m_i}$$

How does an electron-proton gas evolve in time starting from an arbitrary velocity distribution function with T_e and T_i of the same order?

1 $\quad e \to i \Rightarrow \quad f_e \longrightarrow$ isotropic - no preferred direction

$e \to e \Rightarrow \quad f_e \longrightarrow$ Maxwellian

$\sqrt{\frac{\mathbf{m_i}}{\mathbf{m_e}}}$ $\quad \mathbf{i} \to \mathbf{i} \Rightarrow \quad f_i \longrightarrow$ Maxwellian

$\frac{\mathbf{m_i}}{\mathbf{m_e}}$ $\quad i \to e$ and $e \to i \quad \Rightarrow T_e = T_i$

time

Collisions in an Ionized Gas: In Numbers ...

$$\tau_{\mathbf{c}}^{\mathbf{e}\to\mathbf{e}} = \mathbf{1}/\nu_{\mathbf{c}}^{\mathbf{e}\to\mathbf{e}}$$

Density	T=0.1 keV	T = 1keV	T = 10keV
$10^{19}\mathrm{m}^{-3}$	2.4μs	67μs	1.9 ms
$10^{20}\mathrm{m}^{-3}$	0.27μs	7.2μs	0.2 ms

$$\tau_{\mathbf{c}}^{\mathbf{D}\to\mathbf{D}} = \mathbf{1}/\nu_{\mathbf{c}}^{\mathbf{D}\to\mathbf{D}}$$

Density	T=0.1 keV	T = 1keV	T = 10keV
$10^{19}\mathrm{m}^{-3}$	0.2 ms	5 ms	0.13 s
$10^{20}\mathrm{m}^{-3}$	21μs	0.54 ms	14 ms

$$\tau_{\mathbf{E}}^{\mathbf{e}\leftrightarrow\mathbf{D}} = \mathbf{1}/\nu_{\mathbf{E}}^{\mathbf{e}\leftrightarrow\mathbf{D}}$$

Density	T=0.1 keV	T = 1keV	T = 10keV
$10^{19}\mathrm{m}^{-3}$	4.4ms	120 ms	3.4 s
$10^{20}\mathrm{m}^{-3}$	0.49 ms	13 ms	0.37 s

from Wesson, Tokamaks

Collisions in an Ionized Gas: What We Have Learned

- Coulomb collisions are *grazing* and *many*.
- Contrary to a normal gas, in a plasma the collision frequency decreases with the temperature.
- Collisions are responsible for the isotropization of the distribution function and for the energy equipartition.

Random walk and Diffusion Equation

Fokker-Planck Equation

Kinetic Theory

- The aim of the Kinetic Theory (KT) is to describe a gas of many particles to make possible the interpretations/predictions of *macroscopic* quantities (e.g. density, temperature ...) starting from the *microscopic* (position of each particle in phase space) descriptions of the gas.
- This is achieved with a quantity called distribution function, $f(\mathbf{x}, \mathbf{v}, t)$, which has two possible interpretations: 1) (*deterministic*) approximation of the *true* gas density in phase space; 2) (*probabilistic*) *probability* to find the gas in a given configuration in the phase space.
- The interpretation you choose has no consequences on the final results.

Kinetic Theory: Irreversibility

- What we need to describe the motion of particles in a gas are simply the Netwon's laws for the particle motion and the Maxwell's equations for the associated electric and magnetic fields inter-connected via the constitutive relations.
- These equations are all reversible in time, i.e. are invariant with respect to the transformation

$$t \to -t \,, \qquad \mathbf{x} \to \mathbf{x} \,, \qquad \mathbf{v} \to -\mathbf{v}$$

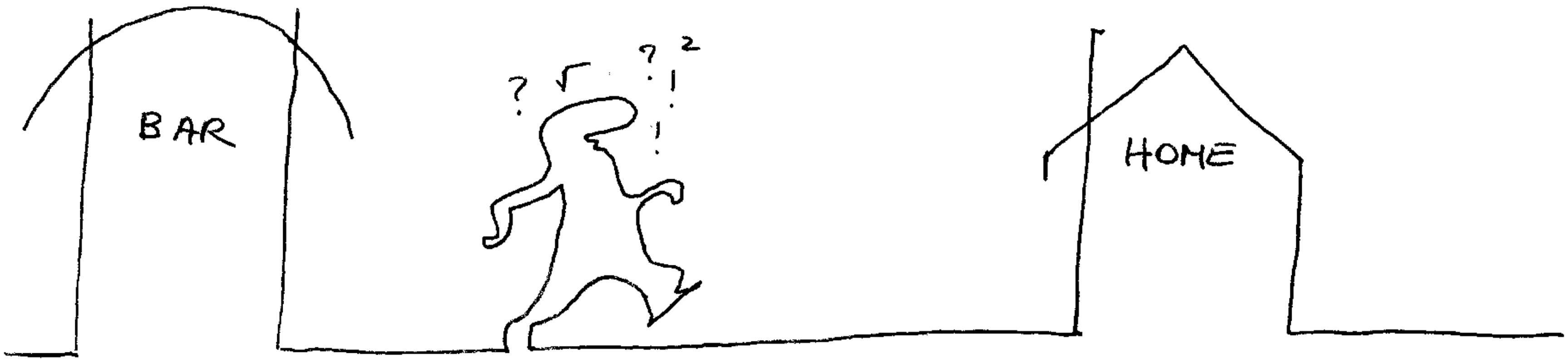
- BUT our experience with normal gases teaches us that the phenomena are typically *irreversible*. How can KT resolve this contradiction?
- For instance, we are used to see equation for f of the type

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{D} \cdot \frac{\partial}{\partial \mathbf{v}} f \right) = S$$

  which is manifestly *not* time-symmetric.

To grasp the flavour of the probabilistic concept of the gas description and how the irreversibility enters, we will investigate the simple **Drunk's problem**.

The Drunk Man Problem



Just imagine a **one-dimensional** drunk man, who, leaving a bar, wants to go home. The *rules* of his wandering are *simply*

- Steps of equal length, i.e. Δx
- At each step the drunk can go either on the right or on the left with *equal probability*.

Question: If home is **m** steps away from the bar, which is the probability that the drunk reaches home after $n(> m)$ steps?

Recall of few Statistical Concepts

- $P(n|m,\ s)$: **conditional probability** – if at time $t=0$ the drunk is $n\Delta x$ away from the Bar, $P(n|m,\ s)$ is the probability to find him $m\Delta x$ away a time $t = s\,\Delta t$.

The equation for $P(n|m,\ s)$ (Smoluchowski's equation - Markoff series)

$$P(n|m,s) = \sum_k P(n|k,s-1)\ P(k|m,1)$$

$\Longrightarrow$ of the whole trajectory only the very last position in the past matters for the present.
In particular, since in the case of the drunk we have the *drunk's rule*

$$P(k|m,1) = \mathbf{\frac{1}{2}}\ \delta_{m,k-1} + \mathbf{\frac{1}{2}}\ \delta_{m,k+1}\ ,$$

we have

$$P(n|m,s) = \frac{1}{2}\ \underbrace{P(n|m+1,s-1)}_{\text{from the right}} + \frac{1}{2}\ \overbrace{P(n|m-1,s-1)}^{\text{from the left}}$$

Diffusion Equation

The Smoluchowski's equation for the discrete random walk is

$$P(n|m,s) = \frac{1}{2}\,P(n|m+1,s-1) + \frac{1}{2}\,P(n|m-1,s-1)$$

By adding $P(n|m,s-1)$ to both sides, we have

$$P(n|m,s) - P(n|m,s-1) = \frac{1}{2}\left[P(n|m+1,s-1) - 2\,P(n|m,s-1) + P(n|m-1,s-1)\right]$$

which can be re-written as

$$\Delta t\;\frac{P(n|m,s) - P(n|m,s-1)}{\Delta t} =$$

$$\frac{(\Delta x)^2}{2}\left[\frac{\frac{P(n|m+1,s-1) - P(n|m,s-1)}{\Delta x} - \frac{P(n|m,s-1) - P(n|m-1,s-1)}{\Delta x}}{\Delta x}\right]$$

Diffusion Equation (con't)

In the limit that $t = s\Delta t$ and $x = m\Delta x$ become **continuous** variables, the Smoluchowski's equation for the discrete random walk,

$$P(n|m,s) = \frac{1}{2}\,P(n|m+1,s-1) + \frac{1}{2}\,P(n|m-1,s-1)\ ,$$

becomes the diffusion equation

$$\boxed{\frac{\partial}{\partial t}P(x_0|x,t) = D\,\frac{\partial^2}{\partial x^2}P(x_0|x,t)}$$

with the constraints that

- The diffusion coefficient,

  $$D = \lim_{\Delta t, \Delta x \to 0} \frac{(\Delta x)^2}{2\,\Delta t}$$ **stays finite**.

