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# Transport in Magnetized Plasmas of Fusion Devices

Clemente Angioni

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C. Bourdelle, J. Candy, R. Dux, E. Fable, X. Garbet, R. Kleiber, A.G. Peeters, G. Pereverzev, F. Ryter, U. Stroth, R.E. Waltz

# Transport in magnetized plasmas

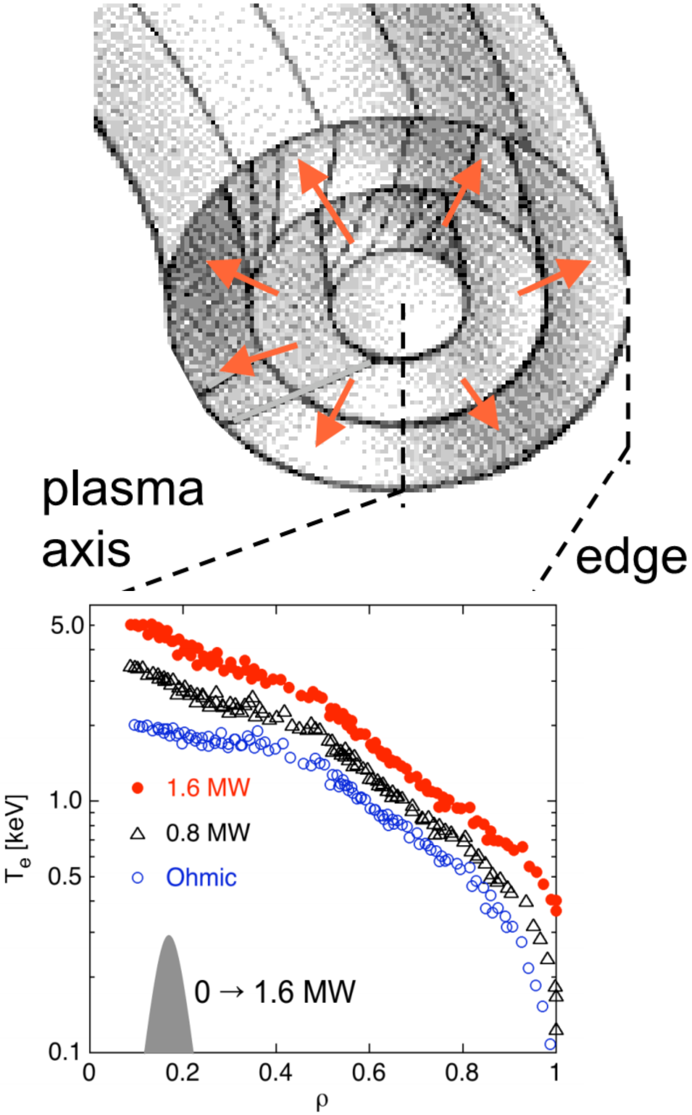
- The topic of “transport” is dedicated to the physical processes by which particles, momentum and energy are moved (transported) in a domain in the real space (or in phase space more in general)
- The general goal is to identify the relationship between the thermodynamic fluxes and the thermodynamic forces
- Thermodynamic fluxes are particle, momentum and energy (heat) fluxes. In fusion devices they are mainly imposed by external means (particle sources, torques and heating powers)
- Thermodynamic forces are the spatial gradients of the particle and momentum densities and of the pressures (temperatures). These describe how plasma reacts to the imposed fluxes, namely the radial profiles of density, rotation and temperature

# Transport in a fusion device

- **As we shall see, temperatures and densities are approx constant over flux surfaces**

- **Radial transport determines the profiles of temperatures and densities (their gradients = thermodynamic forces) in response to heat and particle sources (thermodynamic fluxes)**

- **Present lecture presents some general aspects on the main transport mechanisms in a magnetized plasma and their role in a fusion device**



# Transport mechanisms in magnetized plasmas

- Even in a completely "quiescent" plasma, there is an unavoidable source of transport: collisions
- The basic mechanism of collisional transport is called classical
- Non-uniform and curved magnetic fields produce drifts of the guiding center of the particle trajectories, which cause an additional source of collisional transport, called "neoclassical"
- However collisions are not the only source of transport.
- Drifts produced by non-uniform and curved magnetic fields also couple different fluctuating fields, thereby, under certain conditions, small fluctuations (Larmor radius scale) can become unstable. These micro-instabilities can lead to a turbulent state in the plasma which produces turbulent (anomalous) transport

