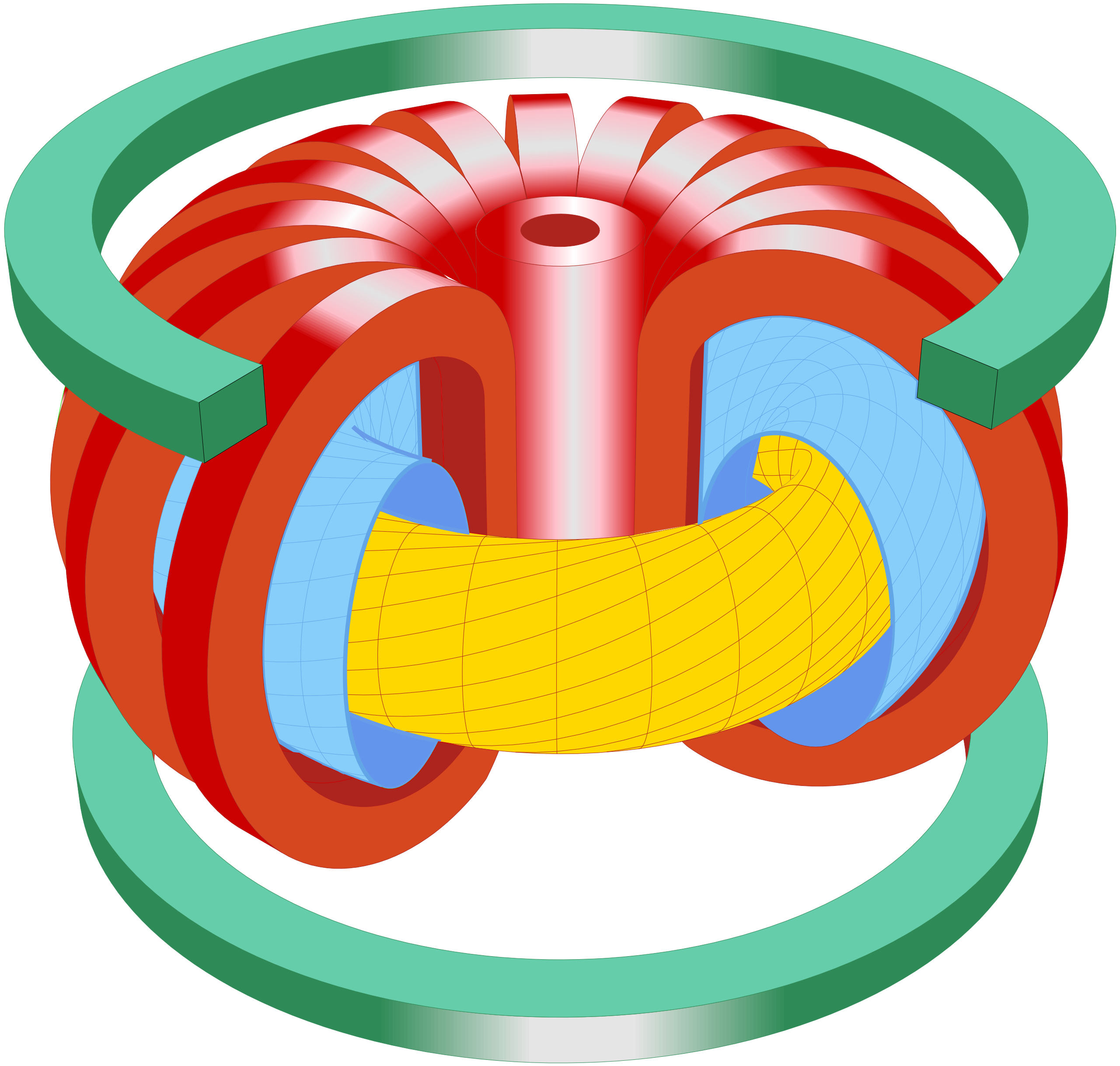
Tokamak Equilibrium and Stability



Wolfgang Suttrop, Max-Planck-Institut für Plasmaphysik, Garching

Definitions ...

- **(Force-) Equilibrium:**
  Plasma configuration in which driving forces
  due to electrical currents, plasma pressure (and possibly others)
  are balanced

- **Stability:**
  A configuration is stable if it restores itself back towards equilibrium
  in response to any possible displacement

- **Tokamak:**
  Axisymmetric toroidal configuration which
  — confines plasma with aid of a toroidal electrical plasma current
  and
  — gains stability from a strong guiding toroidal field

Plasma Force Equilibrium

The most simple way to describe a force equilibrium

Examples of magnetic equilibria — (“natural” and “man-made”)

Why a strong magnetic field helps

Describe plasma as a single, conducting fluid

"Magneto-Hydro-Dynamics", MHD

Describes motion (velocity $\vec{u}$) of fluid volume elements (mass density mn).

Force equation: Reaction = Action (Newton III)

$$\underbrace{m\,\frac{\mathrm{d}}{\mathrm{d}t}(n\,\vec{u})}_{\text{moving frame}} = \underbrace{m\,n\,\frac{\partial\vec{u}}{\partial t}}_{\text{fixed frame}} + \underbrace{m\,n\,(\vec{u}\cdot\nabla)\,\vec{u}}_{\text{advection}} = \rho\vec{E} + \vec{j}\times\vec{B} - \nabla\cdot\overline{\overline{P}}_0$$

Simplifications:

— $\partial/\partial t = 0$

— No advection, but allow $\vec{u} \neq 0$

$\rightarrow$ additional centrifugal force

— Charge neutrality

— Isotropic plasma pressure, no shear stress

$\Rightarrow$ **MHD "equilibrium"**

$$\boxed{\nabla p = \vec{j}\times\vec{B}}$$

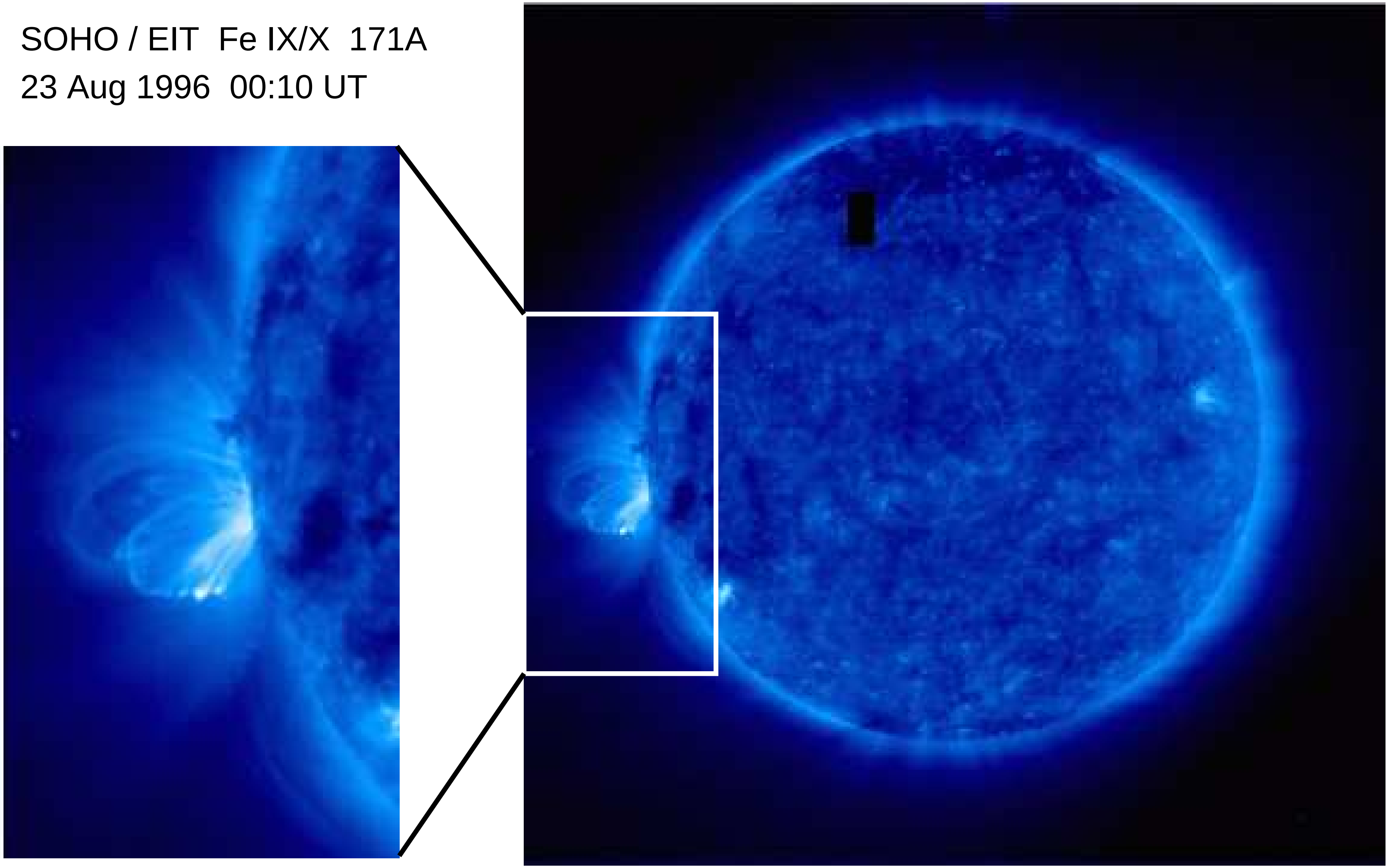
With $\overline{\overline{T}}$: "magnetic pressure tensor"

$$\nabla p = -\nabla\cdot\overline{\overline{T}}$$

Observation of magnetic loops in solar corona

Lifetime: Hours (Inertial time scale: Seconds)

SOHO / EIT Fe IX/X 171A
23 Aug 1996 00:10 UT



Quelle: `http://sohowww.nascom.nasa.gov`

Magnetic Surfaces, Flux Tubes

Force balance ("equilibrium"):

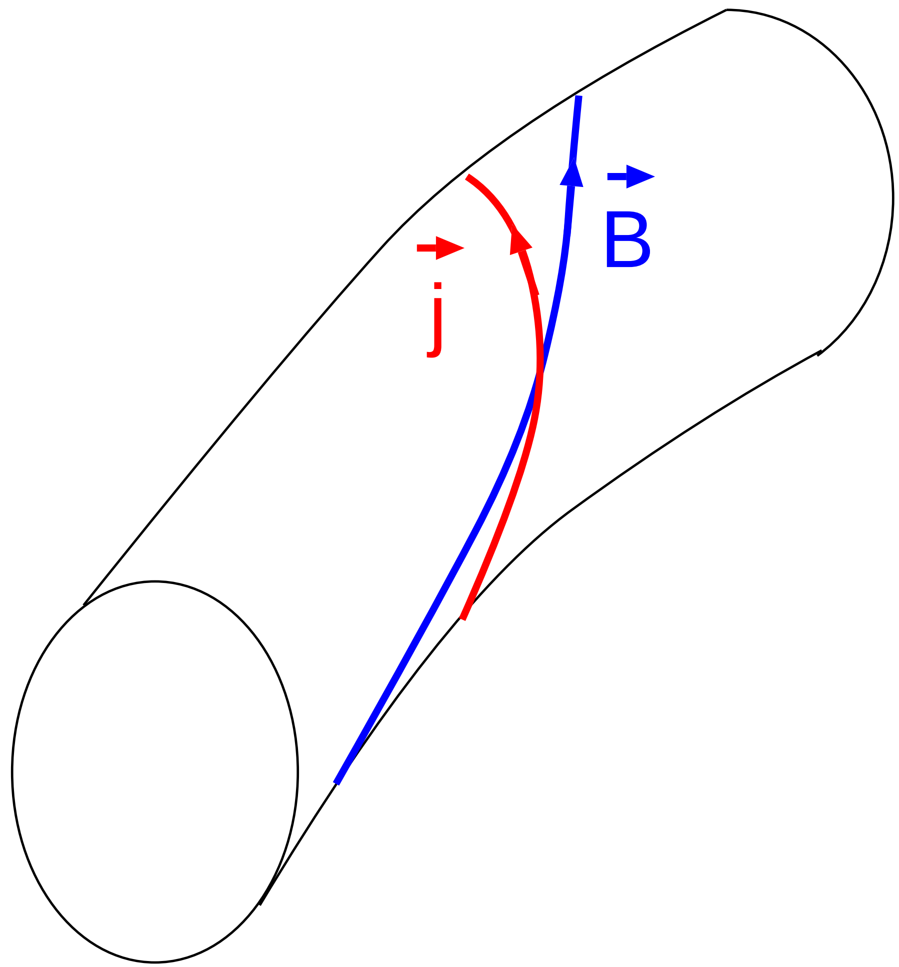
$$\nabla p = \vec{j} \times \vec{B}$$

Dot-product with $\vec{B}$, $\vec{j}$:

$$\nabla p \cdot \vec{B} = (\vec{j} \times \vec{B}) \cdot \vec{B} = 0$$
$$\nabla p \cdot \vec{j} = (\vec{j} \times \vec{B}) \cdot \vec{j} = 0$$

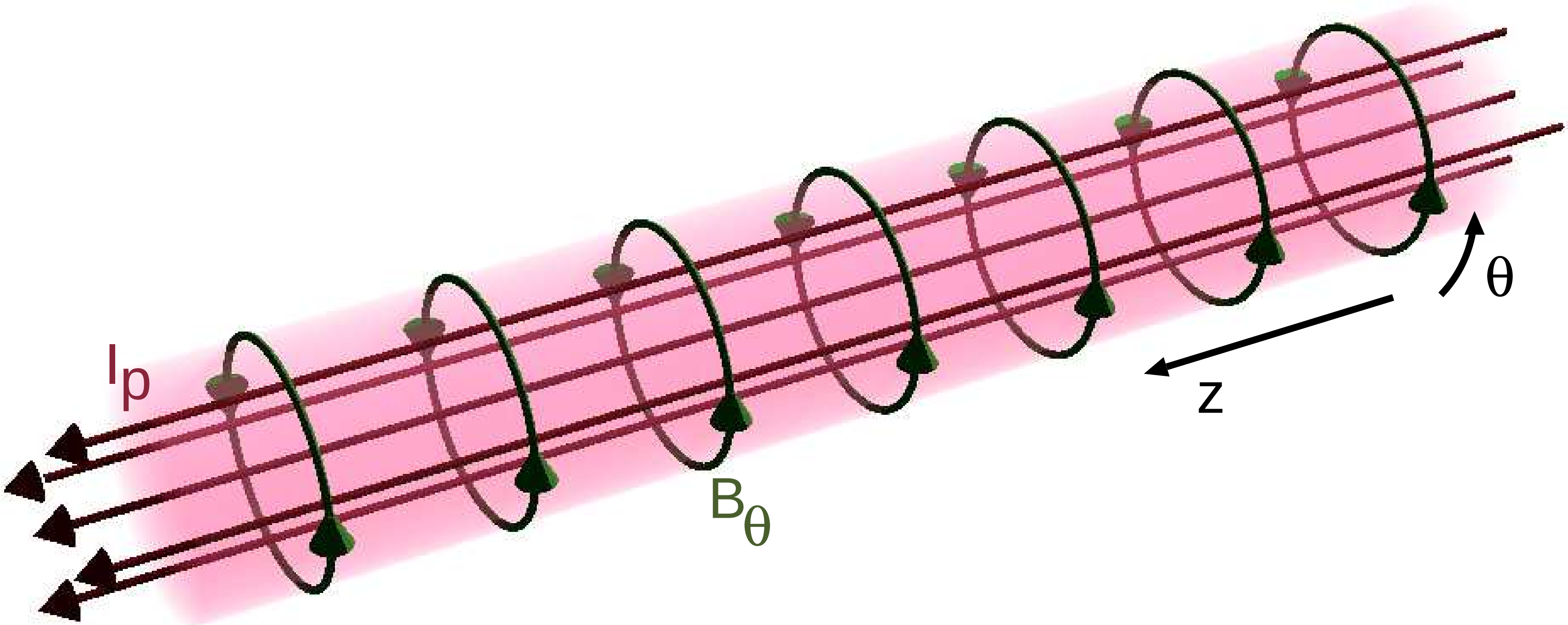
$\Rightarrow \quad \vec{B} \perp \nabla p, \quad \vec{j} \perp \nabla p.$

$\vec{B}$, $\vec{j}$ span surfaces
with constant plasma pressure,
magnetic surfaces.