  (In other words, the spread increases with the square root of time.)

- D does **not** depend on x, so that

  $$\frac{\partial}{\partial t}\int P(x_0|x,t)\mathrm{d}x = 0$$

  the "drunk" does **not disappear**! (Conservation law of the drunk).

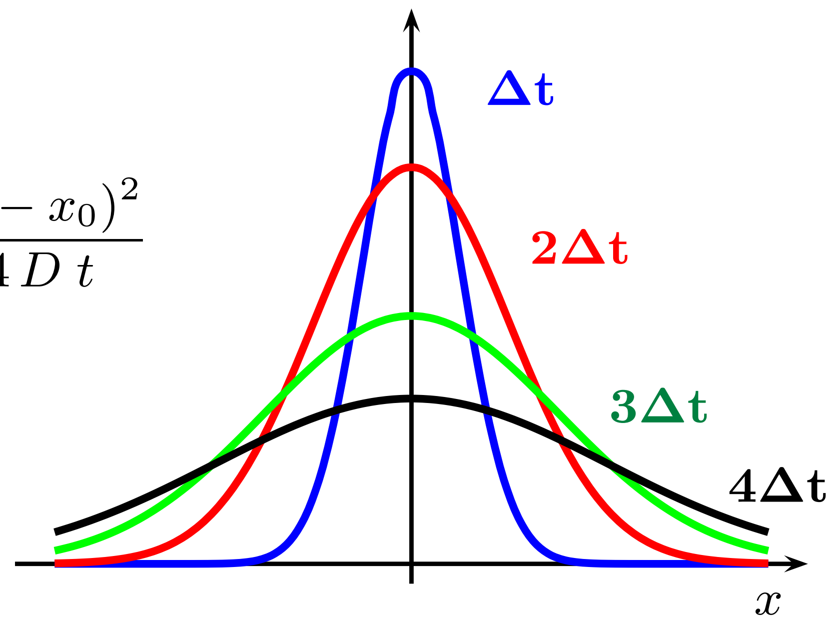
Fundamental Solution of the Diffusion Equation

A last word on the diffusion equation,

$$\frac{\partial}{\partial t} P(x_0|x,t) = D\ \frac{\partial^2}{\partial x^2} P(x_0|x,t)$$

Starting at $t = 0$ with the Dirac (distribution) function $\delta(x - x_0)$ the solution of this parabolic equation is:

$$P(x_0|x,t) = \frac{1}{\sqrt{4\pi\, D\, t}}\ e^{-\dfrac{(x-x_0)^2}{4\,D\,t}}$$



In words, an initially localized function irreversibly smoothed out with time.

Fokker–Planck Equation (I)

Now we relax the constraint that D is constant, i.e. go beyond the *drunk's rule*.

- Start from the time derivative of $P(x|y,t)$

$$\int \mathrm{d}y\; R(y)\,\frac{\partial}{\partial t}P(x|y,t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int \mathrm{d}y\; R(y)\,\Big[P(x|y,t+\Delta t) - P(x|y,t)\Big]$$

  where $R(y)$ is an arbitrary function such that $\lim_{y\to\pm\infty} R(y) = 0$.

- Use the Smoluchowski's equation for the continuous Markovian process

$$P(x|y,t+\Delta t) = \int \mathrm{d}z\; P(x|z,t)\,P(z|y,\Delta t)$$

  to obtain

$$\int \mathrm{d}y\; R(y)\,\frac{\partial}{\partial t}P(x|y,t) =$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\int \mathrm{d}z\; P(x|z,t)\int \mathrm{d}y\, R(y)P(z|y,\Delta t)\; - \int \mathrm{d}y\; R(y)\; P(x|y,t)\right\}$$

Fokker–Planck Equation (II)

- We replace $R(y)$ with its Taylor series around z,

$$\int \mathrm{d}z\ R(z)\ \frac{\partial}{\partial t} P(x|z,t) = \int \mathrm{d}z\ P(x|z,t) \left[A_1(z)\, R'(z) + \frac{1}{2} A_2(z)\, R''(z) + ... \right]$$

with the coefficient of the Taylor series,

$$A_n(z) := \lim_{\Delta t \to 0} \frac{\langle z^n \rangle}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int \mathrm{d}y\, (y-z)^n\ P(z|y,\Delta t)$$

- If we <u>**assume**</u> that:

$$\boxed{A_n(z) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int \mathrm{d}y (y-z)^n\ P(z|y,\Delta t) = \mathbf{0} \quad \text{for} \quad \mathbf{n > 2}}$$

we have

$$\int \mathrm{d}z \left\{ R(z)\ \frac{\partial}{\partial t} P(x|z,t) - P(x|z,t) \left[A_1(z)\, R'(z) + \frac{1}{2} A_2(z)\, R''(z) \right] \right\} = 0$$

Fokker–Planck Equation (III)

$$\int \mathrm{d}z \left\{ R(z)\, \frac{\partial}{\partial t} P(x|z,t) - P(x|z,t) \left[A_1(z)\, R'(z) + \frac{1}{2} A_2(z)\, R''(z) \right] \right\} = 0$$

By integrating by part (recalling that $\lim_{y \to \pm\infty} R(y) = 0$), we have

$$\int \mathrm{d}z\ R(z)\ \left\{ \frac{\partial}{\partial t} P(x|z,t) + \frac{\partial}{\partial z} \Big(A_1(z)\ P(x|z,t) \Big) - \frac{1}{2} \frac{\partial^2}{\partial z^2} \Big(A_2(z)\ P(x|z,t) \Big) \right\} = 0$$

Since this must be satisfied for any $R(z)$, $P(x|z,t)$ must be solution of:

$$\boxed{\frac{\partial}{\partial t} P(x|z,t) = -\frac{\partial}{\partial z} \Big(A_1(z)\ P(x|z,t) \Big) + \frac{1}{2} \frac{\partial^2}{\partial z^2} \Big(A_2(z)\ P(x|z,t) \Big)}$$

This is the **Fokker-Planck** equation whose coefficients depend only on the *first* and *second* order moments of the changes.

1^{st} Remark on the Fokker–Planck Equation

The Smoluchowski equation (Markoff process)

$$P(x|y, t + \Delta t) = \int \mathrm{d}z \; P(x|z, t) \, P(z|y, \Delta t)$$

becomes the Fokker-Planck equation

$$\frac{\partial}{\partial t} P(x|z, t) = -\overbrace{\frac{\partial}{\partial z}\Big(A_1(z)\; P(x|z, t)\Big)}^{\text{friction}} + \overbrace{\frac{1}{2}\frac{\partial^2}{\partial z^2}\Big(A_2(z)\; P(x|z, t)\Big)}^{\text{diffusion}}$$

under the <u>**assumption**</u>

$$\boxed{A_n(z) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int \mathrm{d}y (y - z)^n \; P(z|y, \Delta t) = \mathbf{0} \quad \text{for} \quad \mathbf{n > 2}}$$

which means that in these processes in **small times** the coordinates can only change of **small amounts**.
In general, without this assumption the Smoluchowski equation becomes an integro-differential equation of the same type of the **Boltzmann equation**.

2^{nd} Remark on the Fokker–Planck Equation

- Let us re-name $A_2(z) = 2\ D(z)$, if

$$A_1(z) = -\frac{\mathrm{d}}{\mathrm{d}z} D(z) \qquad (1)$$

then the Fokker-Planck equation can be written as a diffusion equation with non constant D

$$\frac{\partial}{\partial t} f(z,t) = \frac{\partial}{\partial z}\left(D(z)\ \frac{\partial}{\partial z} f(z,t) \right) \qquad (2)$$

Note: All Hamiltonian systems satisfy (1)!

- An important property of the diffusion equation (and of the Fokker-Planck one) is:

If $f > 0$ is solution of (2) and is L^1 integrable, namely

$$I(t) = \int_\Omega f(z,t)\ \mathrm{d}z \quad \text{is finite}$$

then $I(t)$ is constant in time, in other words I is conserved.

Ex.: if f is the plasma density, then the total density is conserved.