Most simple geometry: Linear *z*-pinch

Willard H Bennet, Phys. Rev. **45** (1934) 890, "Magnetically self-focusing streams"



Cylindrical coordinates: j_z, B_θ, $\mathrm{d}p/\mathrm{d}r$

z-pinch equilibrium

z, θ ignorable coordinates

Ampère's law:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \mu_0 j_z$$

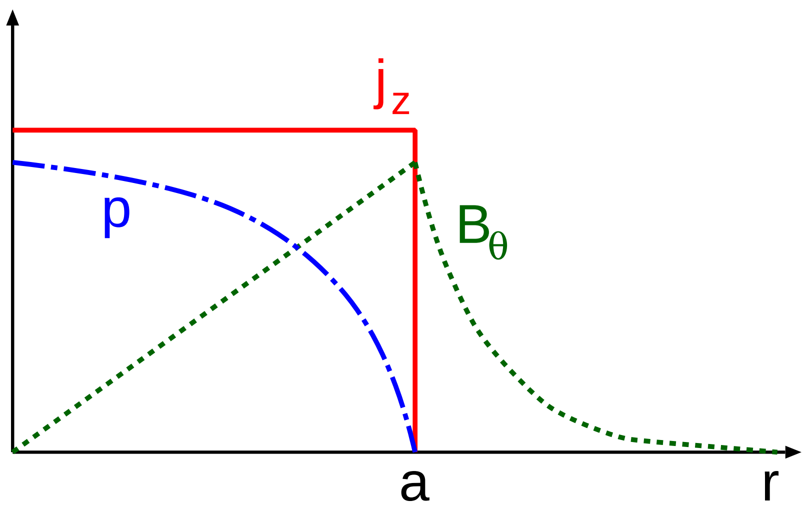
Radial force balance:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -j_z B_\theta$$

z-pinch maximises "Plasma-Beta":

$$\beta = \frac{\text{kinetic pressure}}{\text{magnetic pressure}} = \langle p \rangle / \frac{B_\theta^2(a)}{2\mu_0} = 1$$

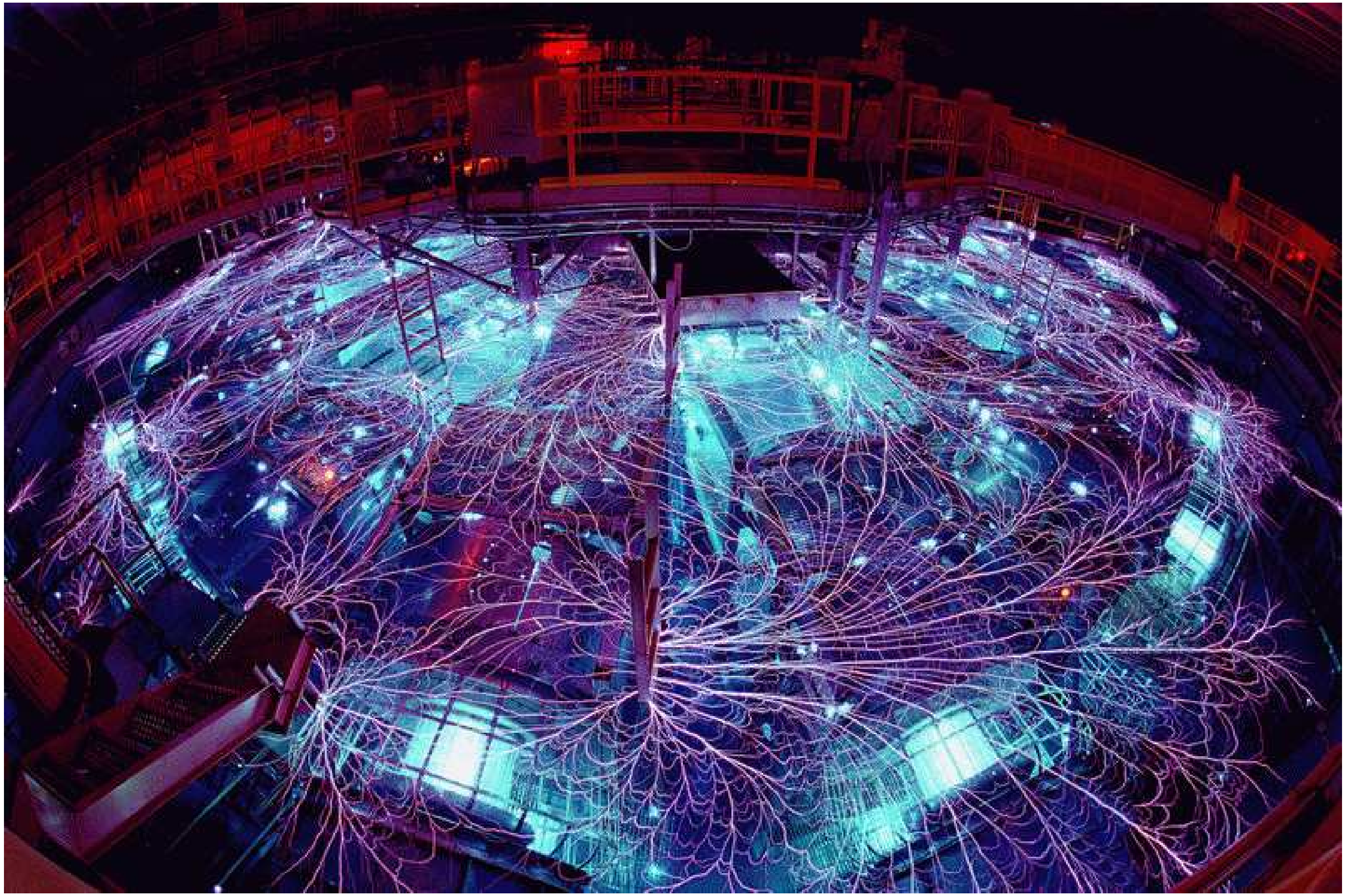
Example: j_z=const.



"Z machine"

Sandia National Laboratories, Albuquerque, NM, U.S.A.

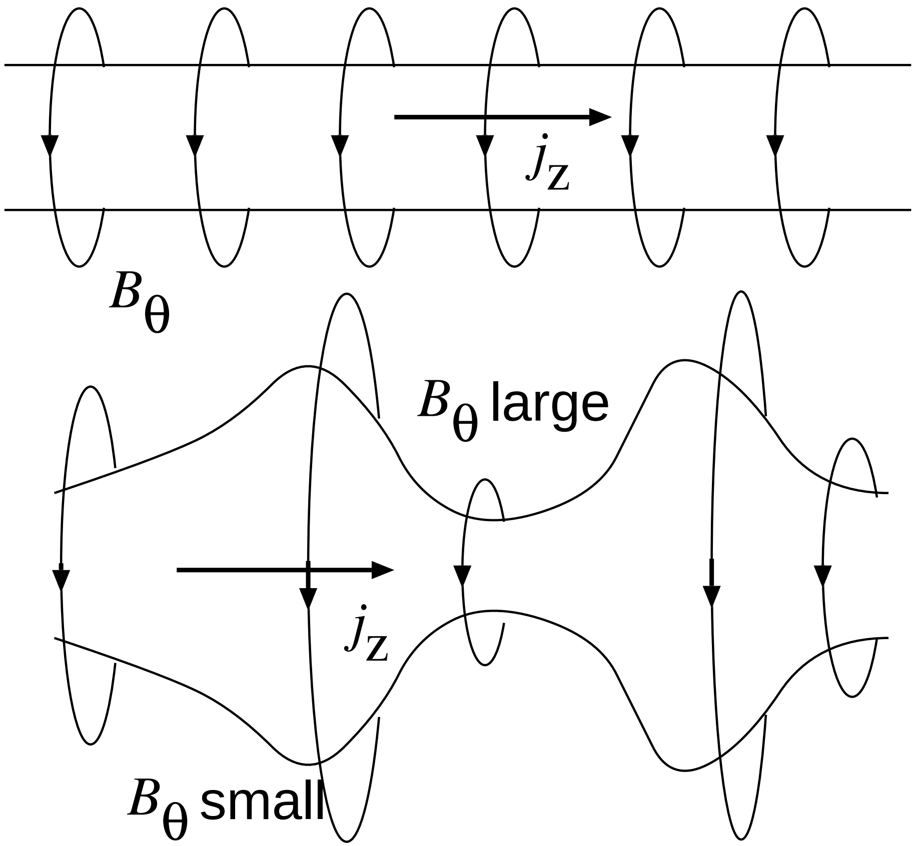
20 MA tungsten wire explosion, strong X-ray source



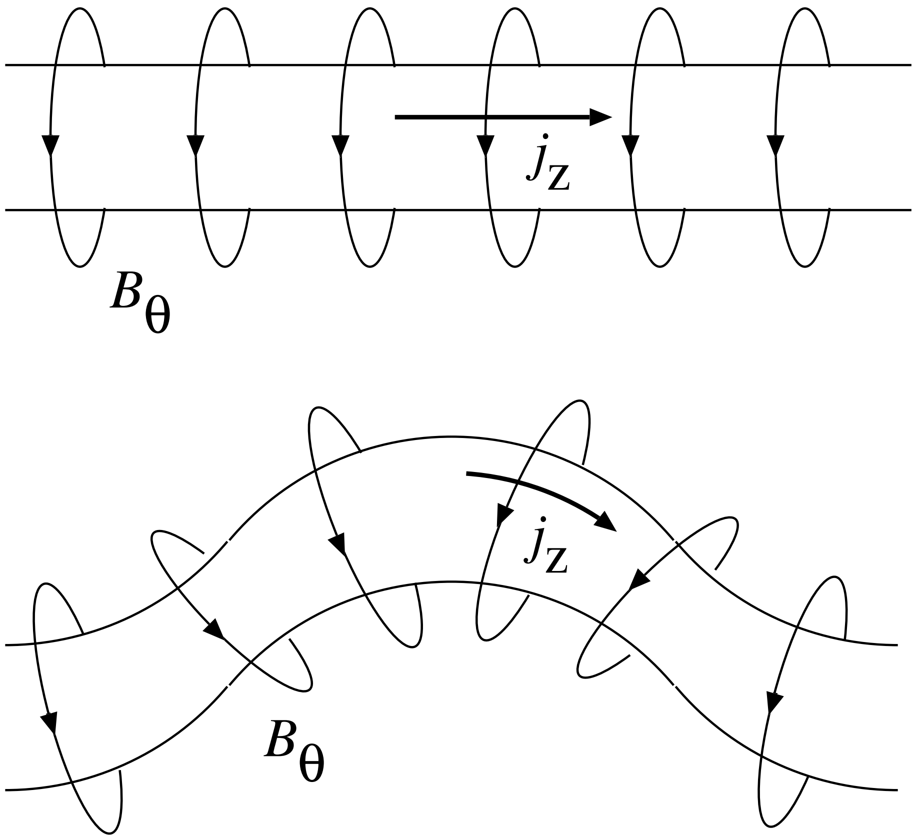
Sources: Wikipedia, Sandia national lab: `http://www.sandia.gov/z-machine`

The Z-pinch is inherently unstable

***Sausage* instability**



***Kink* instability**



Physical origin: Deformation of plasma column

$\rightarrow$ local change of magnetic field pressure $B_\theta^2/2\mu_0$, not balanced by (constant) plasma pressure.

$\rightarrow$ Growing perturbation

(Historical) observation of the *sausage* instability

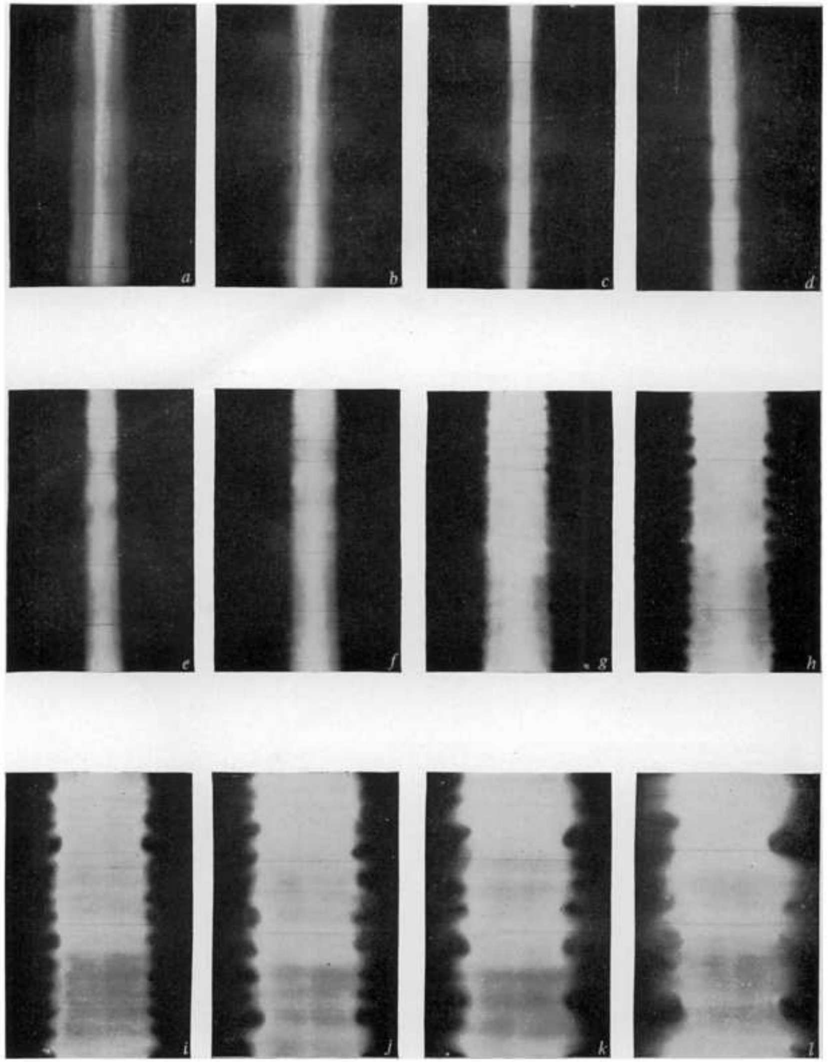


FIGURE 2. Kerr cell photographs (0·4 μs exposure) of a condenser discharge in argon. Circuit capacity 1940 μF; condenser voltage 2·5 kV. Circuit inductance 150 nH; gas pressure. 1000 μ. The discharge tube diameter is equal to the width of the individual frames.

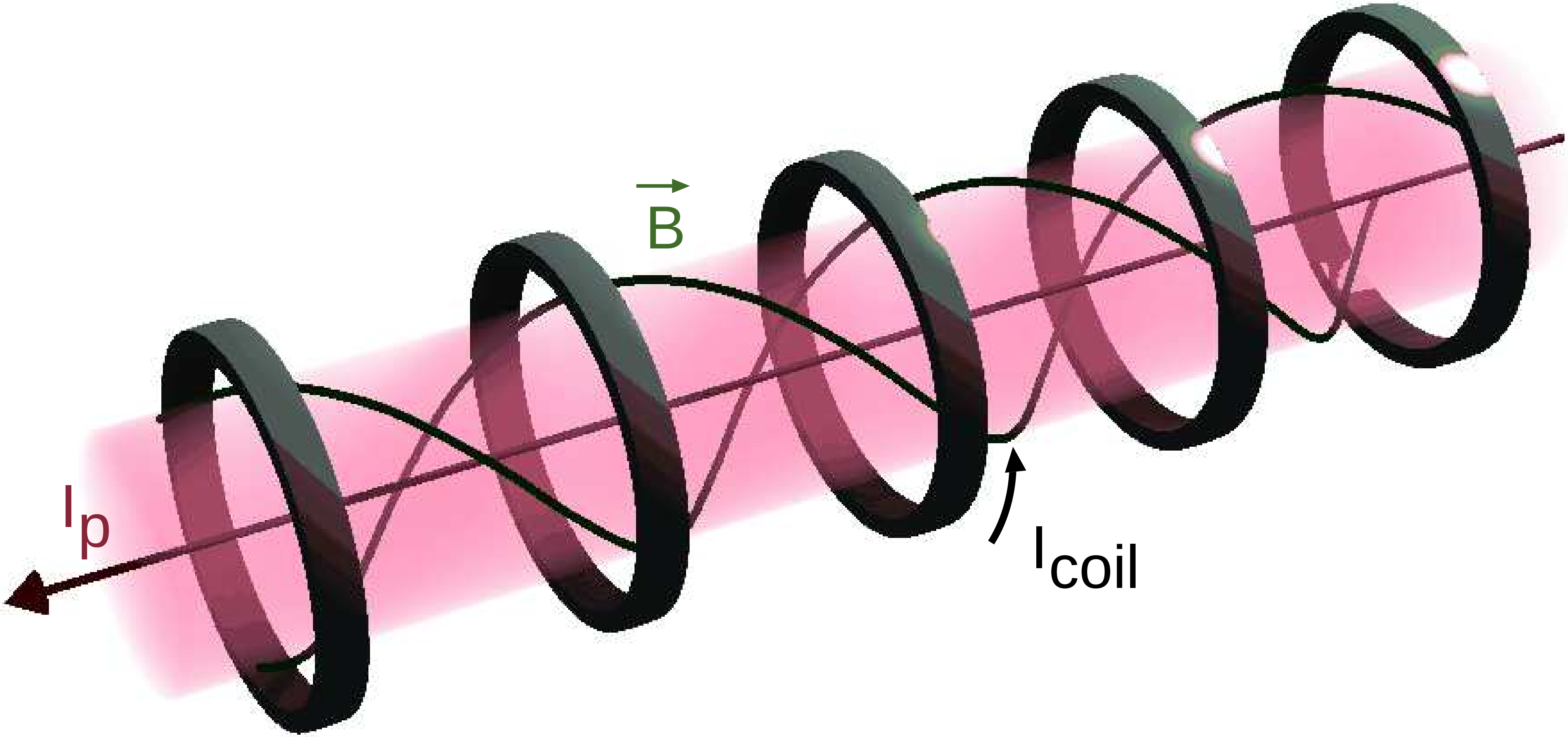
(*a*) t = 18·2 μs	(*b*) t = 19·4 μs	(*c*) t = 21·0 μs	(*d*) t = 21·8 μs
(*e*) t = 22·5 μs	(*f*) t = 23·6 μs	(*g*) t = 24·3 μs	(*h*) t = 26·0 μs
(*i*) t = 27·2 μs	(*j*) t = 29·5 μs	(*k*) t = 31·3 μs	(*l*) t = 34·0 μs

F L Curzon *et al*, Proc. Roy. Soc. A **257** (1960) 386

Short exposure time: Kerr cell (electro-optical polarisation rotation) in between crossed polarisers.

“Guiding” magnetic field improves stability

Add azimuthal external currents (magnets) $\rightarrow$ “screw pinch”



However: Linear configurations suffer from “end losses”

— Magnetic field intersects the wall.