Note: In more dimensions ($d > 1$), equation (1) becomes

$$\frac{\partial}{\partial t} f(z,t) = \frac{1}{J} \frac{\partial}{\partial z^i}\left(J\ D^{ij}(z)\ \frac{\partial}{\partial z^j} f(z,t) \right)$$

with J the Jacobian associated with the coordinate system z^i.

Part II: Some basic concepts presented in a more rigorous fashion

OUTLINE

- **Distribution function** & **kinetic equation**: statistical description;
- **Collision operator**: Landau formulation;
- **Landau Damping**: wave–particle interaction;

Introduction

Distribution Function

Kinetic Equation

Discrete Plasma Description

The **state** of a set of particles mutually interacting and subject to external forces is *completely* determined by a system of coupled Ordinary Differential Equations (ODE), namely the **Newton equations**

$$\boxed{m_i\, \ddot{\mathbf{X}}_i = \mathbf{F}_i\Big(\equiv \sum_{j\neq i} \mathbf{F}_{j\to i}(t) + \mathbf{F}_{\rm ext}\Big)}$$

plus a set of **initial value conditions** $\{\mathbf{X}_i(0), \dot{\mathbf{X}}_i(0) := \mathbf{V}_i(0)\}_{1\leq i\leq N}$.
$\mathbf{F}_{j\to i}$ is the force exerted by particle j on particle i, and $\mathbf{F}_{\rm ext}$ the total external force.

. . . Complexity Issues . . .

- **Initial Knowledge**: the lack of information about the **Initial Conditions** forces us to tackle the problem from a **statistical** point of view, namely as an **ensemble average** over all possible initial conditions.
- **Dimensionality**: the computational effort to follow all the particles of a laboratory plasma ($\approx 10^{22}$) is . . . at the moment . . . technically *prohibitive*.
- **Distillation**: The measured quantities are *average* quantities: a detailed knowledge of the motion of each particle is not only *redundant* but also makes *troublesome* and *lengthy* the interpretation of the numerical results.

AIM

Replace a **very large** number of simple **ODE** equations (one for each particle) with only **ONE Partial differential** equation (PDE) for a **function** which is defined in a **reduced** space of variables and which is able to thoroughly describe the plasma properties (*measurements*).

... in other words ... we need to define/determine

- the **state** of a plasma;
- the **measure** (particle counter) in the space of plasma states;
- the **equation** for this measure.

Dynamical State and its Measure

Let us *double* the number of variables by splitting the second-order ODE for the particle motion in two first-order ODEs,

$$\dot{\mathbf{X}}_i = \mathbf{V}_i, \qquad \text{and :} \quad \dot{\mathbf{V}}_i = \frac{\mathbf{F}_i}{m_i} = \frac{1}{m_i}\sum_{j\neq i}\mathbf{F}_{j\to i}(t) + \frac{1}{m_i}\mathbf{F}_{\rm ext}$$

- Thus, the state of an N-particle system at time t is described by a point in the $(\mathbf{x},\mathbf{v})^N = (\mathbf{x}_1,\ldots,\mathbf{x}_2;\mathbf{v}_1,\ldots,\mathbf{v}_N)$ phase space $\Omega = (\mathbb{T}^3\times\mathbb{R}^3)^N$ ($\mathbb{T}$ stands for the torus, since we assume a confined system with periodic boundary conditions), and this is called the **dynamical** state of the system.
- The most natural **measure** in this phase space is

$$P_N(B\subseteq\Omega;t) = \int_B \left(\prod_{i=1}^N \delta(\mathbf{x}_i - \mathbf{X}_i(t))\ \delta(\mathbf{v}_i - \mathbf{V}_i(t))\right)\ \prod_{i=1}^N \mathrm{d}\mathbf{x}_i\mathrm{d}\mathbf{v}_i$$

where $\delta(x)$ is the Dirac delta function.

- $P_N(B\subseteq\Omega;t)$ = probability that at time t the state of the system is in $B\subseteq\Omega$.
$\Longrightarrow$ Note, $P_N(B\subseteq\Omega;t)$ can be either 0 or 1.

Distribution Function and its Equation

- In the space of the dynamical states, Ω, we can define the **density distribution function** associated with the system,

$$f_N(\mathbf{x}_1, \mathbf{v}_1, \ldots, \mathbf{x}_N, \mathbf{v}_N; t) := \prod_{i=1}^{N} \delta(\mathbf{x}_i - \mathbf{X}_i(t))\, \delta(\mathbf{v}_i - \mathbf{V}_i(t))$$

so thus

$$P_N(B \subseteq \Omega; t) = \int_B f_N(\mathbf{x}_1, \mathbf{v}_1, \ldots, \mathbf{x}_N, \mathbf{v}_N; t) \prod_{i=1}^{N} \mathrm{d}\mathbf{x}_i \mathrm{d}\mathbf{v}_i$$

- For Hamiltonian systems the continuity equation of the density distribution function together with the divergence theorem gives the Liouville equation

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^{N} \left[\mathbf{v}_i \cdot \frac{\partial f_N}{\partial \mathbf{x}_i} + \frac{F_i}{m} \cdot \frac{\partial f_N}{\partial \mathbf{v}_i} \right] = 0$$

What did we learn?

- **state**, $\mathbf{z} := (\mathbf{x}, \mathbf{v})^N \in \Omega$;
- **measure**, $f_N(\mathbf{z}; t)$;
- **equation** of state for f_N.

BUT
All the complexity is still there!

Relaxing the Initial Knowledge

- The use of Dirac delta function, $\delta(\mathbf{z})$ with $\mathbf{z} := (\mathbf{x}, \mathbf{v})$, implies that we *exactly* know the initial conditions of each particle.

- As proxy of the *probabilistic* interpretation of f_N, we can replace $\delta(x)$ with a Gaussian

$$\frac{1}{(\pi\sigma^2)^{1/2}} e^{-\dfrac{x^2}{\sigma^2}} \xrightarrow{\sigma\to 0} \delta(x)$$

  and f_N is an *ensemble average* over the possible initial conditions in phase space.

- $P_N(B \subseteq \Omega; t)$ is not anymore restricted to the values 0 and 1, but can assume values between 0 and 1.

  $f_N(\mathbf{x}_1, \mathbf{v}_1, \ldots, \mathbf{x}_N, \mathbf{v}_N; t) \prod_{i=1}^N \mathrm{d}\mathbf{x}_i \mathrm{d}\mathbf{v}_i$ is the probability of finding the state of the system within the volume $\prod_{i=1}^N \mathrm{d}\mathbf{x}_i \mathrm{d}\mathbf{v}_i$ around $(\mathbf{x}_1, \mathbf{v}_1, \ldots, \mathbf{x}_N, \mathbf{v}_N)$ at time t.

  $f_N(\mathbf{x}_1, \mathbf{v}_1, \ldots, \mathbf{x}_N, \mathbf{v}_N; t)$ is still solution of Liouville equation with initial conditions.

What did we achieve?

Conceptually, we have done a substantial improvement, but the complexity due to high dimensionality ($6N$-dimensional state space) is still there!

Reducing the Dimensionality

- At the price of loosing some of the microscopic details, the dimensionality of the phase space can be reduced to $(\mathbb{T}^3 \times \mathbb{R}^3)^s$, with $s < N$, by defining the measure

$$f_s(\mathbf{x}_1, \mathbf{v}_1; \ldots; \mathbf{x}_s, \mathbf{v}_s|t) = \frac{N!}{(N-s)!} \int f_N(\mathbf{x}_1, \mathbf{v}_1; \ldots; \mathbf{x}_N, \mathbf{v}_N|t) \prod_{i=s+1}^{N} \mathrm{d}\mathbf{x}_i \mathrm{d}\mathbf{v}_i$$

  with f_s called the **reduced distribution function**.

  The combinatorial factor comes from the fact that the particles are identical and thus there are *indistinguishable*.

Distribution Function

Which of the reduced f_s (with: $s \geq 1$) are of interest?

The **macroscopic** description of a system, is usually done in a 6-dim space ($\mathbf{x}_1$ and $\mathbf{v}_1$) $\Longrightarrow$ the best we can do is to reduce the plasma description to:

$$f(\mathbf{x}, \mathbf{v}, t) \equiv f_1(\mathbf{x}_1, \mathbf{v}_1, t) = N \int f_N(\mathbf{x}_1, \mathbf{v}_1; \ldots; \mathbf{x}_N, \mathbf{v}_N | t) \prod_{i=2}^{N} \mathrm{d}\mathbf{x}_i \mathrm{d}\mathbf{v}_i$$

called "**distribution function**".