Magnetic pressure tensor

Using

1. Ampère's law: $\mu_0 \vec{j} = \nabla \times \vec{B}$
2. Vector identity: $\nabla(\vec{F} \cdot \vec{G}) = (\vec{F} \cdot \nabla \vec{G}) + \vec{F} \times (\nabla \times \vec{G}) + (\vec{G} \cdot \nabla)\vec{F} + \vec{G} \times (\nabla \times \vec{F})$

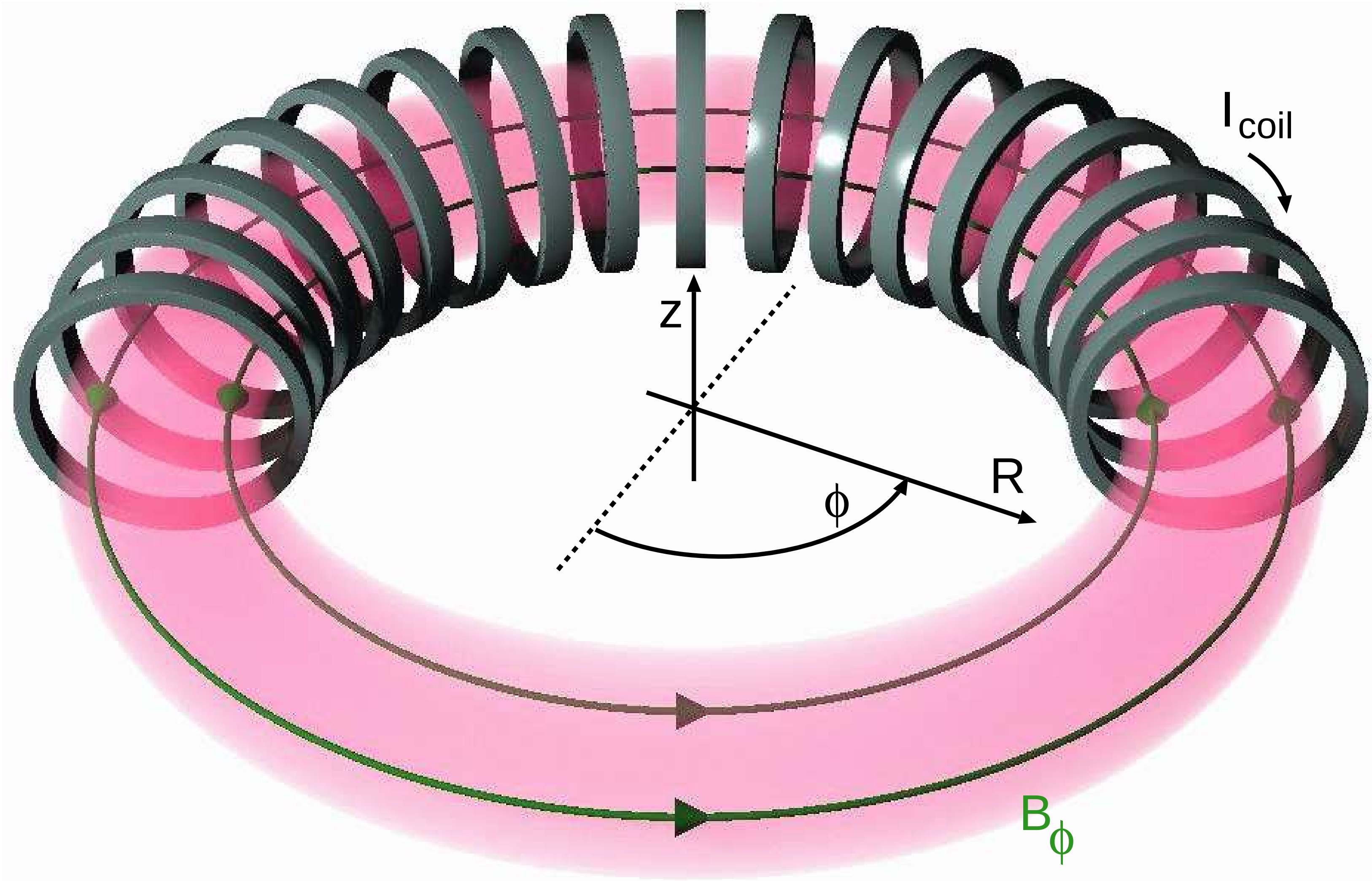
we have

$$\vec{j} \times \vec{B} \;=\; -\frac{1}{\mu_0}\vec{B} \times \left(\nabla \times \vec{B}\right) \;=\; -\nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\left(\vec{B} \cdot \nabla\right)\vec{B} \;\equiv\; -\nabla \cdot \overline{\overline{T}}$$

Magnetic pressure tensor:

$$\overline{\overline{T}} \equiv \underbrace{\left(\frac{B^2}{2\mu_0}\right)}_{\text{isotropic pressure}} \overline{\overline{1}} - \underbrace{\left(\frac{\vec{B}\vec{B}}{\mu_0}\right)}_{\text{tension in B-direction}}$$

Toroidal plasma avoids end-losses



Change coordinate system: Cylinder coordinates with z-axis = *torus axis*

$B_\theta, j_\theta \quad \rightarrow \quad B_p, j_p$ (poloidal direction)

$B_z, j_z \quad \rightarrow \quad B_\phi, j_\phi$ (toroidal direction)

Toroidal Force Equilibrium

Why a poloidal field is needed

Toroidal + poloidal fields = helically twisted magnetic field

Use **fluxes** (ψ, I) instead of **fields** ($\vec{B}$, $\vec{j}$)

Most simple case: **Axisymmetric toroidal equilibrium**

Poloidal field is generated by plasma current

Tokamak:

Axisymmetric toroidal configuration with strong guiding field

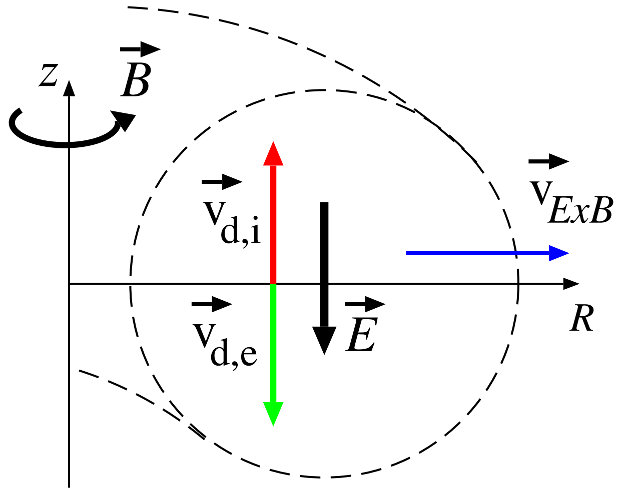
Describe toroidal equilibrium by the **Grad-Shafranov-Schlüter equation**

Vertical field for radial force balance (and plasma shaping)

The need for a poloidal field (particle picture)

Grad-B and curvature drift:

$$\vec{v}_d = \frac{mv_\parallel^2 + mv_\perp^2/2}{q} \quad \frac{\vec{B}\times\nabla B}{B^3}$$

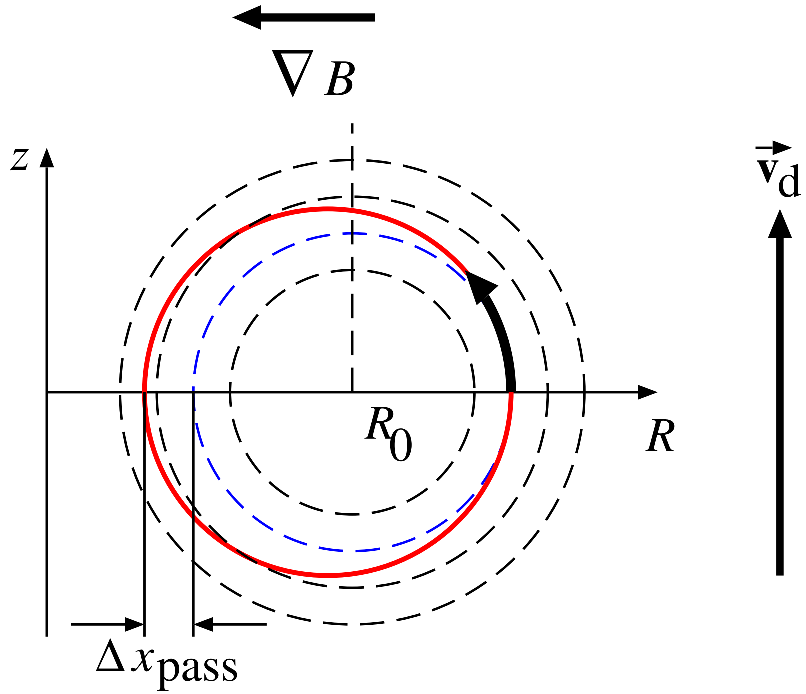


$\rightarrow$ Charge separation, vertical $\vec{E}$.

$\rightarrow$ Outward $E \times B$-drift (along major radius).

Fast radial particle loss

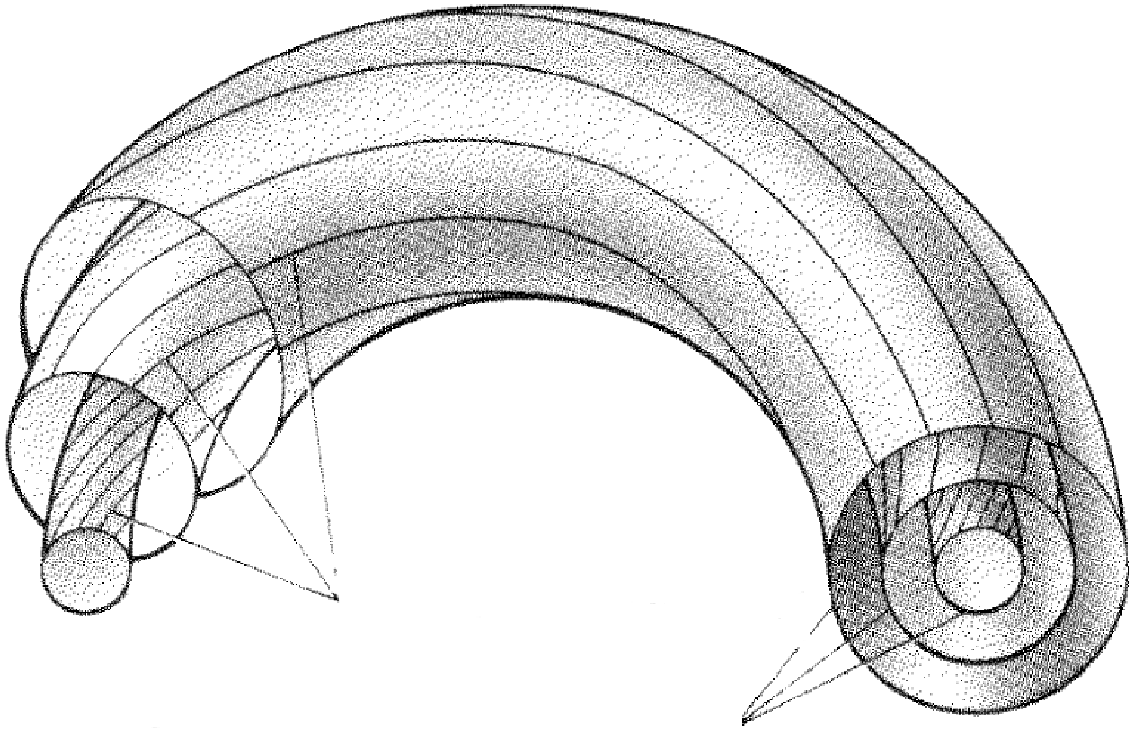
$$\Delta t \sim r/v_d \;\sim\; (r\,L_B)\,\omega_c\,/\,v_{\rm th}^2 \;\sim\; \frac{r\,\omega_c}{R_0\,v_{\rm th}^2}$$



Solution:

additional poloidal field guides particle orbits to above and below equator, cancels drift motion.

Rotational transform (“field line pitch”)



Definition:

$$ \iota\!\!\!-\equiv \frac{\iota}{2\pi} = \frac{\text{Number poloidal passes}}{\text{Number of toroidal passes}} $$

. . . of a field line before it closes back in itself.

Axisymmetry,
Cylindrical approximation ($r \ll R_0$):

$$ \frac{\iota}{2\pi} = \frac{R}{r}\frac{B_p}{B_\phi} $$

The need for a poloidal field (MHD picture)

Force balance: $\nabla p = j_\perp \times B$ — constant on a flux surface.

Consider two positions on the same flux surface: $R_1 < R_2$:
Toroidal field $B_\phi(R_1) > B_\phi(R_2)$.
$\Rightarrow$ Poloidal current density: $j_p(R_2) > j_p(R_1)$, in general not divergence free.

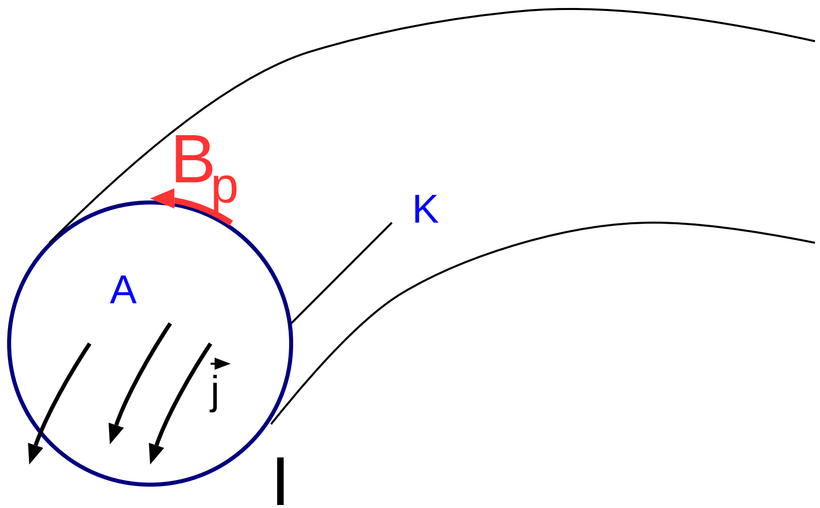
For source-free (= divergence-free) plasma current, $\nabla \cdot \vec{j} = 0$,
need additional toroidal **"Pfirsch-Schlüter" current** $j_{\phi,PS}$:

$$\nabla_\phi \cdot \nabla j_{\phi,PS} + \nabla_p \cdot \nabla j_p = 0$$

N.B. Can make zero total toroidal current $I_\phi = \int_A \vec{j}_\phi \mathrm{d}A$ (e.g. in a stellarator)
but need finite current density j_ϕ locally somewhere.

With axisymmetry, poloidal field is produced by toroidal current

Consider poloidal section:



Stokes & Ampère:

$$\int_A \nabla \times \vec{B}_p \cdot \mathrm{d}\vec{A} = \int_A \mu_0 \vec{j}_\phi \cdot \mathrm{d}\vec{A} = \mu_0 I_\phi = \oint_K \vec{B}_p \cdot \mathrm{d}\vec{s}$$

B_p may not vanish anywhere on path K

(otherwise rotational transform would disappear)

$\Rightarrow$ Finite I_ϕ

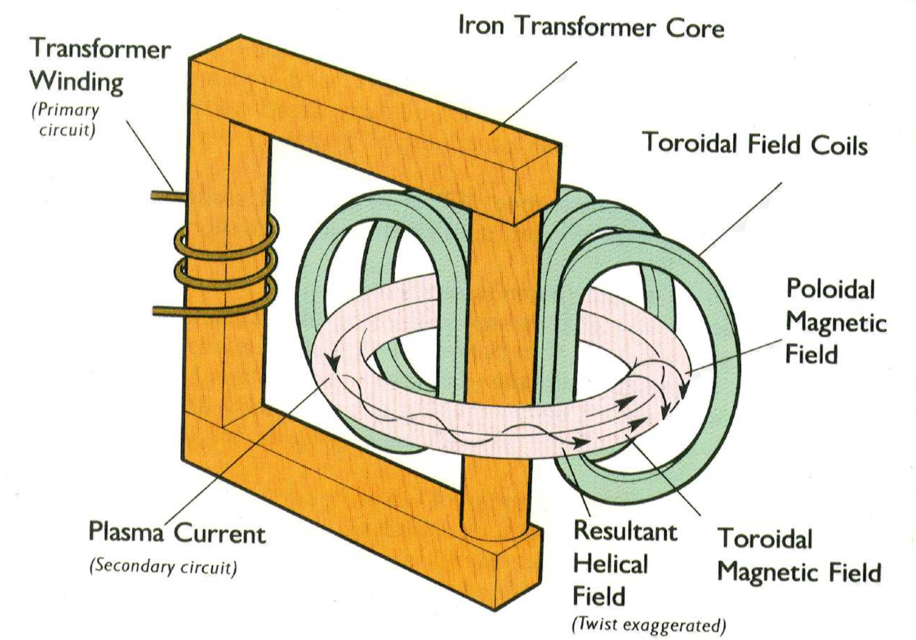
Note:

Not true for *Stellarator* (non-axisymmetric) — B_p pattern can depend on toroidal angle.

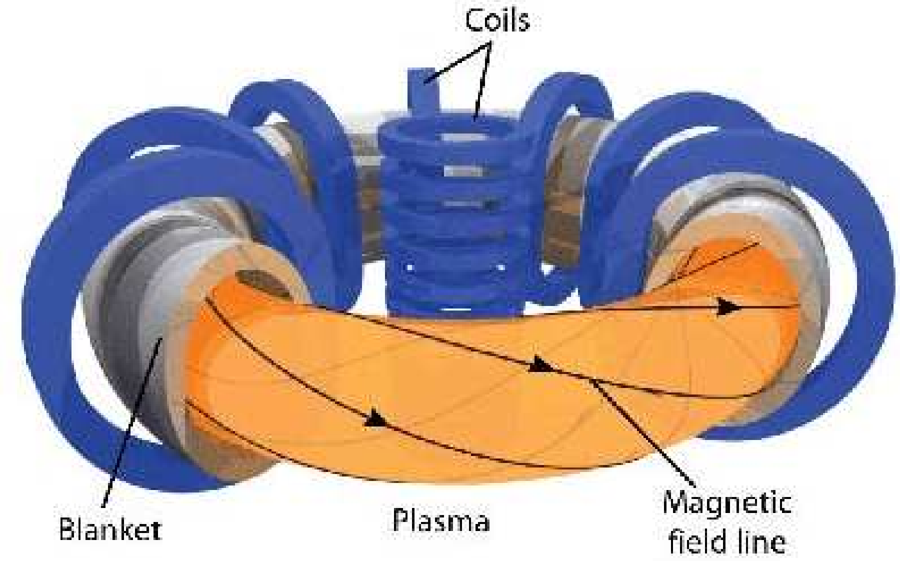
Toroidal current is induced by separate transformer winding

Produce magnetic flux $\Phi(t)$ $\Rightarrow$ *loop voltage* $U_\phi = -\dot{\Phi}$ $\Rightarrow$ $I_\phi = U_\phi / R$,

Iron core transformer:



Central solenoid with air core:

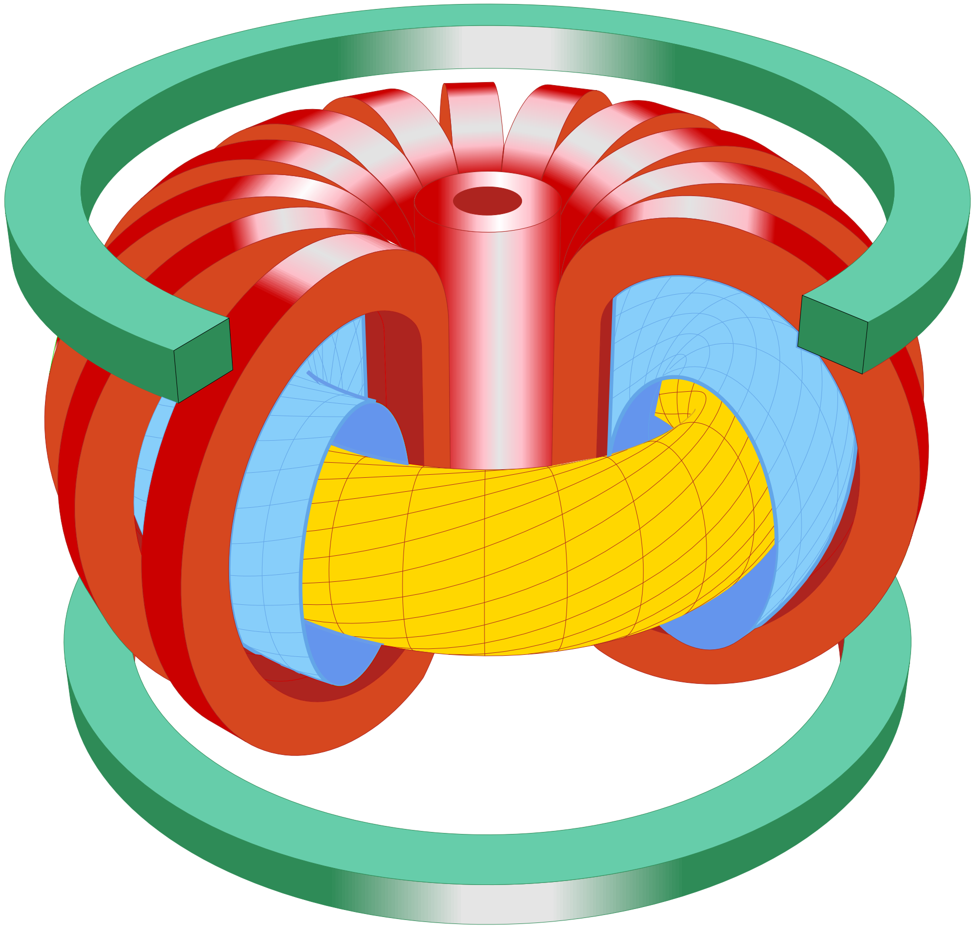


— Technically, current in primary winding is limited

$\Rightarrow$ *Pulsed* operation

(seconds in existing devices, 0.3-3 hours in a fusion plant)