A dual *interpretation* of f_1 is the *deterministic* one,

$\mathbf{f}(\mathbf{x}, \mathbf{v}, \mathbf{t})\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{v}$ = **density** of particles in the (phase-space) volume element $\mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{v}$ centered at $(\mathbf{x}, \mathbf{v})$.

What did we achieve?

The problem dimensionality has been drastically reduced from $(6N + 1)$ to $(6 + 1)$. We need the equation for the evolution of $f(\mathbf{x}, \mathbf{v}, t)$, called **KINETIC EQUATION (KE)**

How to Derive the KE

To derive the **Kinetic Equation** for $f(\mathbf{x}, \mathbf{v}, t)$

$$\mathbf{N} \int \left\{ \frac{\partial}{\partial t} + \sum_{i=1}^{N} \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} + \sum_{i=1}^{N} \frac{\mathbf{F}_i}{m_i} \cdot \frac{\partial}{\partial \mathbf{v}_i} \right\} f_N(\mathbf{x}_1, \mathbf{v}_1; \ldots; \mathbf{x}_N, \mathbf{v}_N | t) \prod_{\mathbf{i=2}}^{\mathbf{N}} \mathrm{d}\mathbf{x_i} \mathrm{d}\mathbf{v_i} = 0$$

we need to

- separate $\mathbf{F}_i$ in an *internal* $\mathbf{F}_i^{\mathrm{int}}$ (interaction with the particles of the same species) and an *external* components $\mathbf{F}_i^{\mathrm{ext}}$ (interaction with particles of other species and external forces).
- recall that for the Lorentz force $\mathbf{F}$ it holds $\nabla_{\mathbf{v}} \cdot \mathbf{F} = 0$
- simplify with the *non-relativistic* limit of the internal **electrostatic** forces

$$\mathbf{F}_i^{\mathrm{int}} = \frac{(Ze)^2}{m} \sum_{j=1, j\neq i}^{N} \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} = -\frac{1}{m} \sum_{j=1, j\neq i}^{N} \frac{\partial \phi_{ij}}{\partial \mathbf{x}_i}$$

- assume that f goes to zero for $|\mathbf{v}| \to \infty$ and with appropriate boundary conditions in $\mathbf{x}$.

The Pair Correlation Function

After some algebra, we obtain:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \frac{\mathbf{F}^{\mathrm{ext}}}{m} \cdot \frac{\partial}{\partial \mathbf{v}_1} \right) f_1(\mathbf{z}_1, t) = \frac{1}{m} \int \frac{\partial \phi_{12}}{\partial \mathbf{x}_1} \cdot \frac{\partial}{\partial \mathbf{v}_1} \mathbf{f_2(z_1, z_2, t)} \, \mathrm{d}\mathbf{z}_2$$

The presence of f_2 on the rhs imposes the problem of **closure**.
Let us **assume**:

$$f_2(\mathbf{z}_1, \mathbf{z}_2, t) = f_1(\mathbf{z}_1, t) \, f_1(\mathbf{z}_2, t) + \mathbf{g(z_1, z_2, t)}$$

$g(1,2,t)$ accounts for the **correlation** between (1) and (2) particles.

$$\left[\frac{\partial}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \frac{1}{m} \left(\mathbf{F}^{\mathrm{ext}} - \int \frac{\partial \phi_{\mathbf{12}}}{\partial \mathbf{x_1}} \mathbf{f}(\mathbf{z_2}) \mathrm{d}\mathbf{z_2} \right) \cdot \frac{\partial}{\partial \mathbf{v}_1} \right] f(\mathbf{z}_1, t) \quad = \quad \frac{1}{m} \int \frac{\partial \phi_{12}}{\partial \mathbf{x}_1} \mathbf{g(z_1, z_2, t)} \, \mathrm{d}\mathbf{z}_2$$

Kinetic Equation

- The **uncorrelated** contribution.

$$-\mathbf{eE}^{\mathrm{self}}(\mathbf{x_1},\mathbf{t}) = -e\frac{\partial\phi^{\mathrm{self}}}{\partial\mathbf{x}_1}$$

with ($n(\mathbf{x},t)$ is the particle density defined later)

$$\phi^{\mathrm{self}}(\mathbf{x}_1,t) := \int \phi_{12}(\mathbf{x}_1,\mathbf{x}_2)f(\mathbf{x}_2,\mathbf{v}_2,t)\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{v}_2 = \int \phi_{12}(\mathbf{x}_1,\mathbf{x}_2)n(\mathbf{x}_2,t)\,\mathrm{d}\mathbf{x}_2$$

is the *average* electric field experienced by one particle due to the other particles considered as a whole.
The potential ϕ^{self} is solution of the Poisson equation and computed assuming that the particles are *uncorrelated*.

- The **correlated** contribution,

$$\left(\frac{\partial\mathbf{f}}{\partial\mathbf{t}}\right)_{\mathbf{c}} = \frac{1}{m}\int\frac{\partial\phi_{12}}{\partial\mathbf{x}_1}\,g(1,2)\,\mathrm{d}\mathbf{x}_2\,\mathrm{d}\mathbf{v}_2$$

is called the **collision term** and is an integral-differential operator. This term gives account of the **discreteness** (binary interaction) of the particle interactions.

Finally,

$$\boxed{\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}} + \frac{1}{m}\left(\mathbf{F}^{\mathrm{ext}} + e\mathbf{E}^{\mathrm{self}}\right)\cdot\frac{\partial}{\partial\mathbf{v}}\right] f = \left(\frac{\partial f}{\partial t}\right)_c}$$

Kinetic Equation - Summary

- The original dimensionality of the problem has been drastically reduced from $(6\,N+1)$ with $N \approx 10^{22}$ to $(6+1)$, 3 for $\mathbf{x}$, 3 for $\mathbf{v}$ and one for time.
- Moreover, we have identified the measure, $f(\mathbf{x},\mathbf{v},t)$, capable to describe the plasma properties in this reduced space, and we know the equation for its evolution,

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{1}{m}\left(\mathbf{F}^{\rm ext} + \mathbf{F}^{\rm self}\right)\cdot\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_c}$$

  - **Nonlinear Vlasov operator** (left-hand side)
    Account for the *uncorrelated* particle interactions plus external forces.
  - **Collisions: Boltzmann operator** (right-hand side)
    Account for the *correlated* particle interactions.

The distribution function $f(\mathbf{x},\mathbf{v},t)$ has a *natural deterministic* interpretation, namely plasma density in the phase space, and

$$n(\mathbf{x},t) = \int f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}$$

is the plasma density in *real* space. Something *measurable*!!!

Vlasov Equation - Conservation

Since the Lorentz force satisfies $\nabla_{\mathbf{v}} \cdot \mathbf{F} = 0$, the Vlasov equation can be written

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{z}} \cdot (\mathbf{V} f) = 0, \quad \text{with :} \quad \mathbf{V} := \left(\mathbf{v}, \frac{q}{m} \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \right)^t, \ \mathbf{z} := (\mathbf{x}, \mathbf{v})$$

since the flux $\mathbf{V}$ in phase space is **incompressible**, i.e. $\nabla_{\mathbf{z}} \cdot \mathbf{V} = 0$.
This is the *conservative* form of the Vlasov equation: variation in time of the density in a volume $\mathcal{V}$ of the phase-space is equal to the *balance* between what enters and what leaves the volume through its boundary $\partial\mathcal{V}$

$$\frac{\partial}{\partial t} \int_{\mathcal{V}} f \mathrm{d}\mathbf{z} + \int_{\partial\mathcal{V}} f \, \mathbf{V} \cdot \hat{\mathbf{n}} \, \mathrm{d}s = 0$$

If we define the **mean velocity** on the same footing as the density,

$$n(\mathbf{x}, t) \, \mathbf{u}(\mathbf{x}, t) = \int \mathbf{v} \, f(\mathbf{x}, \mathbf{v}, t) \, \mathrm{d}\mathbf{v}$$

then the incompressibility of the flow in phase space translates into the **continuity equation** for the plasma density

$$\frac{\partial n}{\partial t} + \nabla_{\mathbf{x}} \cdot (n \, \mathbf{u}) = 0$$

Kinetic Model

In a *kinetic model*, each plasma species (ions and electrons) is described by its distribution function $f_i(\mathbf{x}, \mathbf{v}, t)$, solution of the *kinetic equation*

$$\frac{\partial f_i}{\partial t} + \mathbf{v} \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{q_i}{m_i} \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \cdot \frac{\partial f_i}{\partial \mathbf{v}} = \left(\frac{\partial f_i}{\partial t} \right)_c$$

where the electromagnetic field satisfies the Maxwell equations,

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \cdot \mathbf{E} = 4\pi\,\rho$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}, \qquad \nabla \cdot \mathbf{B} = 0$$

The *self-consistent* electromagnetic fields enter naturally as contribution to charge and current via the **constitutive relations**

$$\rho(\mathbf{x}, t) = \sum_i q_i \int f_i(\mathbf{x}, \mathbf{v}, t) \mathrm{d}\mathbf{v}$$

$$\mathbf{J}(\mathbf{x}, t) = \sum_i q_i \int f_i(\mathbf{x}, \mathbf{v}, t)\ \mathbf{v}\ \mathrm{d}\mathbf{v}$$

The mutual interaction among plasma particles is contained in the collision operator and in the constitutive relations.