The Tokamak



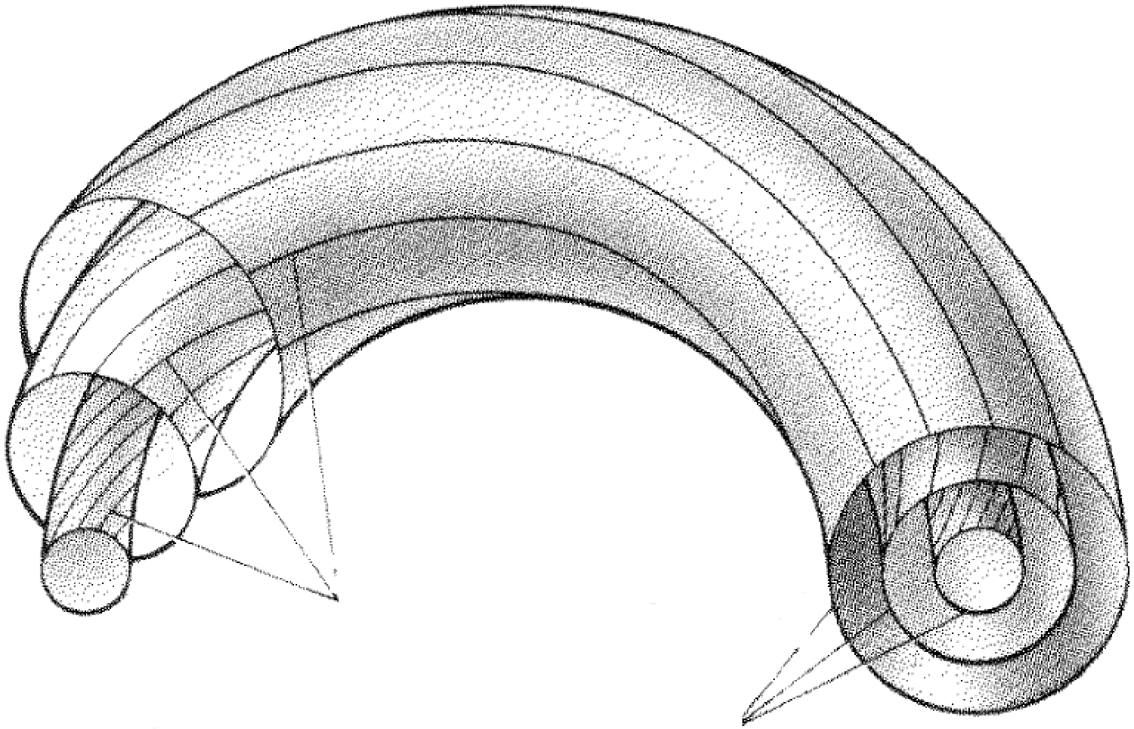
toroidalnaja **ka**mera (toroidal chamber)

magnitnaja **k**atushka (with magnetic coils)

Tamm, Sakharov (1952)

- Toroidal axisymmetric plasma
- Toroidal field from poloidal windings
- Central solenoid:
  $\rightarrow$ Toroidal plasma current I_ϕ
  $\rightarrow$ Ohmic heating: $P_{\mathrm{OH}} = U_\phi \, I_\phi$
- Vertical field from toroidal coils
  ... to balance "hoop force"
  ... for plasma shaping

Safety factor



Definition:

$$q = \frac{\text{Number of toroidal passes}}{\text{Number poloidal passes}}$$

…of a field line before it closes back in itself.

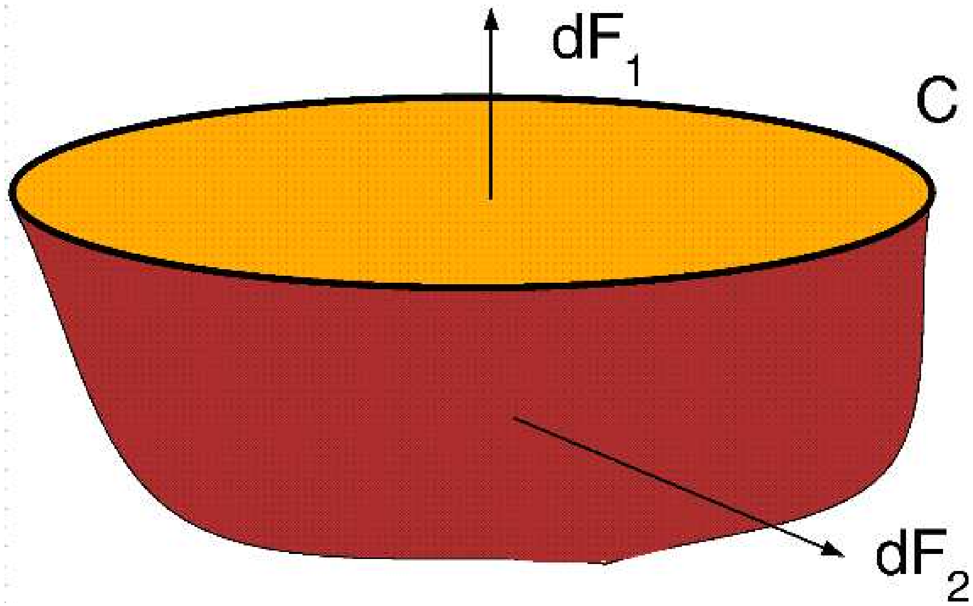
Cylindrical approximation ($r \ll R_0$):

$$q = \frac{r}{R}\frac{B_\phi}{B_p}$$

(Kink) stability requirements:

$$\boxed{q(r) > 1, \quad q(a) > 2}$$

Magnetic flux, electrical current flux



Maxwell: $\nabla \cdot \vec{B} = 0$,

No sources of electrical charge: $\nabla \cdot \vec{j} = 0$

$$\int_{F_1} \vec{B}\, \mathrm{d}\vec{A} + \int_{F_2} \vec{B}\, \mathrm{d}\vec{A} = \int_{F_1+F_2} \vec{B}\, \mathrm{d}\vec{A}$$

$$\underbrace{=}_{\text{Gauss' theorem}} \int_V \nabla \cdot \vec{B}\, \mathrm{d}V = 0$$

$\Rightarrow$ Fluxes

$\psi \equiv \int_F \vec{B}\, \mathrm{d}\vec{A}, \quad I \equiv \int_F \vec{j}\, \mathrm{d}\vec{A}$

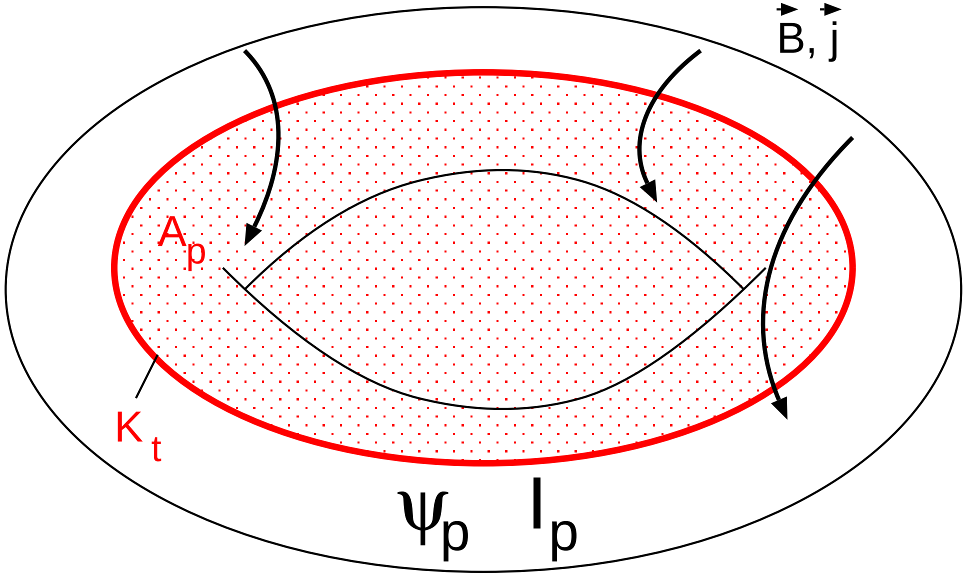
depend only on boundary of the corresponding area (not its shape).

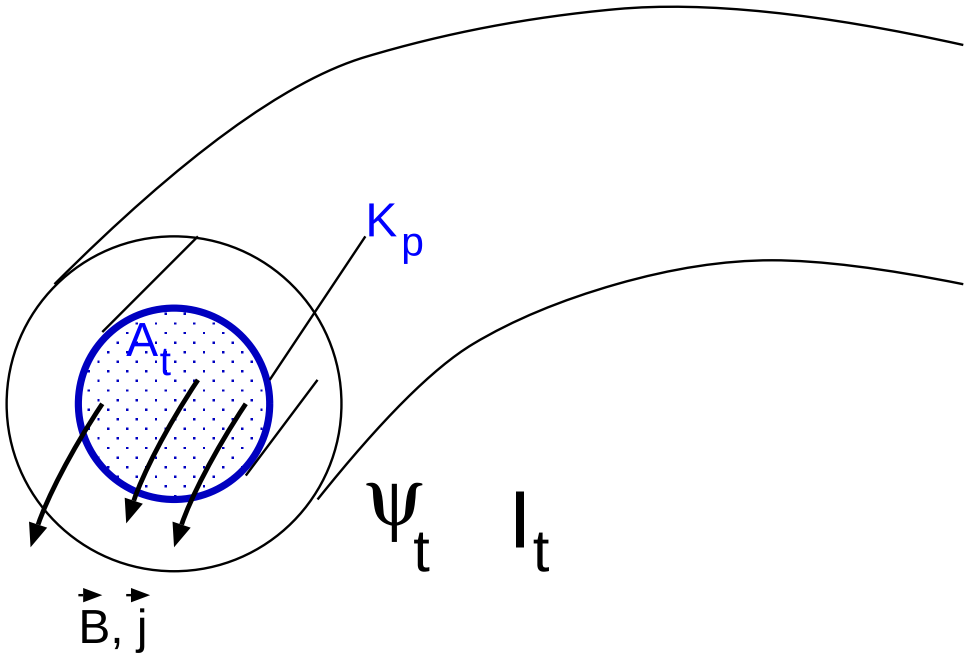
Poloidal und toroidal fluxes

Two ways of defining boundary in toroidal geometry:

Poloidal flux through surface with
– poloidal normal A_p
– toroidal boundary K_t

Toroidal flux through surface with
– toroidal normal A_t
– poloidal boundary K_p





Magnetic surfaces characterised by p, ψ_p und ψ_t. $\rightarrow$ "flux surface quantities"

By convention, choose poloidal fluxes (ψ_p, I_p).

Axisymmetric "equilibrium": Grad-Shafranov-Schlüter equation

$\vec{B}$ components can be expressed by fluxes:

$$B_R = -\frac{1}{2\pi R}\frac{\partial \psi_p}{\partial z}, \quad B_z = \frac{1}{2\pi R}\frac{\partial \psi_p}{\partial R}, \quad B_\phi = \frac{\mu_0}{2\pi R} I_p$$

Note: As I_p is a flux quantity, so is RB_ϕ

Ampère:

$$j_\phi = -\frac{1}{2\pi\mu_0}\left[\frac{1}{R}\frac{\partial^2\psi_p}{\partial z^2} + \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi_p}{\partial R}\right)\right]$$

or, with Δ^* ("*Elliptic differential operator*")

$$\boxed{j_\phi = -\frac{1}{2\pi\mu_0}\,\frac{1}{R}\,\Delta^*\psi_p}$$

Radial force balance $(\nabla p)_R = (\vec{j}\times\vec{B})_R$

…in cylindrical coordinates:

$$\frac{\partial p}{\partial R} = j_\phi B_z - j_z B_\phi$$

— Replace field by fluxes (ψ_p, I_p)

— divide by $\partial\psi_p/\partial R$

— use notation: $' \equiv \partial/\partial\psi_p$

$\rightarrow$ GSS equation for poloidal flux ψ_p

$$\boxed{\Delta^*\psi_p = -4\pi^2\mu_0 R^2 p' - \mu_0^2 I_p I_p'}$$

Non-linear differential equation in ψ_p!

Vertical magnetic field needed for toroidal solution

Consider Solovjev's analytical ansatz with constant $C_1 \dots C_5$, $C_1 = \mu_0^2 I_p I_p'$, $C_2 = 4\pi^2 \mu_0 p'$

$$\psi_p = \frac{C_1}{2} z^2 + \frac{C_2}{8} R^4 + \underbrace{C_3 + C_4 R^2 + C_5 \left(R^4 - 4R^2 z^2\right) + \dots}_{\text{solutions of homogeneous equation}}$$

Since

$$B_z = \frac{1}{2\pi R} \frac{\partial \psi_p}{\partial R}$$

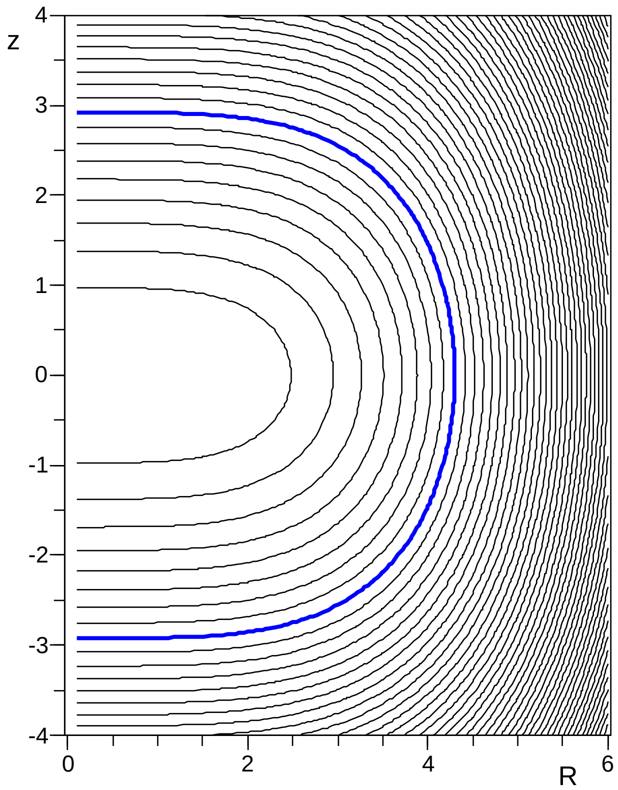
a homogeneous vertical field has

$$\psi_p \propto R^2$$

(Coefficient C_4 in above ansatz)

Without vertical field:

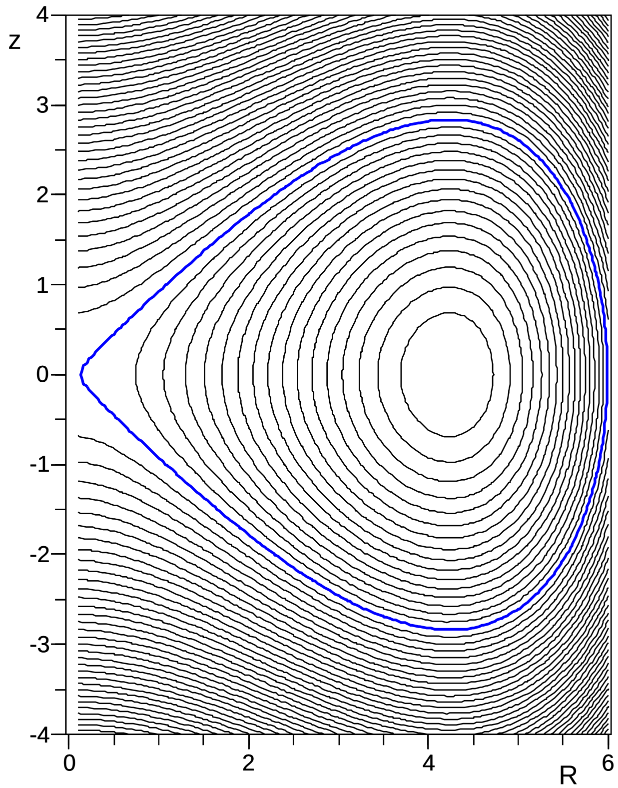
Spheroid



$C_1 = -10,\ C_2 = -1,\ C_3 = C_5 = 0$ $\quad C_4 = 0$

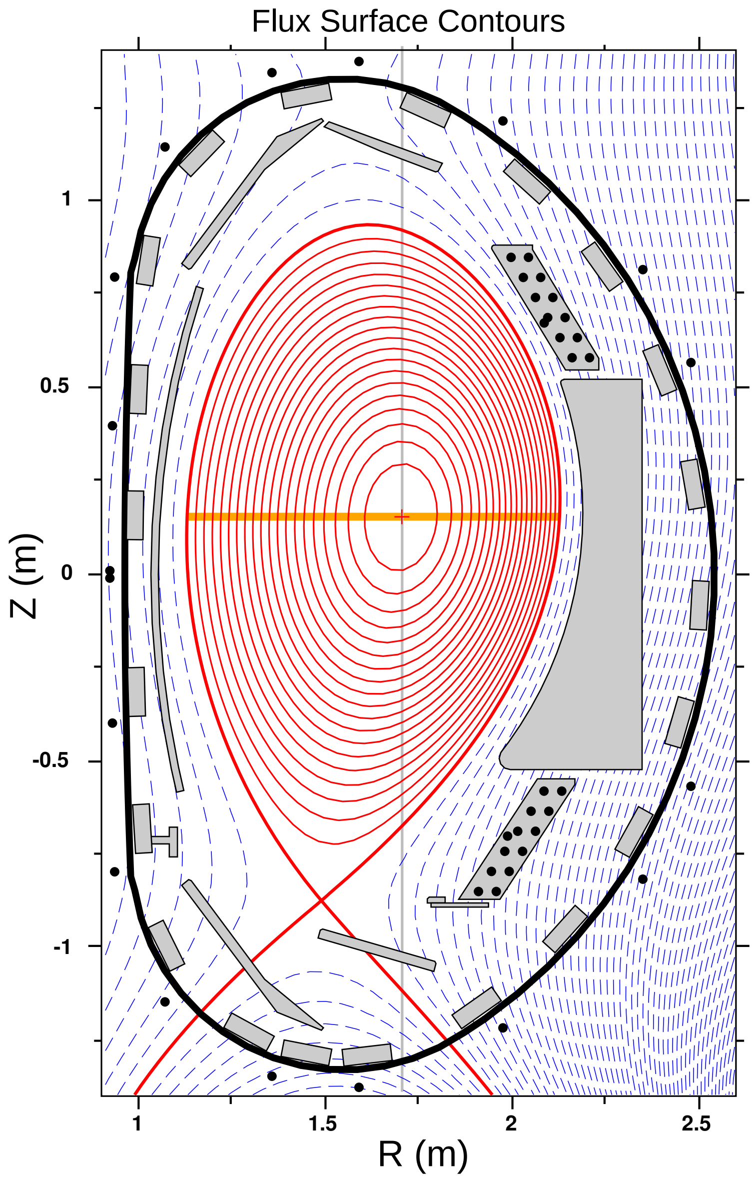
With vertical field:

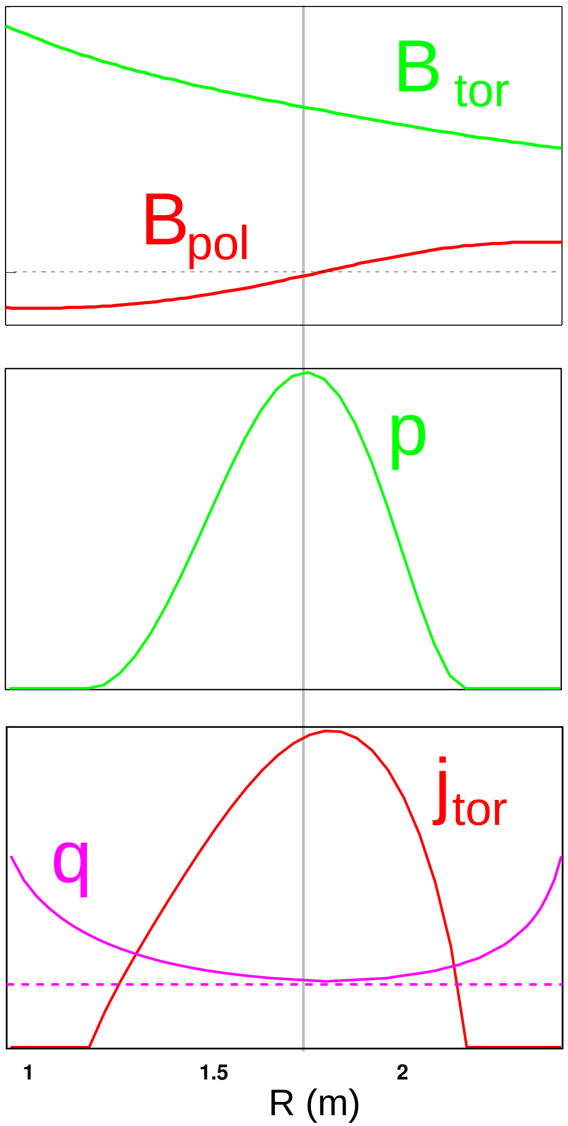
Torus



$C_4 = 4.49$

Tokamak configuration





GSS - numerical solution

Magnetic shear q':

— "built-in"

(j_{tor}, B_{pol} profiles)

— provides stability

"Divertor":

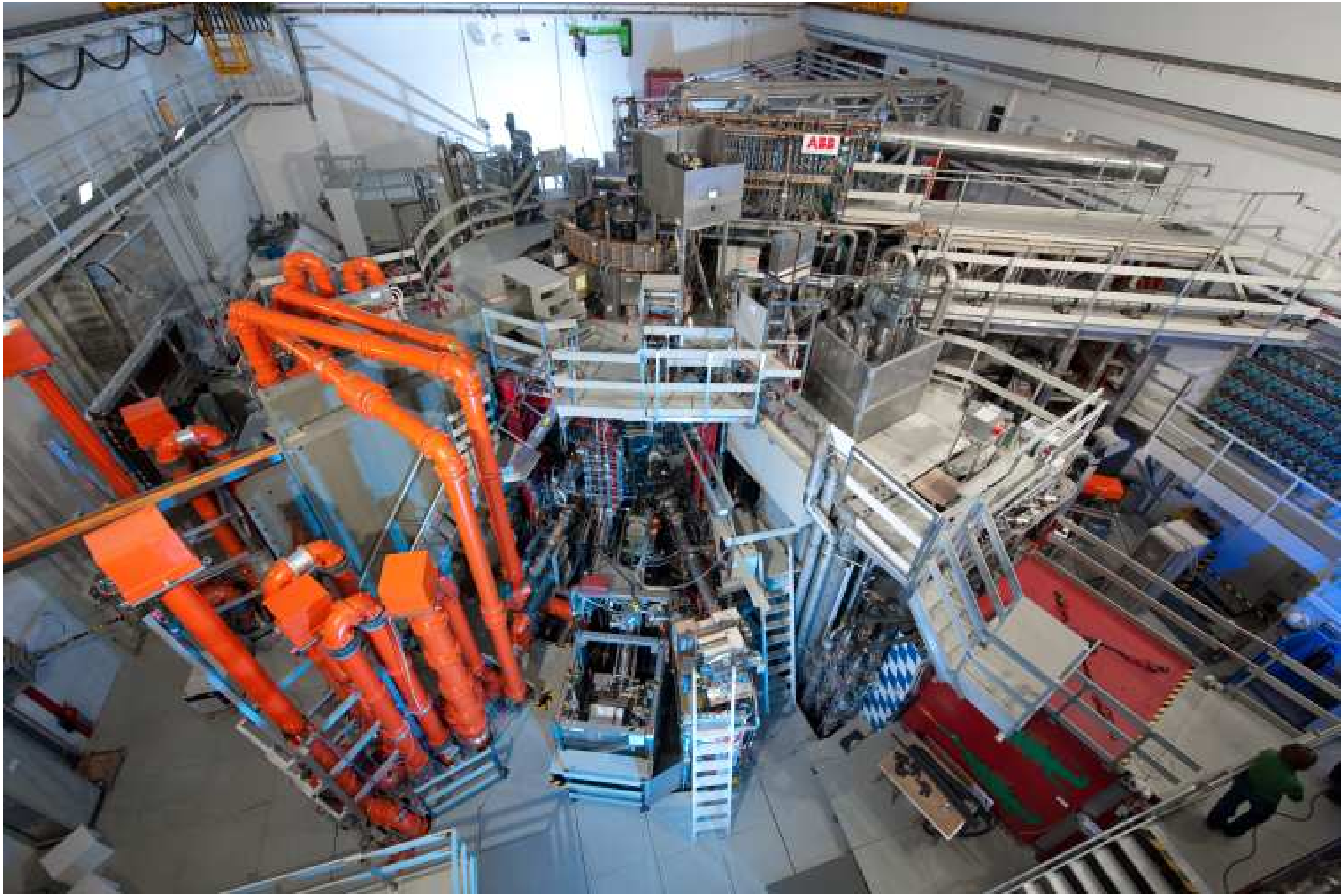
Plasma current

+ external coil:

"X-line" (zero $B_p = 0$)

$\rightarrow$ long (parallel) connection length to wall, favourable for power exhaust

Axial-Symmetric Divertor EXperiment (ASDEX) Upgrade

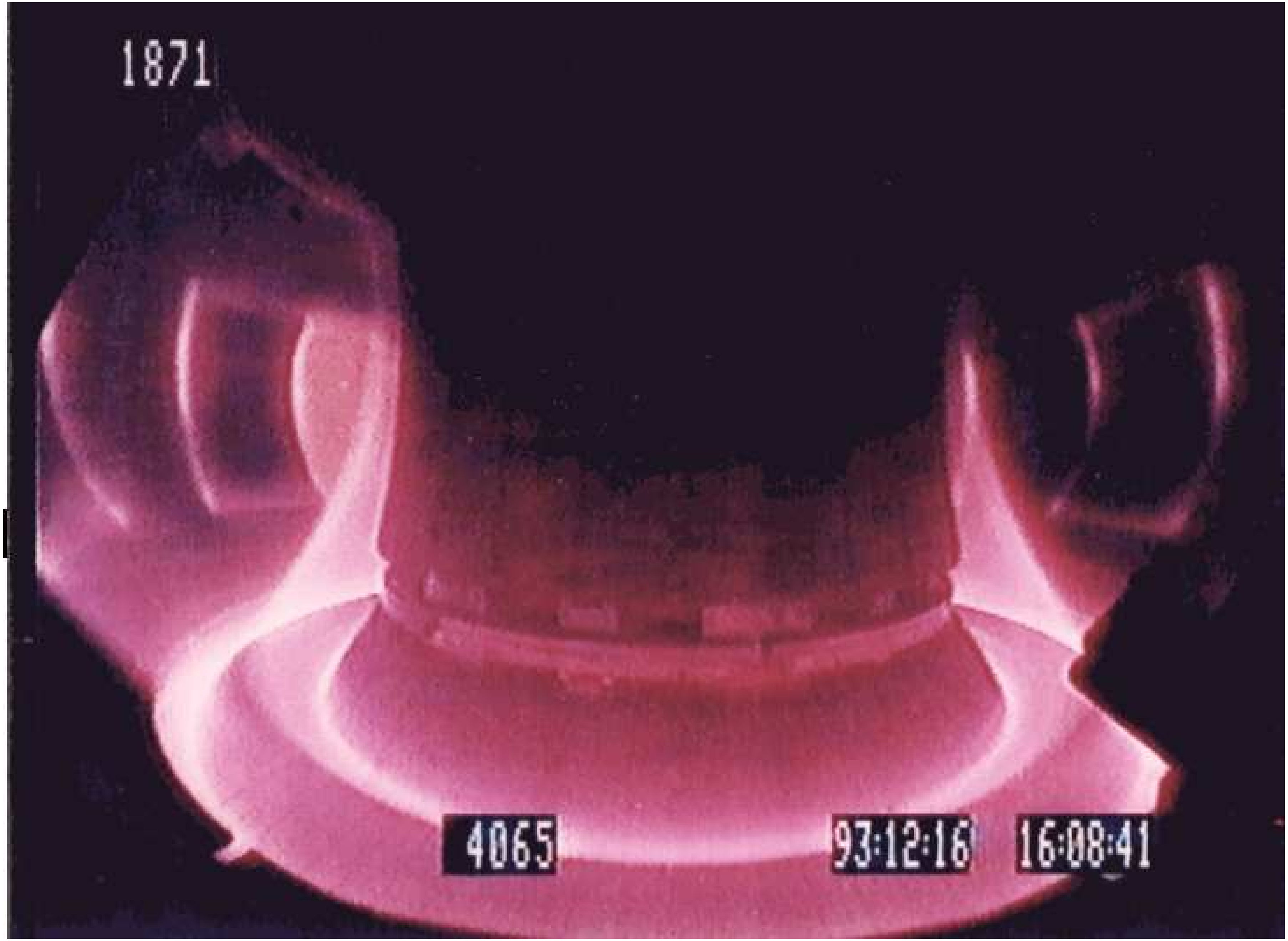


Max-Planck-Institut für Plasmaphysik (Garching)

ASDEX Upgrade, view into vacuum vessel

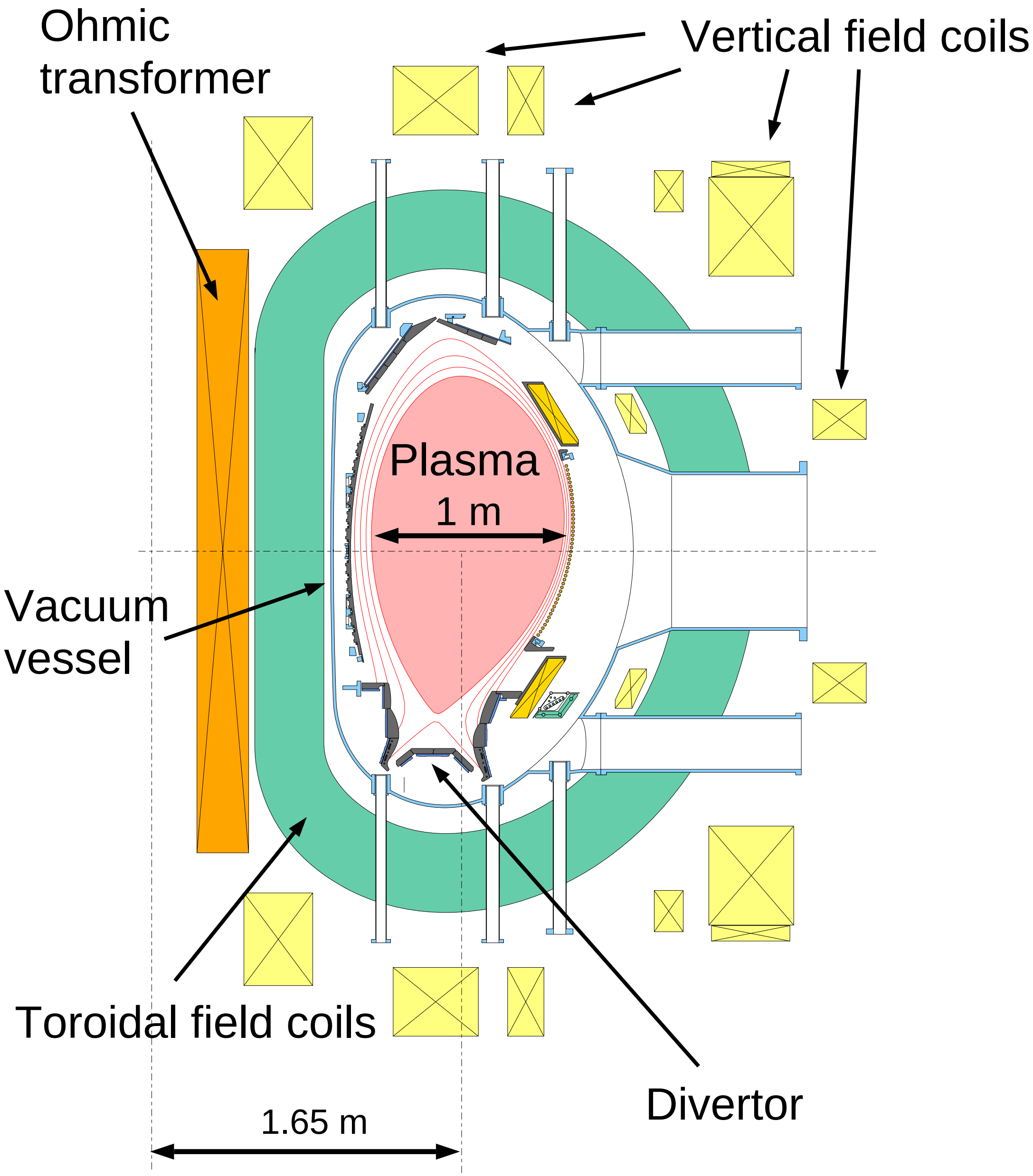
Max-Planck-Institut für Plasmaphysik (Garching)

ASDEX Upgrade, view of plasma



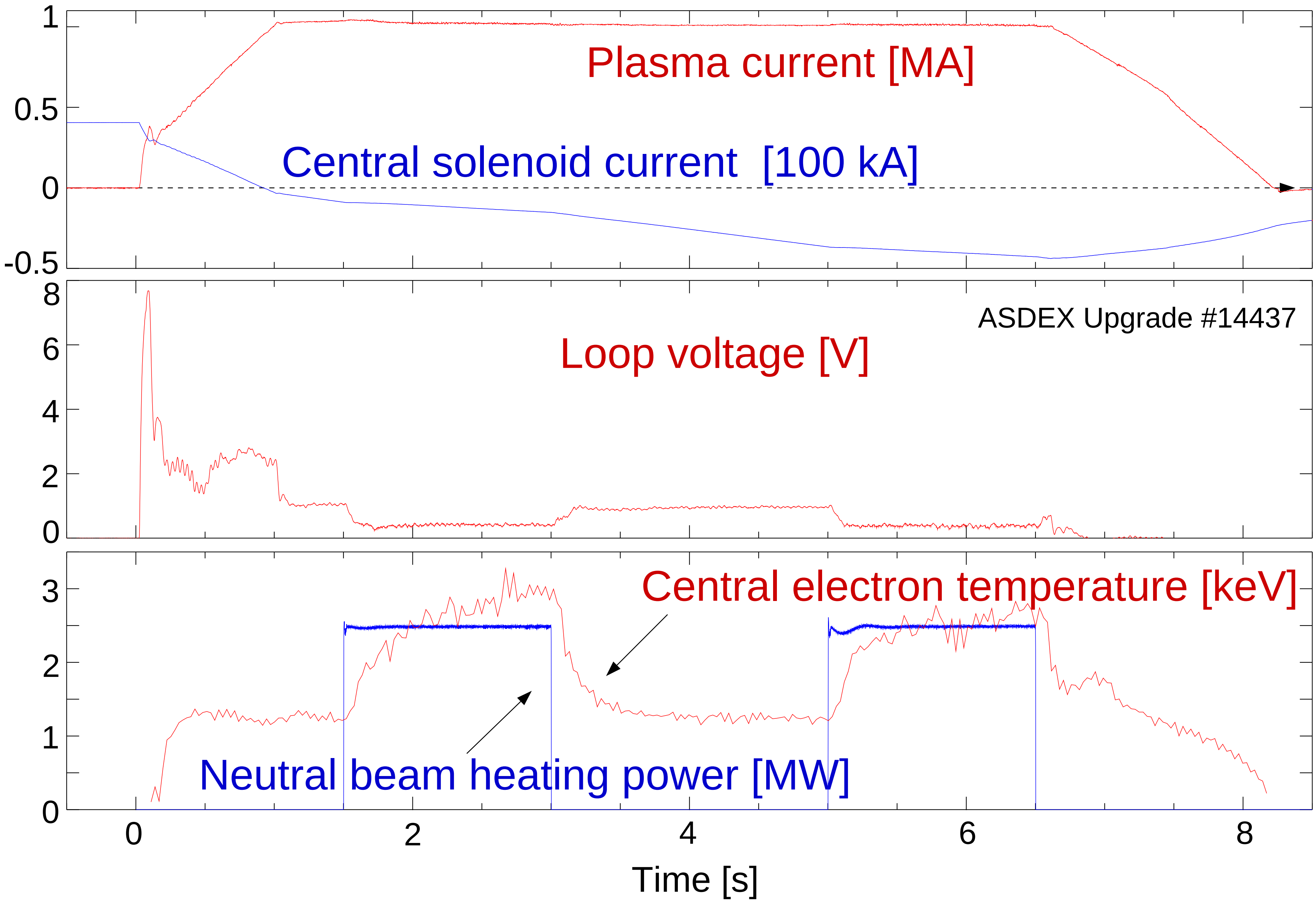
Max-Planck-Institut für Plasmaphysik (Garching)