Kinetic Models - Summary

Although the dimensionality of the problem has been *tremendously reduced* (from $(6N + 1)$ to $(6 + 1)$!!!), the numerical solution of a full *kinetic* model is still challenging, since

- The *kinetic* and *Maxwell* equations are *nonlinearly* coupled.
- Different time and length scales are covered, for instance gyromotion vs drift motion $\Longrightarrow$ **Multiscale** problem.

One can analytically simplify the equations and introduce *gyrokinetic*, *gyrofluid*, and *fluid* (etc.) models. However, at each level of simplification, the domain of validity (physical phenomena that can be still correctly described) becomes narrower: developing simplified models mainly requires

- derivation of the equations,
- identify the validity domain.

Kinetic Models $\Longrightarrow$ Fluid Models

A *fluid* is characterized by

$$\text{density } (u^0): \quad n(\mathbf{x},t) := \int f(\mathbf{x},\mathbf{v},t)\,\mathrm{d}\mathbf{v}$$

$$\text{mean velocity } (u^1): \quad n(\mathbf{x},t)\mathbf{u}(\mathbf{x},t) := \int f(\mathbf{x},\mathbf{v},t)\;\mathbf{v}\;\mathrm{d}\mathbf{v}$$

$$\text{Pressure tensor } (u^2): \quad \mathbf{P}_{kl}(\mathbf{x},t) := \int f(\mathbf{x},\mathbf{v},t)\;\left(\mathbf{v}-\mathbf{u}(\mathbf{x},t)\right)_k\;\left(\mathbf{v}-\mathbf{u}(\mathbf{x},t)\right)_l\mathrm{d}\mathbf{v}$$

$$\text{Pressure scalar } (u^2): \quad p(\mathbf{x},t) := \frac{1}{3}\mathrm{Tr}(\mathbf{P}(\mathbf{x},t)) = \int f(\mathbf{x},\mathbf{v},t)\;|\mathbf{v}|^2\;\mathrm{d}\mathbf{v}$$

If you take the moments 0^{th}, 1^{st}, and 2^{nd} of the kinetic equation and use a closure for the 3^{rd} moment you get the fluid equations, a set of equations for n, $\mathbf{u}$, and p.
The constitutive equations becomes simply,

$$\rho(\mathbf{x},t) = \sum_i q_i\, n_i(\mathbf{x},t), \qquad \mathbf{J}(\mathbf{x},t) = \sum_i q_i\, n_i(\mathbf{x},t)\,\mathbf{v}(\mathbf{x},t)$$

Collisions

Boltzmann Equation

Landau Collision Operator

Boltzmann Equation

(Landau, 1936.)

- Let us introduce the number of collisions (per unit time) between α-species particles with momentum p and β-species particles with momentum p'

$$w(p+\frac{\Delta}{2}, p'-\frac{\Delta}{2}; \Delta)\; f_\alpha(p)\; f_\beta(p')\; d^3p'\; d^3\Delta$$

  $w(p+\Delta, p'+\Delta')$ = probability per unit time that in a collision the particles' momenta are changed from (p, p') to $(p+\Delta, p'+\Delta')$.
  We have explicitly used the momentum conservation $\Delta' = -\Delta$.

- $\partial f_\alpha(p)/\partial t$ = balance between the particles that leave and enter the volume d^3p around p:

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = \sum_\beta \int d^3p'\; d^3\Delta \quad w(p+\frac{\Delta}{2}, p'-\frac{\Delta}{2}; \Delta)$$
$$\left[f_\alpha(p)\; f_\beta(p') - f_\alpha(p+\Delta)\; f_\beta(p'-\Delta)\right]$$

  where $\quad w(p+\Delta/2, p'-\Delta/2) = w(p-\Delta/2, p'+\Delta/2) \quad$ it has been used.
  This is the **Boltzmann collision operator**.

Landau Approach

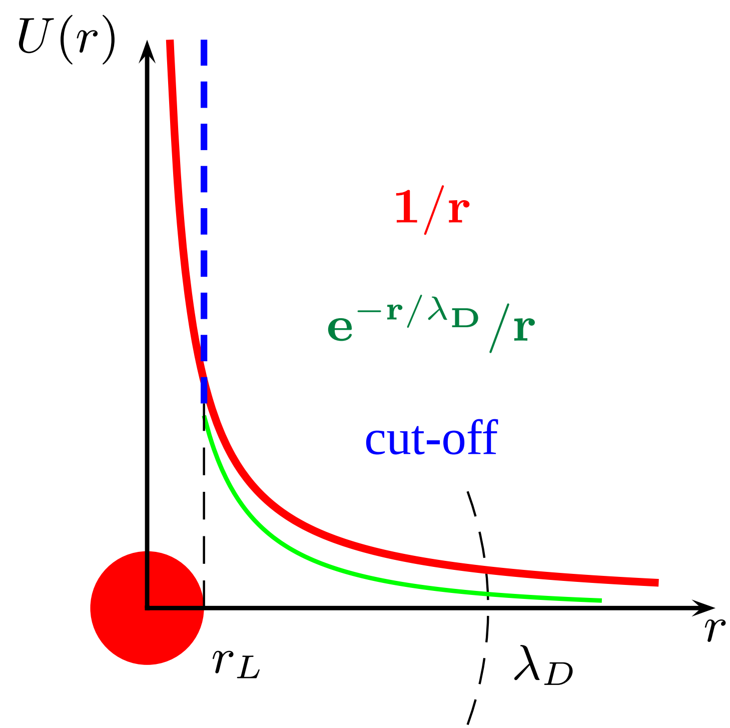
The **collisional kernel** for Coulomb interactions is given by the **Rutherford formula**,

$$w\left(p+\frac{\Delta}{2}, p'-\frac{\Delta}{2};\Delta\right) d^3\Delta = |v_\alpha - v'_\beta| \mathrm{d}\sigma = \left(\frac{Z_\alpha Z_\beta e^2}{m_{\alpha\beta}}\right)^2 \frac{\sin\theta}{\sin^2(\theta/2)} \frac{1}{|v_\alpha - v'_\beta|^3} \mathrm{d}\theta$$

with θ the deviation angle, i.e. the angle between $v_\alpha - v'_\beta$ and $(v_\alpha + \Delta_{v,\alpha}) - (v'_\beta - \Delta_{v,\alpha})$

Singularities of the collision kernel (CK)

- *grazing collision singularity*:
  The CK diverges as $\theta \longrightarrow 0$
  Cure: Debye–Hückel screening:
  $$\phi_D(r) = \frac{Z_\alpha Z_\beta e^2}{r} e^{-r/\lambda_D}$$
- *head-on collision singularity*:
  The CK diverges as $\theta \longrightarrow 2\pi$
  Cure: Cut off at the *Landau distance* (length) r_L defined by



$$\frac{Z_\alpha Z_\beta e^2}{r_L} = E_{\mathrm{th}} \Longrightarrow r_L = \frac{Z_\alpha Z_\beta e^2}{T}$$

where $r_L \ll n^{-1/3} \ll \lambda_D$.

Landau Collision Operator

Since *grazing* collisions, characterized by many small changes of the particle momentum, are the *dominant* binary interactions, we can apply the same procedure discussed for the drunk problem to derive the collision operator which is of the Fokker-Planck type.

- Expand the integrand of the Boltzmann operator in a Taylor series of powers of Δ_i and up to the second order.
- Observe that the zeroth and first order terms are equal to zero.

Then, finally one obtains:

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = -\frac{\partial}{\partial p_i}\sum_\beta \int d^3p' \left(f_\alpha(p)\frac{\partial f_\beta(p')}{\partial p'_k} - f_\beta(p')\frac{\partial f_\alpha(p)}{\partial p'_k}\right) \int \mathbf{w}\,\frac{\mathbf{\Delta_i \Delta_k}}{\mathbf{2}} \mathrm{d}\mathbf{\Delta}$$

with (after some algebra):

$$B_{ik}^{\alpha\beta}(p,p') = \int w\,\frac{\Delta_i\Delta_k}{2}\mathrm{d}\Delta = 2\pi \ln\Lambda\,\frac{(Z_\alpha Z_\beta e^2)^2}{|v-v'|}\left[\delta_{\alpha,\beta} - \frac{(v_i - v'_i)(v_k - v'_k)}{|v-v'|^2}\right]$$

$$\text{with}: \qquad \ln\Lambda = \ln\frac{\lambda_D}{r_L}, \qquad \text{known as the Coulomb logarithm}$$

Properties of Landau Collision Operator

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = -\nabla_p \cdot S_c^\alpha = -\nabla_p \cdot \sum_\beta C^{\alpha/\beta}(f_\alpha, f_\beta)$$

$$= -\frac{\partial}{\partial p_i} \sum_\beta \int d^3p' \left(f_\alpha(p) \frac{\partial f_\beta(p')}{\partial p'_k} - f_\beta(p') \frac{\partial f_\alpha(p)}{\partial p'_k} \right) B_{ij}^{\alpha\beta}(p, p')$$

- **CONSERVATION OF**:
  - **Particles**, independently for each species:
    $\int \mathrm{d}^3 p \, (\frac{\partial f_\alpha}{\partial t})_c = \int \mathrm{d}p \, \frac{\partial}{\partial p^i} S^i = 0$
  - **Momentum** for each pair of particle species:
    $\int \mathrm{d}^3 p \, [p \, C^{\alpha/\beta} + p \, C^{\beta/\alpha}] = 0$
  - **Energy** for each pair of particle species:
    $\int \mathrm{d}^3 p \, [(p^2/m_\alpha) \, C^{\alpha/\beta} + (p^2/m_\beta) \, C^{\beta/\alpha}] = 0$
- **BOLZTMANN H THEOREM**:
  $\frac{\partial s}{\partial t} = \sum_{\alpha,\beta} \int \mathrm{d}^3 p \, \ln f_\alpha \, C^{\alpha/\beta} \geq 0, s = \sum_\alpha \int \mathrm{d}p \, f_\alpha \ln f_\alpha$
  - Collisions increase the entropy
  - An equilibrium state exists, the Maxwellian.
  - Collisions act to smear out irregularities of the distribution function: they drive the distribution towards the Maxwellian.

Wave-Particle Interaction

Landau Damping

Wave-Particle Interaction

With a hand-waving approach it is possible to build a physical picture of the Landau damping containing many of its peculiarities.

One-Dimensional problem: Let us consider an electron in the electric field of an electrostatic wave

$$E = E_0 \ \cos(kx - \omega t)$$

- We **assume** that the electron and the wave stay in resonance for a time Δt **short enough** to justify the **linearization** of the equation of the motion $\Delta v \ll v$:

$$x(t) \approx x_0 + v_0\, t \qquad \text{and :} \qquad v(t) \approx v_0 + \Delta v$$

- The velocity change Δv is

$$\begin{aligned} \Delta v &= \frac{eE_0}{m_e} \int_0^t \cos\left[kx_0 + (kv_0 - \omega)t'\right] \,\mathrm{d}t' \\ &= 2\,\frac{eE_0}{m_e}\,\frac{1}{k\,v_0 - \omega}\ \sin\left(\frac{k\,v_0 - \omega}{2} t\right)\ \cos\left(k\,x_0 + \frac{k\,v_0 - \omega}{2} t\right) \end{aligned}$$

Landau Damping: Diffusion

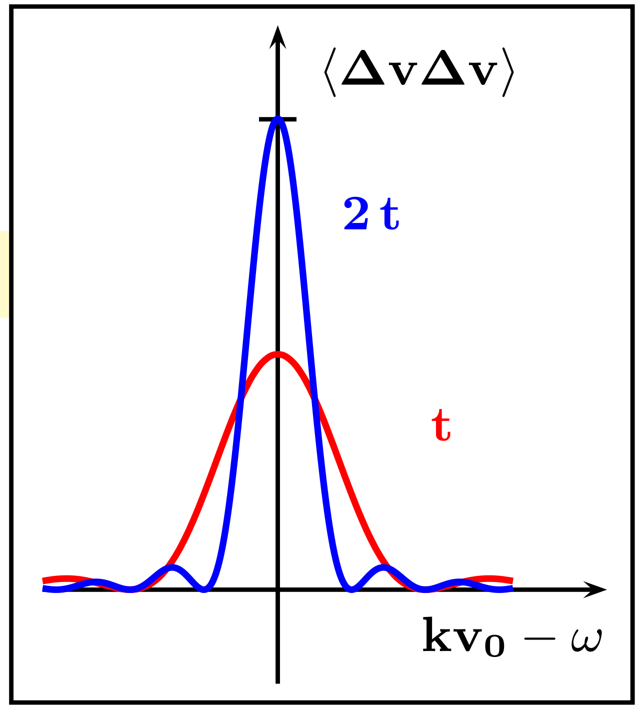
- We average over the initial phase $k\,x_0$ (**assumed random**) to obtain

$$\begin{aligned}\langle \Delta v\ \Delta v\rangle &\equiv \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}(kx_0)\ \Delta v\ \Delta v\\ &= 2\left(\frac{eE_0}{m_e}\right)^2\left[\frac{\sin((k\,v_0-\omega)t/2)}{(k\,v_0-\omega)}\right]^2\end{aligned}$$

As t increases the *average* resonant particle picks up more energy per unit time (*height*) **but** it stays in resonance for a shorter time (*width*):

the (area) energy picked up in the t time

does not depend on t !!!



$$t \longrightarrow \infty \quad \boxed{\frac{\pi}{\omega}\left(\frac{eE_0}{m_e}\right)^2\ \delta(k\,v_0-\omega)}$$

where $\sin\,(t\ x)/x \stackrel{t\to\infty}{\longrightarrow} \pi\delta(x)$ has been used.

Landau Damping: Friction

There is a contradiction to be solved between

- $\langle \Delta v \rangle = 0$ stated two slides before
- and a theorem that for Hamiltonian systems states

$$
\begin{aligned}
\langle \Delta v \rangle &= \frac{1}{2} \frac{\partial}{\partial v} \langle \Delta v \Delta v \rangle \\
&= 2 \left(\frac{eE_0}{m_e} \right)^2 \frac{k}{(kv_0 - \omega)^2 t} \left[-\frac{1 - \cos((kv_0 - \omega)t)}{kv_0 - \omega} + \frac{t}{2} \sin((kv_0 - \omega)t) \right] \\
&\neq 0
\end{aligned}
$$

What is wrong? The theorem is correct ... ideas?

Ans.: To evaluate $\langle \Delta v \rangle$ one has to expand to second order in the time, since $\langle \Delta v \Delta v \rangle$.
$\Longrightarrow$ This is the typical example of an inconsistency that appears when the orderings are not properly (consistently) taken into account.

Landau Damping: Fokker-Planck Eq.

- The electron distribution function in presence of wave-particle interaction is described by a **Fokker–Planck equation**

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v}(\langle \Delta v \rangle \, f) + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left(\langle \Delta v \; \Delta v \rangle \, f \right)$$

the energy per unit time $\mathcal{K}$ exchanged by electrons with the wave is:

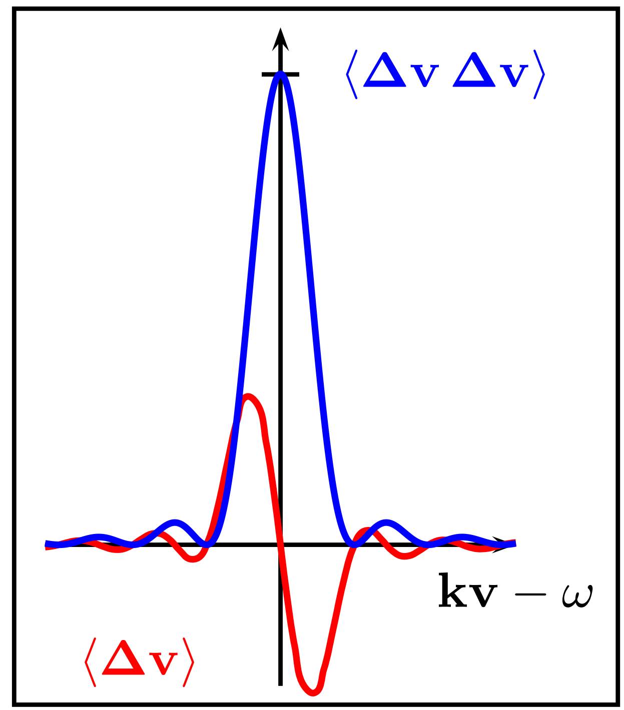
$$\frac{\partial \mathcal{K}}{\partial t} = \int \mathrm{d}v \, \frac{m}{2} v^2 \, \frac{\partial f}{\partial t} = \int \mathrm{d}v \left[\frac{m}{2} \, v \, \langle \Delta v \rangle \right] f$$

with : $\qquad \langle \Delta v \rangle = \frac{1}{2} \frac{\partial \langle \Delta v \Delta v \rangle}{\partial v}$