ASDEX Upgrade, Technical data

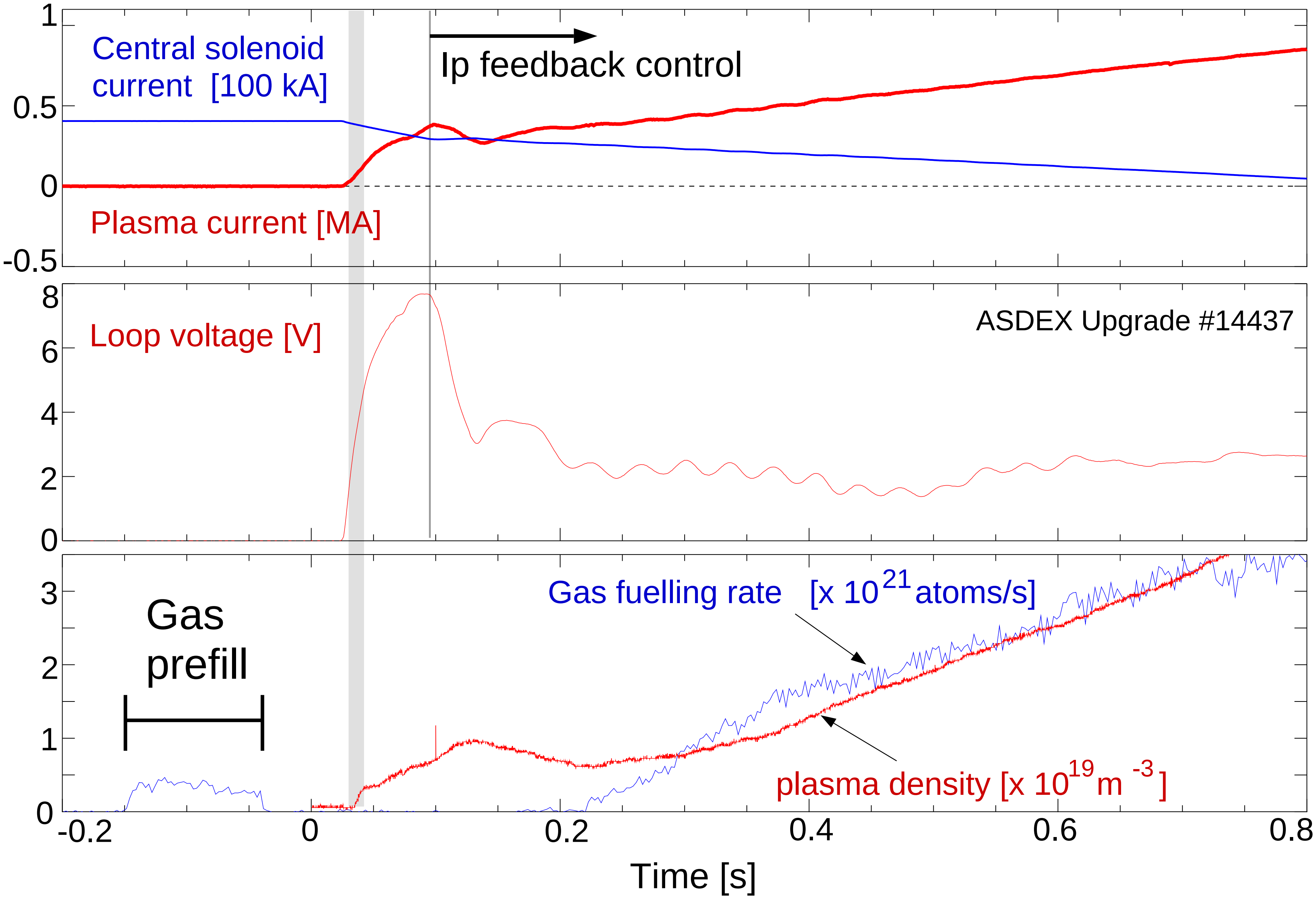


Major plasma radius	R	1.65	m
minor plasma radius	a	0.5	m
tor. plasma current	I_ϕ	≤ 1.6	MA
toroidal field	B_ϕ	≤ 3.2	T
vacuum base pressure	p	$\leq 1 \times 10^{-7}$	mbar
plasma heating			
Neutral beam injection	P_{NBI}	≤ 20	MW
Ion cyclotron heating	P_{ICRH}	≤ 8	MW
Electron cyclotron heating	P_{ECRH}	≤ 4	MW
typ. plasma performance			
confinement time	τ_E	≤ 0.15	s
stored kinetic energy	W_{kin}	≤ 1	MJ
poloidal beta	β_p	≤ 3	

Time traces of a plasma discharge



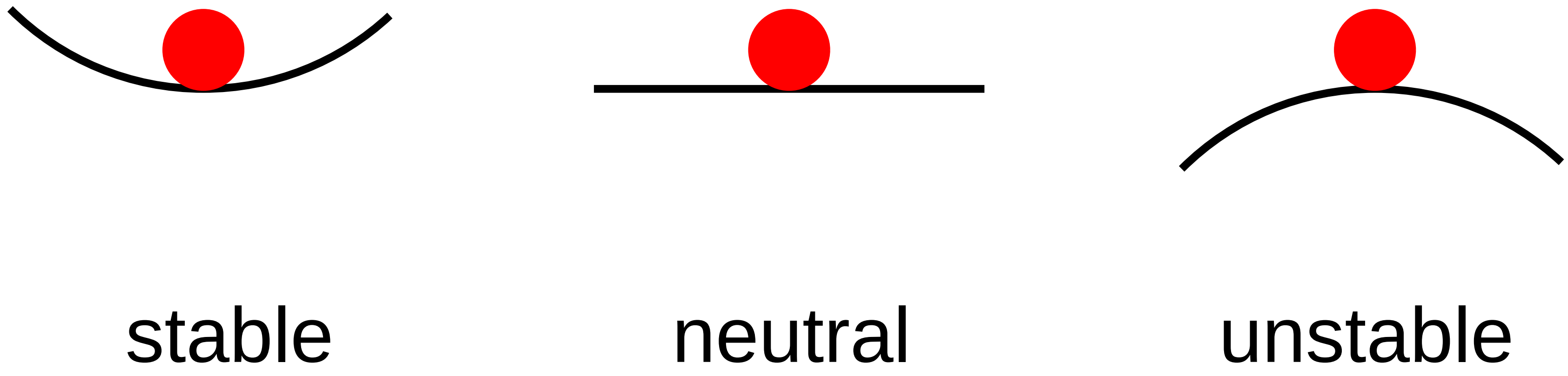
Start-up of a plasma discharge



More experimental results in a separate talk!

Comments on tokamak stability

Force equilibria can be ...



Consider potental energy W.

Force balance: $\mathrm{d}W/\mathrm{d}\xi = F_{\mathrm{net}} = 0$

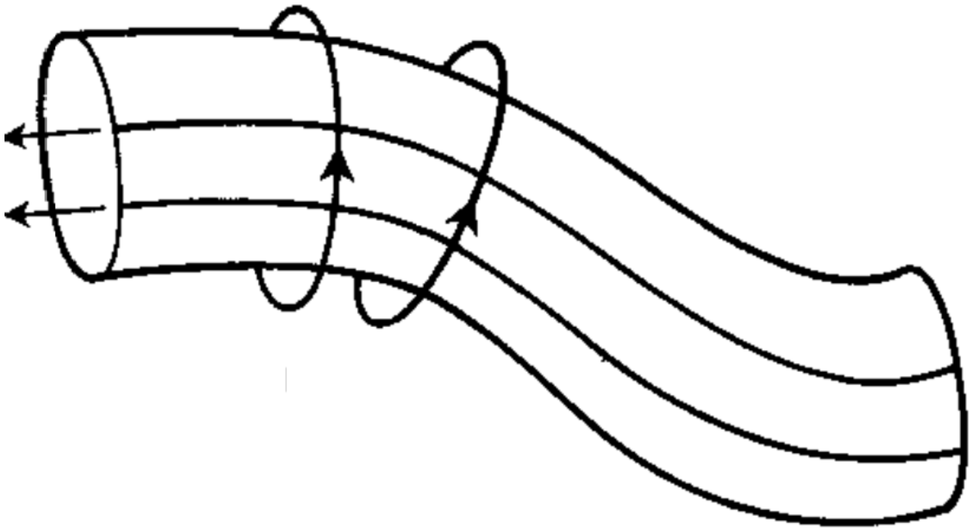
The configuration is

— **Stable**, if $\mathrm{d}^2W/\mathrm{d}\xi^2 > 0$ for all possible displacements ξ

— **Unstable**, if $\mathrm{d}^2W/\mathrm{d}\xi^2 < 0$ for one displacement ξ

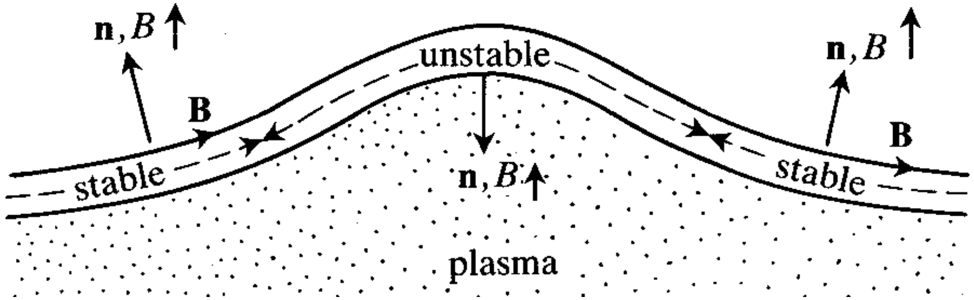
Instability drive (free energy source)

Plasma currents



Example: Kink instability

Pressure gradient



Example: Interchange Instability

— Instabilities can limit tokamak operational space (q, β) → next talk

— Plasma displacement breaks axisymmetry

Toroidal and poloidal periodicity ⇒ discrete modes (m, n)

— Magnetic topology can change (“magnetic reconnection”) or not

Thought experiment: Periodic field perturbation in a slab

"toroidal" direction (periodic):

$$z = 0 \ldots 2\pi$$

"poloidal" direction (periodic):

$$x = -\pi \ldots \pi$$

"radial" direction:

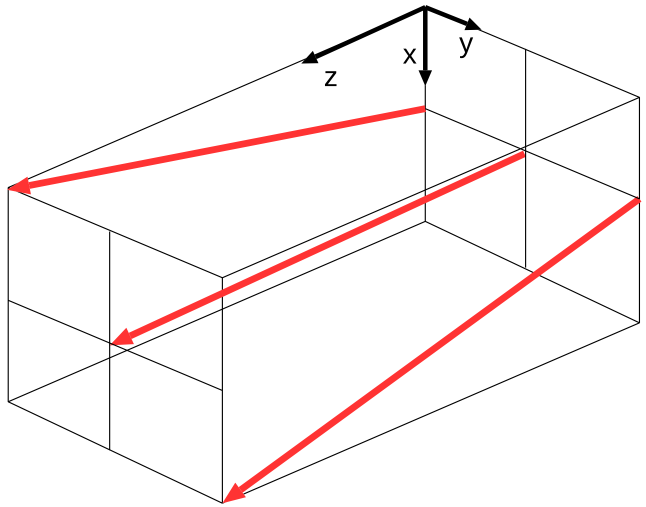
$$y = -1 \ldots 1$$

Backgound "toroidal" field:

$$B_z = \text{const.}$$

Sheared "poloidal" field:

$$B_x = B_x' y$$



Magnetic perturbation:

(a "radial" field that normally would not exist)

$$B_y = B_{y,0} \cos(mx - nz)$$

Notes:

— Ignore $\nabla \cdot \vec{B} = 0$ for now

— n: No. of periods in "toroidal" z-direction

— m: No. of periods in "poloidal" x-direction

— *Globally* imposed perturbation

(i.e. does not depend on "radius" y)

Follow field lines (solve ODE, see appendix):

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{B_x}{B_z}, \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{B_y}{B_z}$$

Result of field line tracing: Magnetic island ...

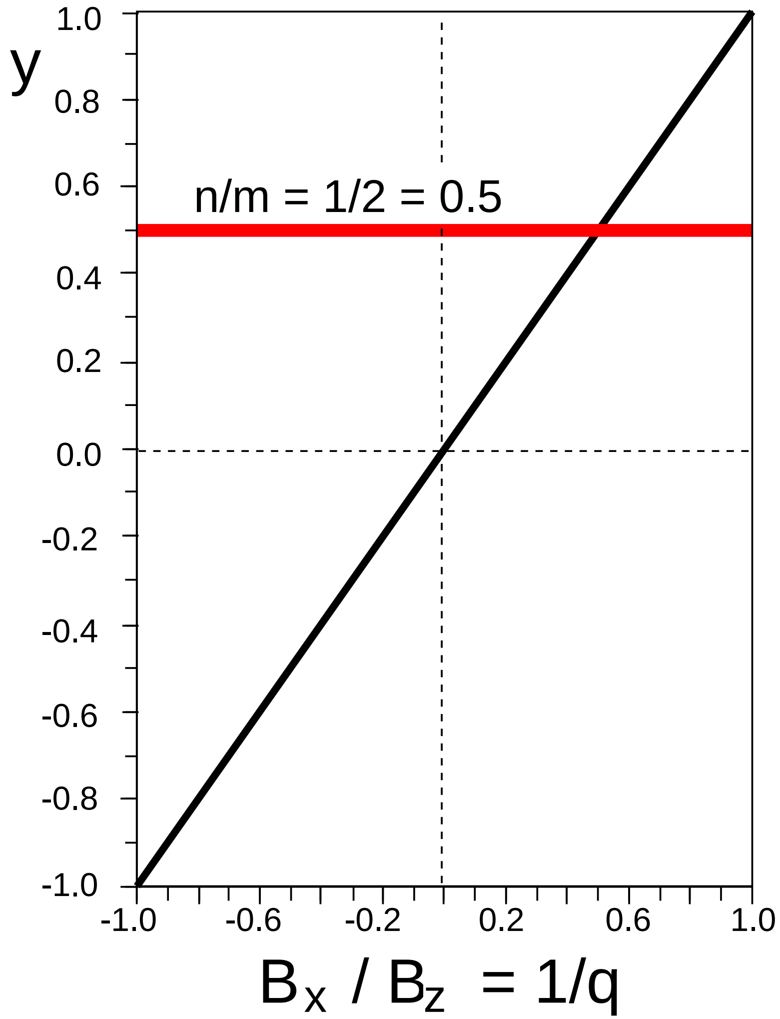
... at the ***resonant surface***, i.e. where $B_x/B_z = 1/q = n/m$,

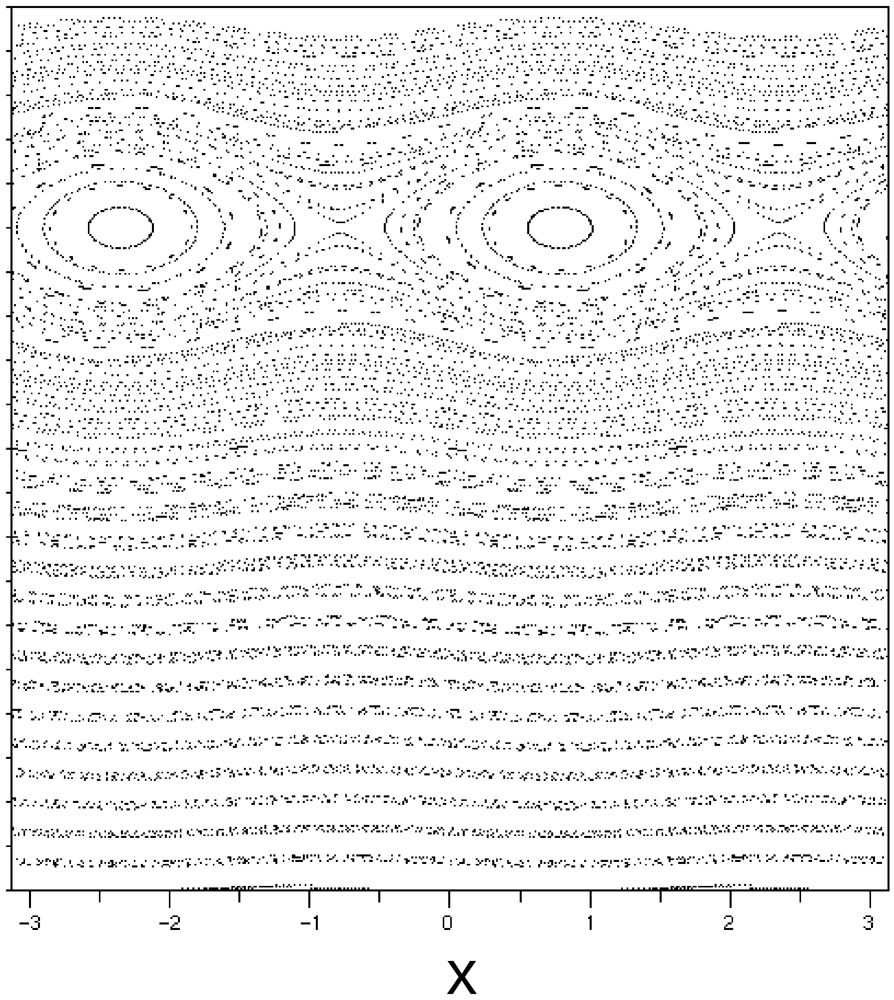
... and only field line bending everywhere else

$n = 1$

$m = 2$

$B_{y,0} =$
$0.025\ B_z$





Note: “Vacuum” perturbation — no helical structure of plasma current needed

Result of field line tracing: Magnetic island . . .