- $\langle \Delta v \rangle$ **asymmetric** around $v_{\mathrm{ph}} = \omega / k$
  - $\mathcal{K}\uparrow$ $\quad$ <u>if</u> $f(v_{\mathrm{ph}} - \delta v) > f(v_{\mathrm{ph}} + \delta v)$

    or $(\partial f / \partial v)_{\mathrm{v_{ph}}} < 0$
  - $\mathcal{K}\downarrow$ $\quad$ <u>if</u> $f(v_{\mathrm{ph}} - \delta v) < f(v_{\mathrm{ph}} + \delta v)$

    or $(\partial f / \partial v)_{\mathrm{v_{ph}}} > 0$



Landau Damping: Damping Rate

- We start from the Fokker-Planck equation in the diffusion form:

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial}{\partial v}\Big[\langle \Delta v\ \Delta v\rangle\, \frac{\partial f}{\partial v}\Big]$$

and evaluate $\partial \mathcal{K}/\partial t$ with $\langle \Delta v\ \Delta v\rangle = \frac{\pi}{\omega}\big(\frac{eE_0}{m_e}\big)^2\ \delta(k\,v_0 - \omega)$:

$$\frac{\partial \mathcal{K}}{\partial t} = \int \mathrm{d}v\, \frac{mv^2}{2}\, \frac{\partial f}{\partial t} = -\int \mathrm{d}v\, \frac{m_e}{2}\, v\ \langle \Delta v \Delta v\rangle\, \frac{\partial f}{\partial v} = \frac{\pi}{2}\, \frac{e^2\, E_0^2}{m_e}\, \frac{\omega}{|k|\, k}\Big(-\frac{\partial f}{\partial v}\Big)_{\omega/k}$$

- To conserve the total energy, it must be satisfied:

$$\frac{\partial \mathcal{K}}{\partial t} + \frac{\partial \mathcal{W}}{\partial t} = 0$$

where $\mathcal{W}$ is the wave energy density $\mathcal{W} \approx E_0^2/8\pi$.
The wave energy density changes at the rate:

$$\frac{\partial \mathcal{W}}{\partial t} = 2\omega_i\ W$$

with $\omega_i \equiv \gamma_\mathrm{L}$ the imaginary part of the oscillation frequency.
Combining the three equations, we obtain:

$$\omega_i \equiv \boxed{\gamma_\mathrm{L} = \frac{\pi}{2}\omega_r\, \frac{\omega_\mathrm{pe}^2}{k^2}\, \Big(-\frac{\partial f}{\partial v}\Big)_{\omega/k}}$$

the same result obtained by Landau with the **linearized Vlasov** equation.

Validity of the Linearization

- The validity of the asymptotic limit has already been discussed: **the change of energy of resonant particles must not depend on Δt**. Thus, the time the particle stays in resonance with the wave disappears from the final result!

- When is the linearization of the equations of motion justified?
  We recall the approximation we have done:

$$x(t+\Delta t) \approx x_0 + v\,\Delta t + \int_0^{\mathbf{\Delta t}} \mathrm{d}\mathbf{t}\ \mathbf{\Delta v(t)} \approx x_0 + v\,\Delta t + \frac{\mathbf{1}}{\mathbf{2}}\frac{\mathbf{eE_0}}{\mathbf{m_e}}\,(\mathbf{\Delta t})^{\mathbf{2}}$$

In the phase of the wave field the first correction to the linear equation is small if:

$$k\ \delta\mathbf{x} \ll 1 \Rightarrow \boxed{\Delta t \ll \sqrt{\frac{\mathbf{m}}{\mathbf{2e\,E_0\,k}}}}$$

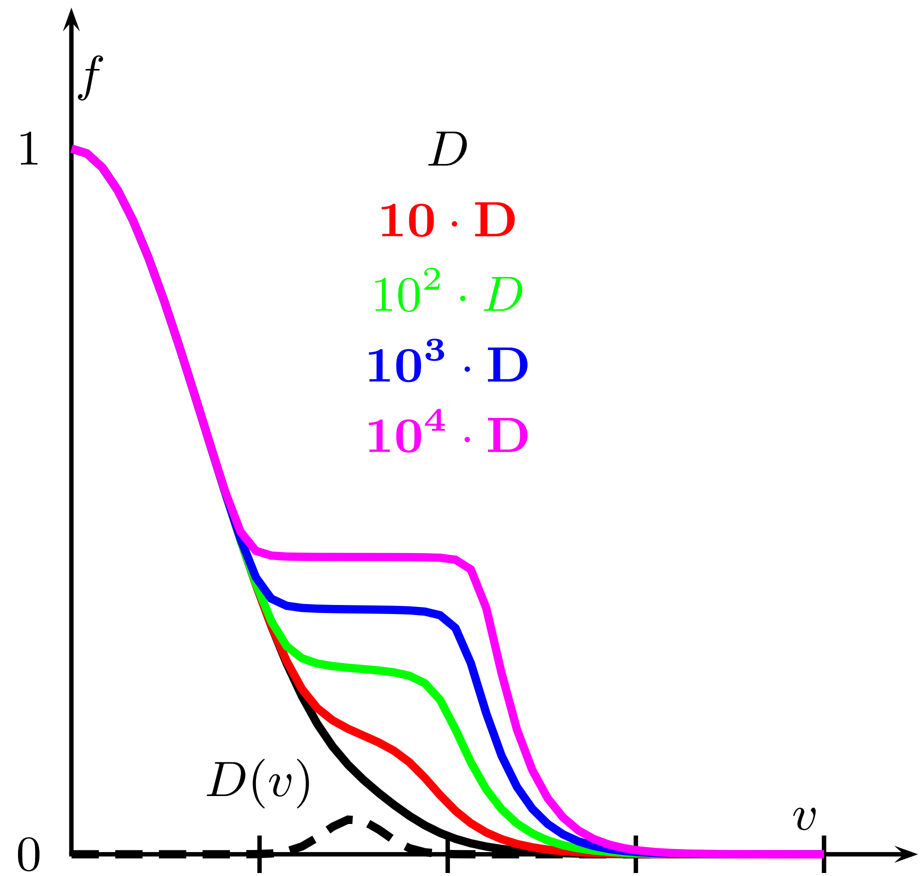
Bouncing time of electrons trapped near the minima of the wave potential

This is also true for the linearization of the Vlasov equation!

Collisions + Waves

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial}{\partial v}\left[D\,\frac{\partial f}{\partial v}\right] + \left(\frac{\partial f}{\partial v}\right)_c$$

- Waves try to form a plateau $\partial \mathbf{f}/\partial \mathbf{v} = \mathbf{0}$.
- Collisions drive f back to Maxwellian $\partial \mathbf{f}/\partial \mathbf{v} < \mathbf{0}$.



Bump-on-tail micro-instability

What does happen when $\partial \mathbf{f}/\partial \mathbf{v} > \mathbf{0}$?

Let us consider the so-called **bump-on-tail instability**: some of the bulk electrons are moved into the distribution tail by an external source.