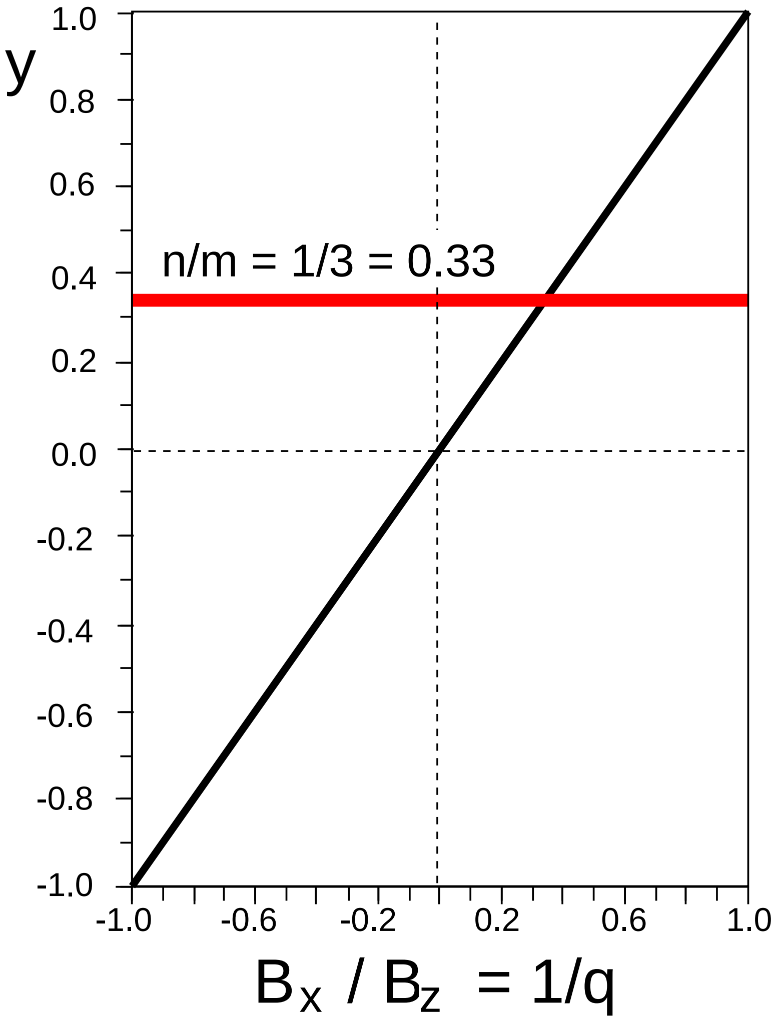
. . . at the ***resonant surface***, i.e. where $B_x/B_z = 1/q = n/m$,

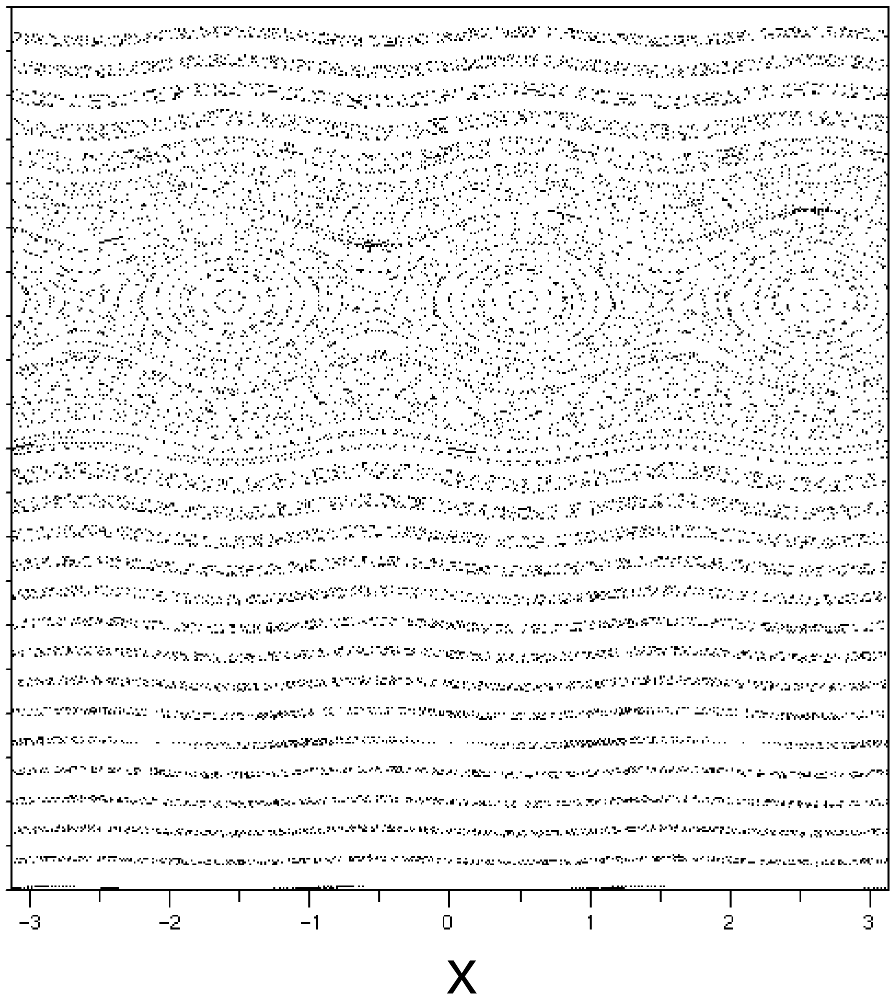
. . . and only field line bending everywhere else

$n = 1$

$m = 3$

$B_{y,0} =$
$0.025\ B_z$





Note: "Vacuum" perturbation — no helical structure of plasma current needed

Result of field line tracing: Magnetic island ...

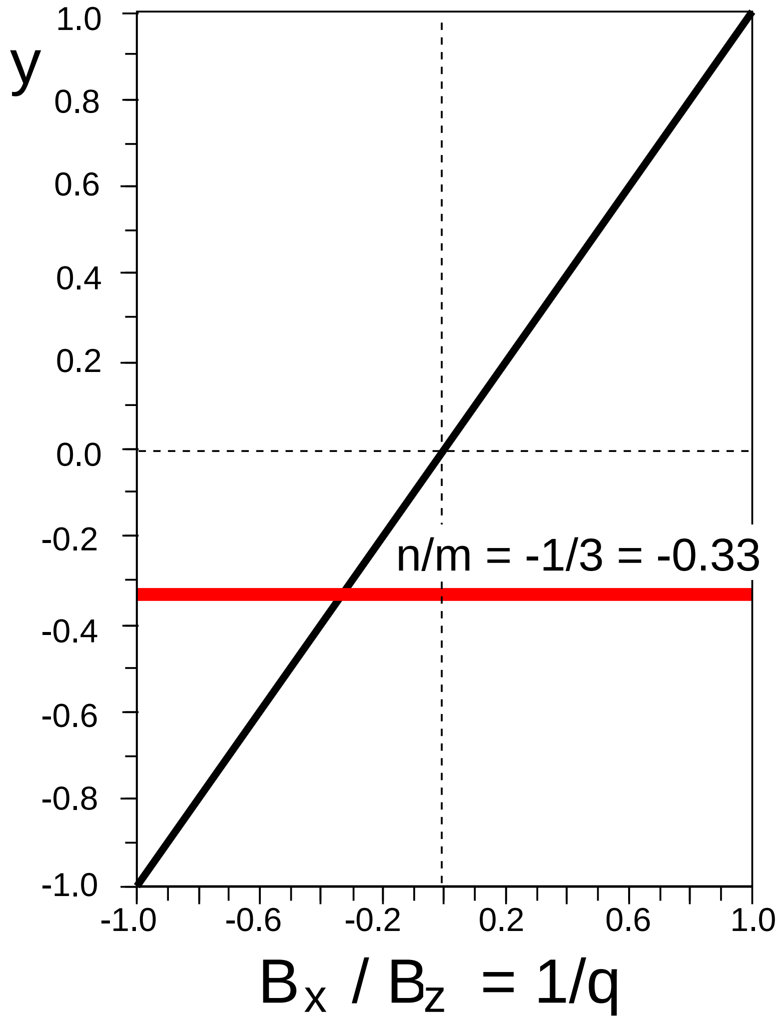
... at the ***resonant surface***, i.e. where $B_x/B_z = 1/q = n/m$,

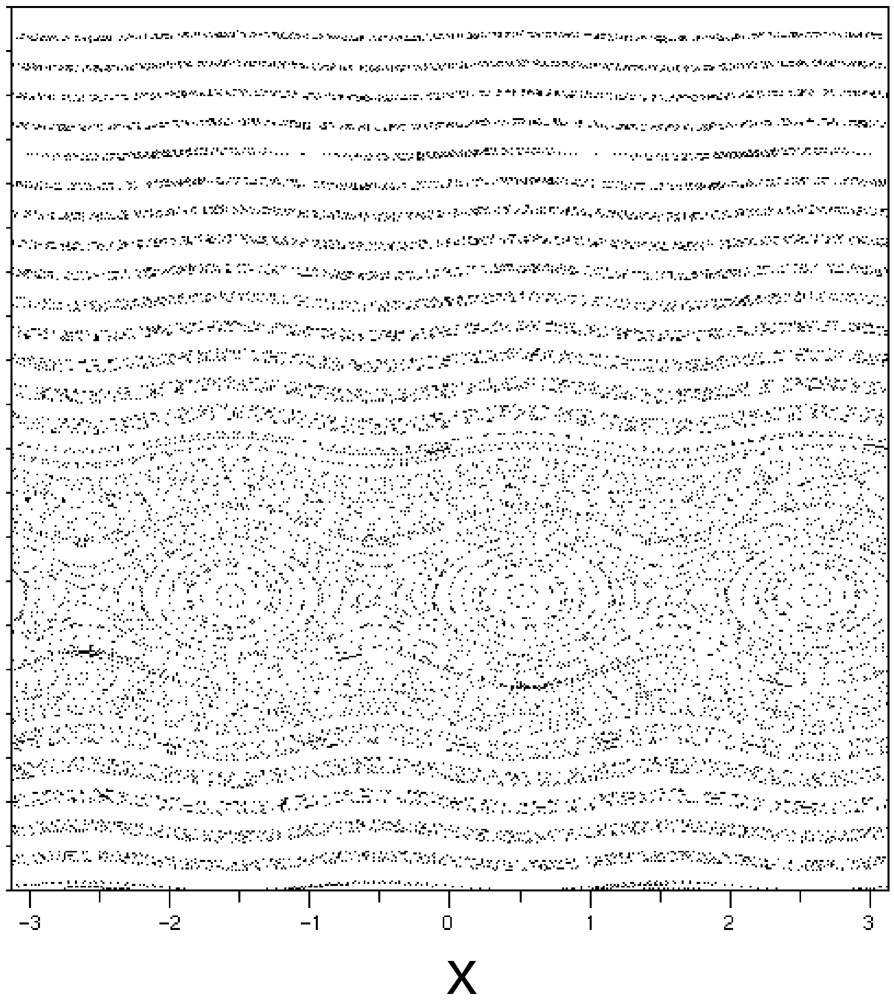
... and only field line bending everywhere else

$n = 1$

$m = -3$

$B_{y,0} = 0.025\ B_z$





Note: “Vacuum” perturbation — no helical structure of plasma current needed

Result of field line tracing: Magnetic island . . .

. . . at the ***resonant surface***, i.e. where $B_x/B_z = 1/q = n/m$,

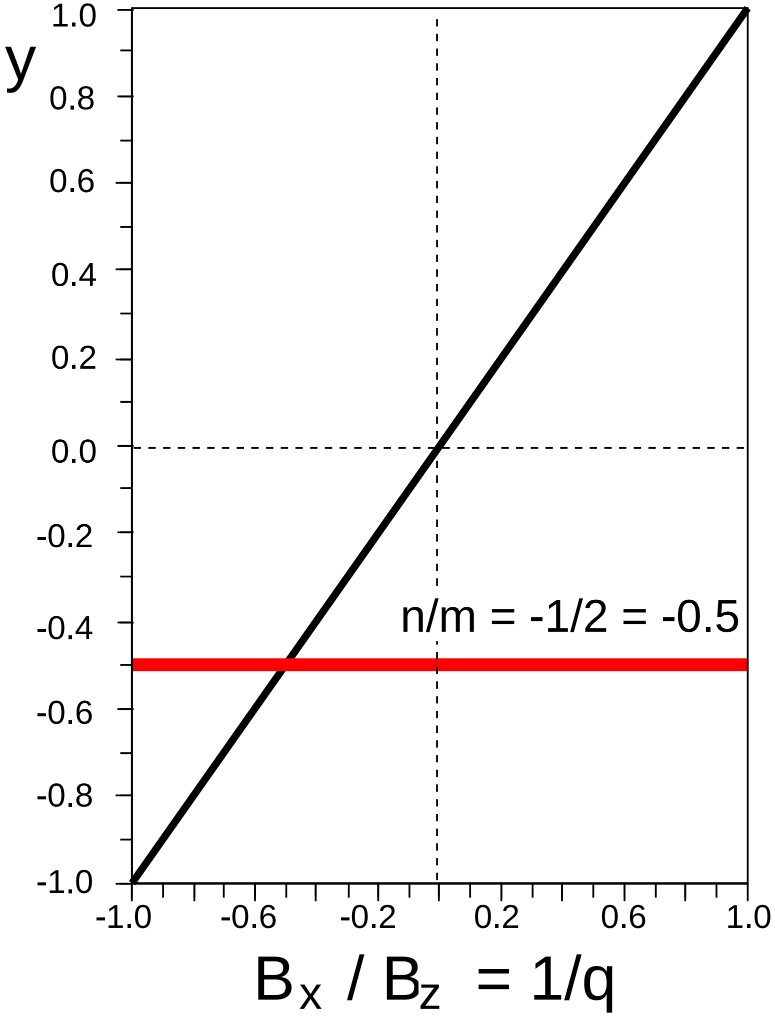
. . . and only field line bending everywhere else

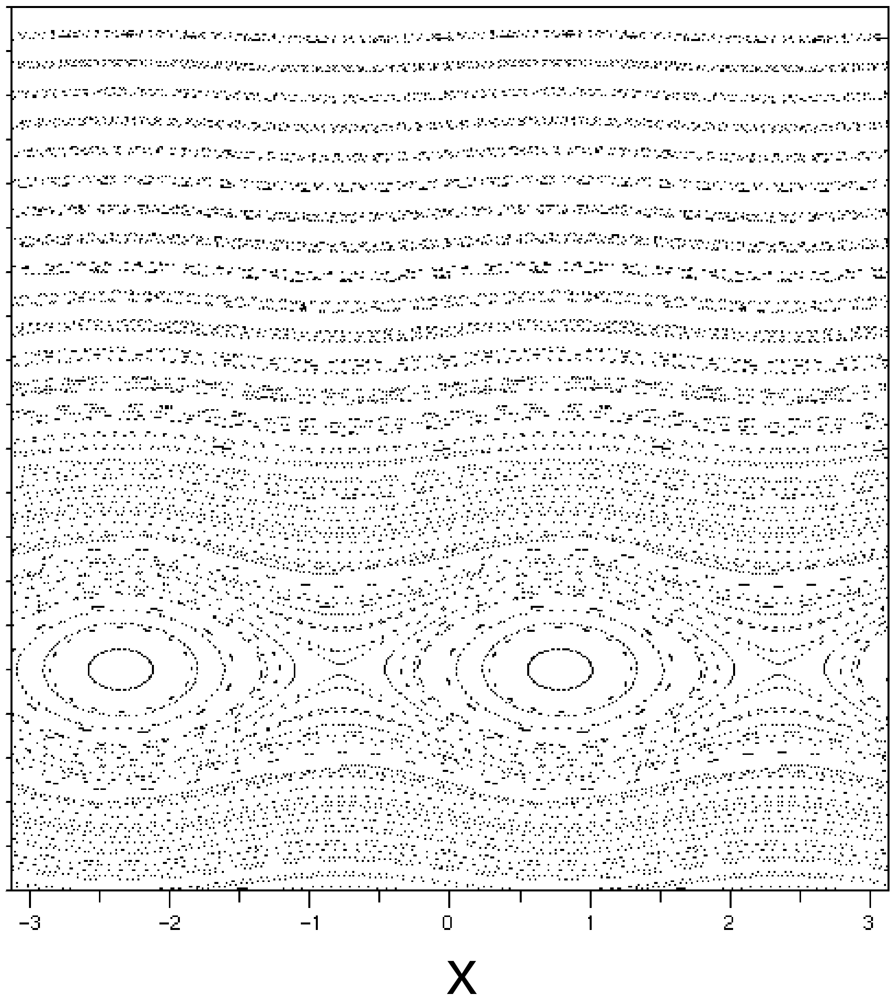
$n = 1$

$m = -2$

$B_{y,0} =$

$0.025\ B_z$



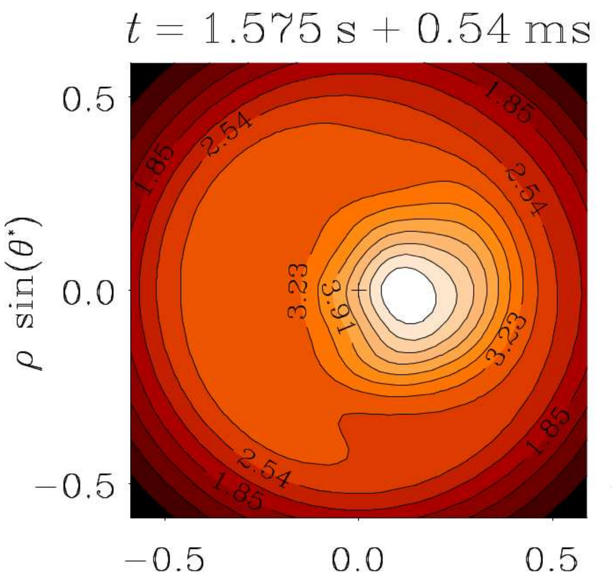


Note: “Vacuum” perturbation — no helical structure of plasma current needed

Magnetic islands can occur in tokamaks . . .

. . . under normal conditions.

- Helical deformation of plasma core
  $n = 1$, $m = 1$ mode if $q = 1$ surface exists



Rotational tomographic reconstruction (poloidal section)
of Soft X-ray emission in AUG #7962,
M. Sokoll, Ph.D. 1997

. . . under abnormal conditions.

- **Tearing instability**
- Other instabilities that produce helical perturbation currents
  ("Neoclassical tearing mode", Radiation instability → next talk)
- Driven by external "error" field

Tearing instability

Faraday

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

Ohm (1-fluid MHD)

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j}$$

Ampère

$$\vec{j} = \frac{1}{\mu_0} \nabla \times \vec{B}$$

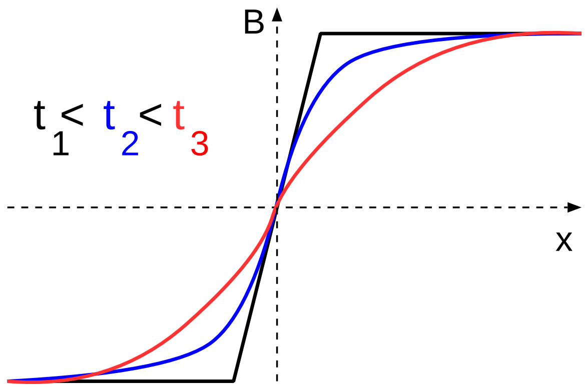
Vector algebra, $\nabla \cdot \vec{B} = 0$

$$\nabla \times \nabla \times \vec{B} = -(\nabla \cdot \nabla) \vec{B}$$

$\Rightarrow$ Magnetic field evolution:

$$\frac{\partial \vec{B}}{\partial t} = \underbrace{\nabla \times \left(\vec{u} \times \vec{B}\right)}_{\text{convection}} - \underbrace{\frac{\eta}{\mu_0} (\nabla \cdot \nabla) \vec{B}}_{\text{diffusion}}$$

Diffusion smoothes “sharp corners”:



Magnetic free energy $W = \int \mu_0 B^2/2$ drops

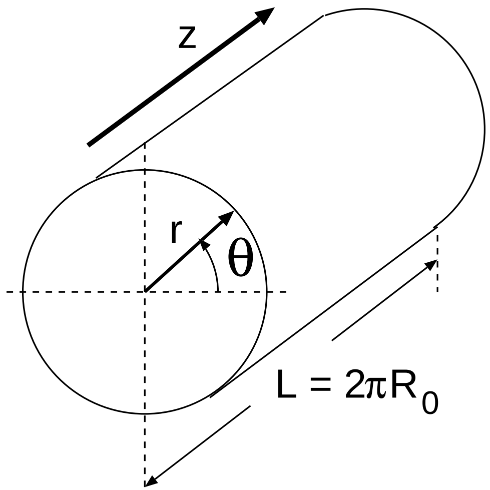
Perturbed quantities$_1$ $\ll$ equilibrium quantities$_0$,
$\Rightarrow$ Linearisation (keep first order), $u_0 = 0$

$$\frac{\partial \vec{B}_1}{\partial t} = \nabla \times \left(\vec{u}_1 \times \vec{B}_0\right) - \frac{\eta}{\mu_0} (\nabla \cdot \nabla) \vec{B}_1$$

Tearing in straight cylinder ("screw pinch") geometry

Cylinder coordinates (r, θ, z)

Periodic in θ and z



Sheared field with safety factor

$$q(r) = \frac{r}{R_0}\frac{B_{0,z}}{B_{0,\theta}}$$

Equilibrium magnetic field

$$\vec{B}_0(r) = B_{0,\theta}\left(0,\ 1,\ q(r)\frac{R_0}{r}\right)$$

Radial component of B_1:

$$\frac{\partial \vec{B}_{1,r}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial\theta}(u_{1,r}B_{0,\theta}) - \frac{\partial}{\partial z}(u_{1,r}B_{0,z}) + \frac{\eta}{\mu_0}\frac{1}{r}\frac{\partial^2}{\partial r^2}(rB_{1,r})$$

Spatial Fourier-Ansatz, exponential growth in time:

$$B_{1,r} \equiv b_r\, e^{i(m\theta - kz)}\, e^{\gamma t}, \qquad u_{1,r} \equiv u_r \ldots$$

where $kz = n\phi$, $n = kR_0$.

$$\gamma\, rb_r = iu_r B_{0,\theta}(m - nq(r)) + \frac{\eta}{\mu_0}\frac{\partial^2}{\partial r^2}(rb_r)$$

— For $q \rightarrow m/n$ left r.h.s. term disappears.