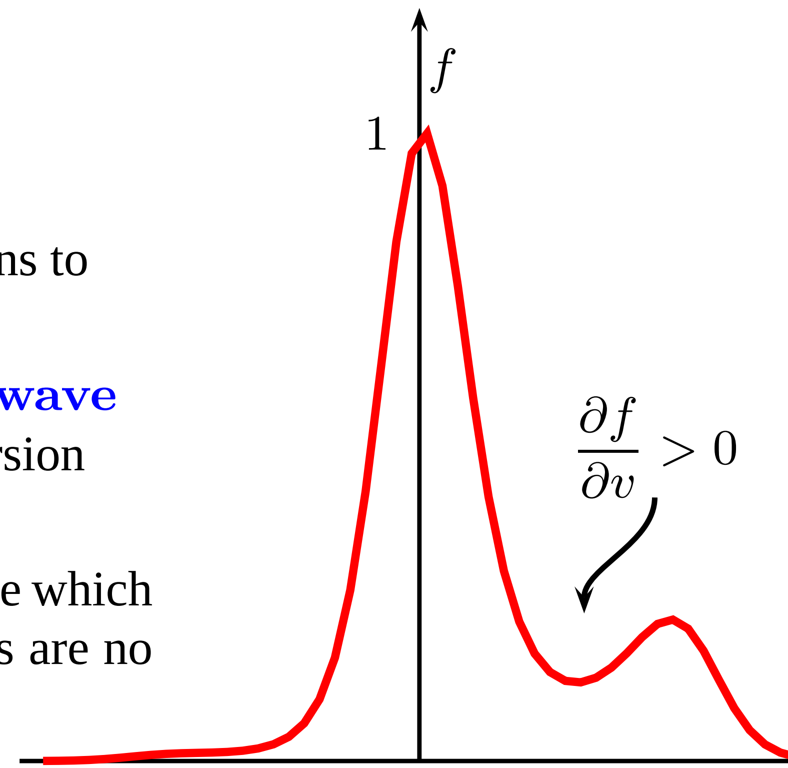
- According to Landau Damping γ,

$$\gamma_{\mathrm{L}} = \frac{\pi}{2}\omega_r \frac{\omega_{\mathrm{pe}}^2}{k^2}\left(-\frac{\partial f}{\partial v}\right)_{\omega/k}$$

  where $\partial f/\partial v > 0$ the energy flows from electrons to waves.

- But it is not sufficient to have $\partial f/\partial v > 0$: **the wave must also exist**, i.e. satisfy the plasma dispersion relation!

- The evolution leads to a diffusion in velocity space which results in a flattening of f, after which the waves are no longer growing in amplitude.



Note: with micro-instability one refers to instability driven by the interaction of a wave with only a relative small fraction of particle population, those in resonance with the wave itself.

A few hints for further reading

- The derivation of the kinetic equation is described in many textbooks, for instance: T.J.M. Boyd and J.J. Sanderson, “The Physics of Plasmas”, Cambridge University Press.
- Many ideas come from Krommes’ notes.
- Numerical solution of the kinetic equation
  E. Sonnendrücker’s lecture,
  http://www-m16.ma.tum.de/foswiki/pub/M16/Allgemeines/NumMethVlasov/Num-Meth-Vlasov-Notes.pdf
- Landau Damping:
  L. Landau, J. Phys. U.S.S.R., **10** (1946) 25.
  Villani’s lecture: http://smai.emath.fr/cemracs/cemracs10/PROJ/Villani-lectures.pdf
- Couloumb collisions:
  L. Landau, Phys. Z. Sovjet, **10** (1937) 203.
  Villani’s review,
  http://cedricvillani.org/wp-content/uploads/2012/07/B01.Handbook.pdf
- An enjoyable book on the history of the fusion research:
  Daniel Clery, “A Piece of the Sun: The Quest for Fusion Energy”, Gerald Duckworth & Co Ltd

Acknowledgments to O. Maj for enlightening discussions in the preparation of these slides.

Appendix - Phase Mixing
An Amplifier of Irreversibility

Phase Mixing: Free Transport Equation

Let us consider the simplest one-dimensional *free* transport problem (here it does not real matter what f represents):

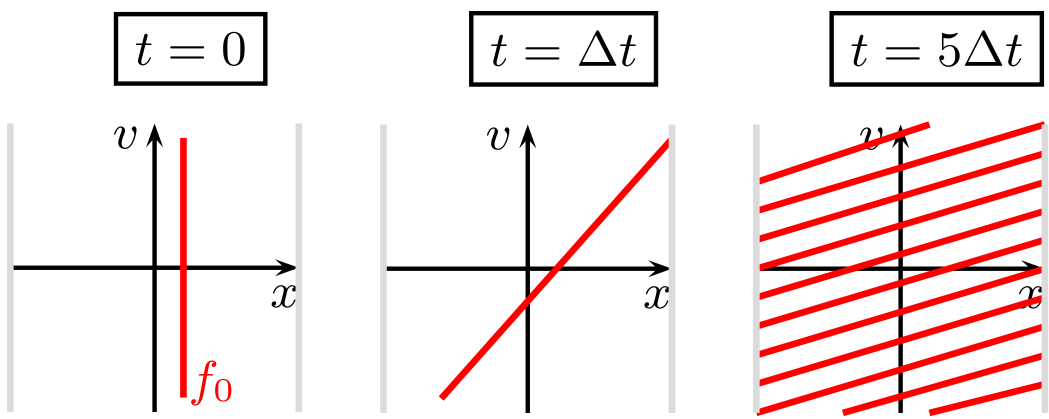
$$\frac{\partial}{\partial t} f(t,x,v) + v \cdot \frac{\partial}{\partial x} f(t,x,v) = 0$$

If the initial condition is $f(0,x,v) = f_0(x,v)$, the solution is simply

$$f(t,x,v) = f_0(x - v\,t, v)$$

Phase mixing is an intrinsic properties of all mechanical systems that once expressed in action-angle variables have the angular velocities of the angle variables depend on the action variables.

In our case, x and v are the angle and action variables, respectively, and $\dot{x} = v$.



Phase Mixing: Cascade in Velocity

From the position-velocity Fourier transform of the solution of the free transport equation

$$\hat{f}(t,k,\eta) = \int\int f_0(x-vt,v)\, e^{-2\pi\, i\, k\, x}\; e^{-2\pi\, i\, \eta\, v} \mathrm{d}x\, \mathrm{d}v$$

$$\overset{x'=x-v\,t}{=} \int\int f_0(x',v)\, e^{-2\pi\, i\, k\, x'}\; e^{-2\pi\, i\, (\eta+k\,t)\, v} \mathrm{d}x'\, \mathrm{d}v$$

$$= \hat{f}_0(k, \eta + k\, t)$$

we infer

- The uniform ($k=0$) spatial mode is preserved in time: $\hat{f}(t,0,\eta) = \hat{f}_0(0,\eta)$
- There is a *cascade* from low to high velocity modes η.
- Riemann-Lebesgue lemma states that if $g(x)$ is L^1 integrable, then $\hat{g}(k) \xrightarrow{|k|\to\infty} 0$.
  Because of the cascade in velocity modes, for $k \neq 0$ it holds $\hat{f}_0(k,\eta+k\,t) \xrightarrow{t\to\infty} 0$.

- In confined mechanical systems, the **recurrence time** is *finite*.
- In the present case, the recurrence time is *infinite*, although the dynamics is reversible. Why?
  (Ans.: Our problem is stated as an *infinite-dimensional* system, and the recurrence time is *finite* only for *finite* confined mechanical systems!)

Phase Mixing: Irreversibility Amplifier

$\Longrightarrow$ Diffusion in velocity space (an irreversible process) increases with η^2,

$$D\,\frac{\partial^2}{\partial v^2}\left(\int\int \hat{f}_i(k,\eta)\, e^{-2\pi\, i\, k\, x}\; e^{-2\pi\, i\, \eta\, v} \mathrm{d}k\; \mathrm{d}\eta\right) =$$

$$-\,D \int\int \underbrace{\boldsymbol{(2\pi\eta)^2}}_{\text{enhancement}}\; \hat{f}_i(k,\eta)\, e^{-2\pi\, i\, k\, x'}\; e^{-2\pi\, i\, \eta\, v} \mathrm{d}k\; \mathrm{d}\eta$$

$\Longrightarrow$ *Velocity cascade* due to phase-mixing "increases" with time the effective $\eta_{\mathrm{eff}} \propto k\ t$.

$\Longrightarrow$ In a periodic box of length L, there is a natural *infrared cutoff* ($k \geq k_{\mathrm{cutoff}} = 2\pi/L$) that enhances the *global* effects of the *velocity cascade*, since the cascade rate is $\propto k$.

BUT

The *velocity cascade* due to the phase mixing is NOT a source of **irreversibility** for the system (entropy does not increase because of phase mixing!),

INSTEAD

it enhances the effectiveness of the sources of irreversibility.

Note: The infinite recurrence time due to the infinite-dimensional description of the system plus the velocity cascade due to phase mixing are often *confused* with *irreversibility*!