Any (small) resistivity matters in a thin *resistive layer*!

— Stability depends on sign of right r.h.s. term

Tearing stability criterion

Define helical flux

(stream function for perturbation)

$$\Psi_1 \equiv \psi_1 \, e^{i(m\theta - kz)} \, e^{\gamma t},$$

so that

$$b_r = \frac{1}{r}\frac{\partial}{\partial\theta}\psi_1$$

and

$$\gamma\psi_1 = u_r B_{0,\theta}\left(1 - \frac{n}{m}q(r)\right) + \frac{\eta}{\mu_0}\psi_1''$$

Resistive layer, $r = r_s - \delta_r \ldots r_s + \delta_r$,

$$\gamma \sim \frac{\eta}{\mu_0}\frac{\psi_1''}{\psi_1}$$

Integrate over resistive layer:

$$\Delta' \equiv \int_{r_s-\delta_r}^{r_s+\delta_r} \frac{\psi_1''}{\psi_1} \mathrm{d}r \sim \frac{1}{\psi_1}\left.\frac{\partial\psi_1}{\partial r}\right|_{r_s-\delta_r}^{r_s+\delta_r}$$

Instability grows depending on curvature of ψ_1 at r_s.

— This depends on ψ_1 in the region <u>outside</u> the resistive layer.

— At small η, fluid motion is slow and ideal force balance can be used outside the resistive layer,

Recipe: (not much detail here, sorry)

$$0 = \nabla\times\nabla p = \nabla\times\left(\vec{j}\times\vec{B}\right) = \left(\vec{B}\cdot\nabla\right)\vec{j} - \left(\vec{j}\cdot\nabla\right)\vec{B}$$

Linearise, $b_\theta = -\partial\psi_1/\partial r$:

$$B_{0,\theta}\left(m - nq(r)\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_1}{\partial r}\right) - \frac{m^2}{r^2}\psi_1\right] - m\psi_1\mu_0\frac{\partial j_{z,0}}{\partial r} = 0$$

Solve this diffeq for ψ_1^- from $r = 0 \ldots r_s - \delta_r$,

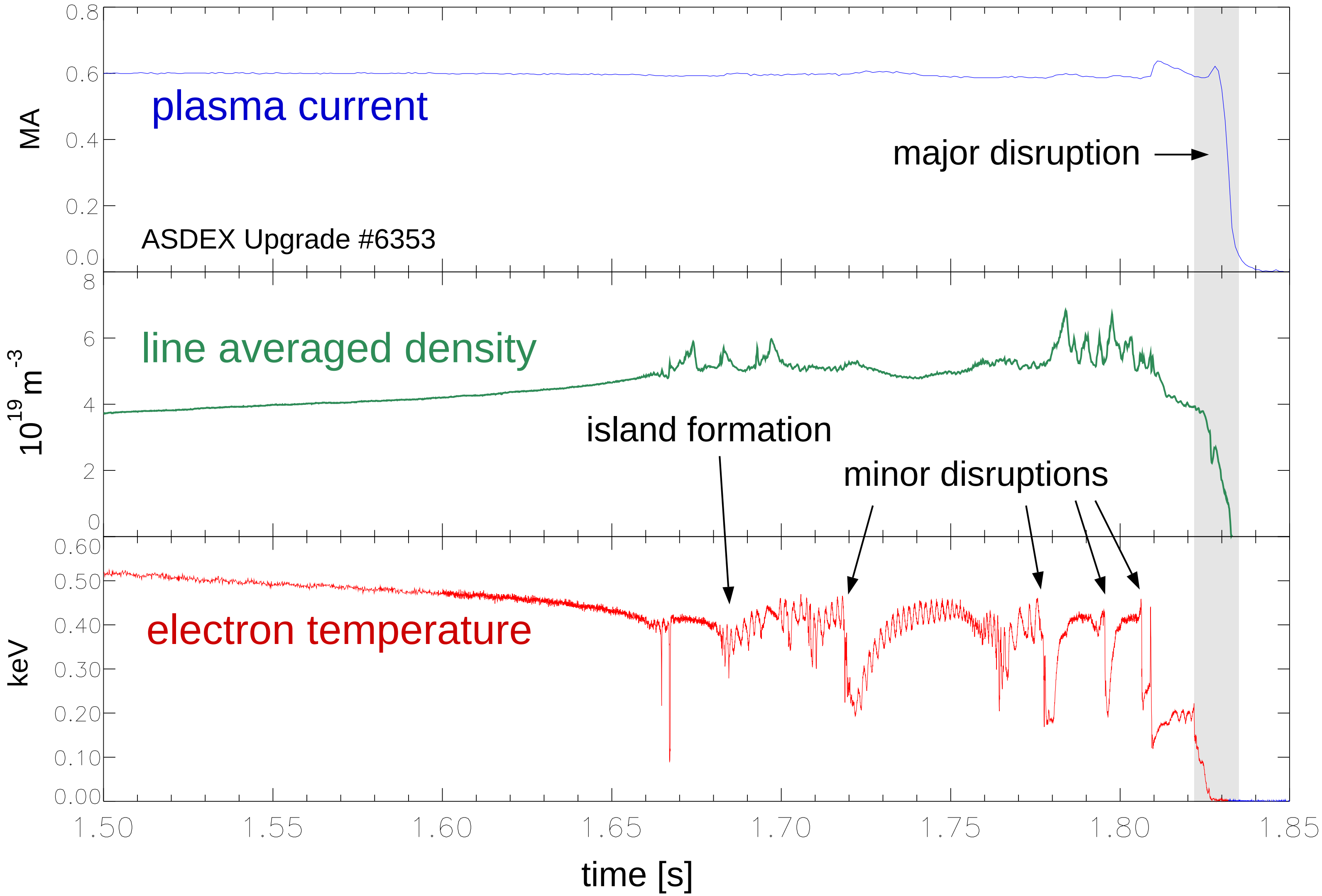
and ψ_1^+ from $r = a \ldots r_s + \delta_r$,

$$\Delta' \sim \delta_r^{-1}\frac{\psi_1^+ - \psi_1^-}{\psi_1^+ + \psi_1^-}$$

Magnetic islands at the density limit

When approaching the densit limit (→ next talk), plasma cools down and islands grow.

Rotating islands are easy to diagnose:



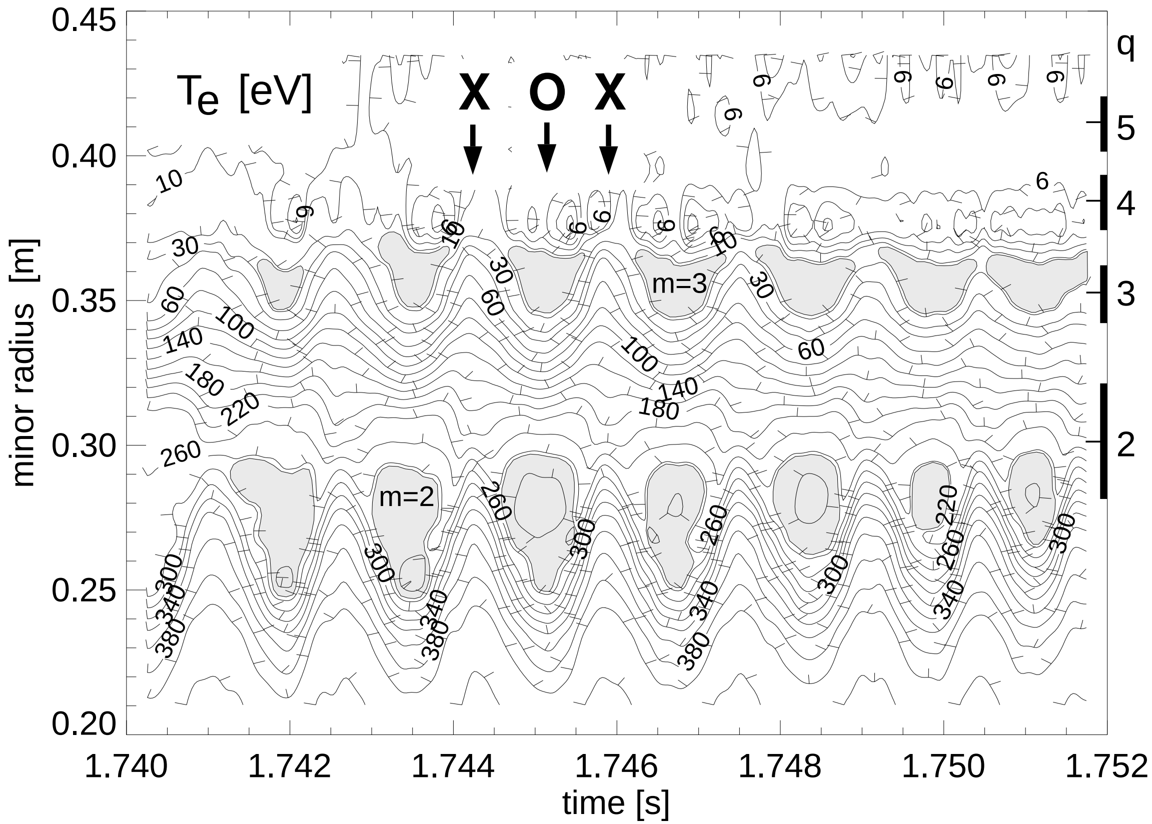
Rotating islands can be diagnosed by T_e measurements

Very fast heat conduction $\|\vec{B}$ makes T_e = const. on flux surfaces.

Island formation (topology change!) can be seen as islands rotate in front of radial diagnostics sightline.

At least two island chains ($q = 2$, $q = 3$).

T_e is flat across island
— confinement reduction



W Suttrop *et al* Nucl. Fus. **37** (1997) 119

Summary

Tokamak equilibrium

- Ideal MHD force balance: $\nabla p = \vec{j} \times \vec{B}$ (stationary configuration)
- Toroidal $\vec{B}$ field: No end losses
- Poloidal field ("rotational transform"):
  Compensation of torus drifts
  Tokamak: Inductive toroidal current
- Axisymmetry (tokamak): No collisionless particle losses
- Vertical field: Torus topology, Radial position control
- Poloidal magnetic flux ψ_p described by Grad-Shafranov (-Schlüter) equation

... and stability

- Configurations without large guiding field possible (z-pinch, reversed field pinch) but largely unstable
- Tokamak: Stability from guiding toroidal field $q(a) \sim (rB_\phi)/(RB_p) > 2$
- Instabilities set operational limits
  — Kink instability (β, j_ϕ)
  — Tearing instability (density limit)

Appendices

- Field line tracing in a slab
- Overlapping islands, ergodic magnetic field

Field line tracing in a slab (for `Scilab`)

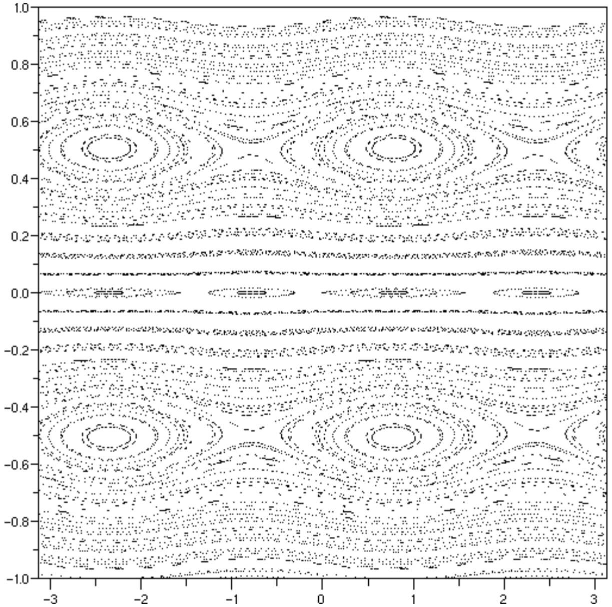
```
m=2; n=1;
dBx_dy=1.0; By0=0.025;
global dBx_dy By0 m n

function [dxy] = Dxy_dz (z,xy)
  global dBx_dy By0 m n
// 1 Island
  dxy = [dBx_dy*xy(2); By0*cos(m*xy(1) - n*z)];
// 2 Islands
//  dxy = [dBx_dy*xy(2); ..
//  2*By0*cos(m*xy(1))*cos(n*z)];
endfunction
```

```
nx=20; ny=30; nz=30; dz=2*%pi;
plot2d([0,0],[0,0],rect=[-%pi,-1,%pi,1],frameflag=1);
for i=0:nx-1
  x = 2*%pi*(i - nx/2)/nx;
  for j=0:ny-1
    y = 2*(j - ny/2)/ny;
    xy=[x; y]; z=0;
    for k=0:nz-1
      z1=z+2*%pi;
      xy = ode (xy, z, z1, Dxy_dz);
      if xy(1)>%pi  then, xy(1)=xy(1) - 2*%pi; end
      if xy(1)<-%pi then, xy(1)=xy(1) + 2*%pi; end
      plot2d1([xy(1)],[xy(2)],frameflag=0,style=0);
      z=z1;
    end
  end
end
```

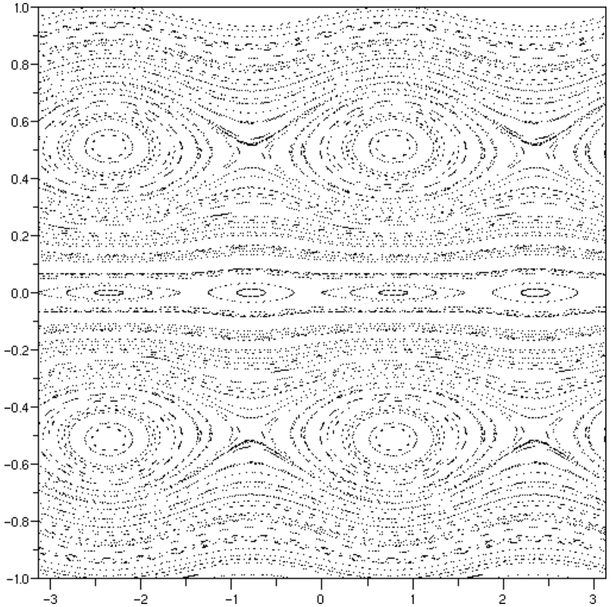
Overlapping islands cause stochastic field line diffusion

In our slab model, introduce **two** simultaneous modes.
Island widths grow with increasing $B_{y,0}$.
As islands overlap, field lines diffuse radially.
Expect strong radial transport $\|\vec{B}$!

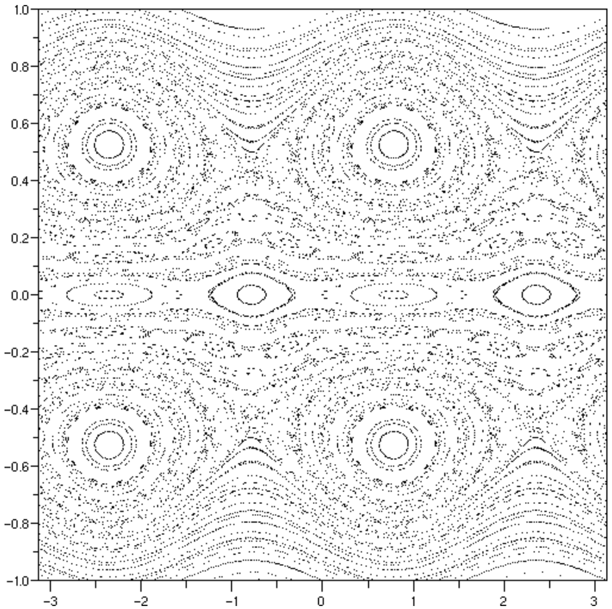
$B_{y,0} = 0.0125 B_z$



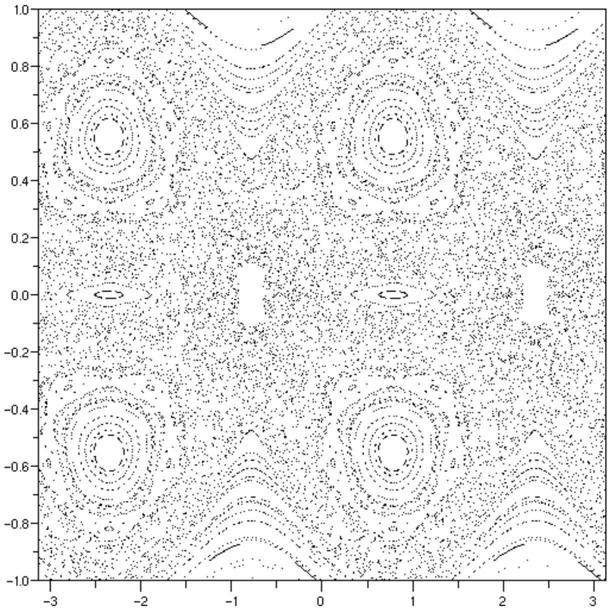
$B_{y,0} = 0.025 B_z$



$B_{y,0} = 0.05 B_z$



$B_{y,0} = 0.1 B_z$



Further reading on ergodisation:
B. V. Chirikov, “A universal instability of many oscillator systems”,
Phys. Rep. **52** (1979) 263