# The role of transport in fusion devices

- **The minimization of transport is an essential element to achieve practical fusion energy**

- **In fact, the lowest is the transport (the highest is the confinement), the highest is the plasma energy for a given amount of heating**

- **Fusion power increases with plasma energy**

$$P_{fus} \sim n_D n_T \langle \sigma v \rangle \sim n_e^2 T^2 \sim W_{plasma}^2$$

- **A reactor must maximize the power multiplication factor Q (ignition), producing a large amount of fusion power**

$$Q = P_{fus}/P_{ext}$$

- **This is why understanding, predicting, and controlling transport in plasmas is one of the highest priorities**

# Outline

- Collisional transport
  - Classical
  - Neoclassical (in fusion devices)
- Turbulent transport
  - Linear micro-instabilities
  - Turbulence and transport, generic properties and comparison with experiments
  - Mechanisms of turbulence reduction and suppression
  - ( Particle and momentum transport )

# Orderings

- Orderings are an essential element of theory of plasma transport
- Depending on the orderings which are assumed, one can obtain very different consequences and compute different transport processes starting from the same equations

- Generic ordering for magnetized plasmas: B is Big ! that is cyclotron frequency is large and Larmor radius is small

$$\delta \equiv \rho/l = \omega/\Omega \ll 1$$

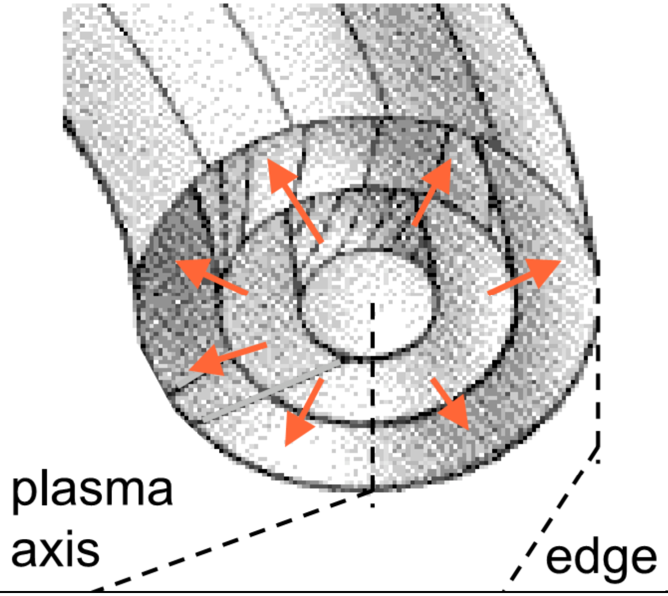
$$\rho \equiv v_{\text{th}}/\Omega = m v_{\text{th}}/(eB) \;;\quad \omega \equiv v_{\text{th}}/l \;;\; l \cong |\nabla \ln p|^{-1}$$

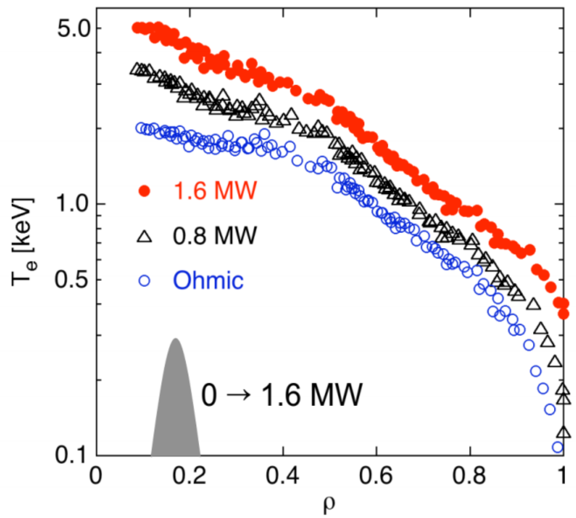
- In addition, collisional transport theory orders small the time derivatives (no rapid fluctuation)

$$\partial \ln p/\partial t = O(\delta^2 \omega)$$

# Zero order in $\delta \equiv \rho / l$ : no variation along B

- It can be shown that with these orderings the distribution function at the lowest order is a local Maxwellian distribution
- Note: this is NOT an assumption, but follows from the orderings
- Orderings also imply that at the lowest order temperatures and densities are constant on the magnetic flux surfaces (lowest order distribution function has a gradient only in the radial direction)



# Classical transport

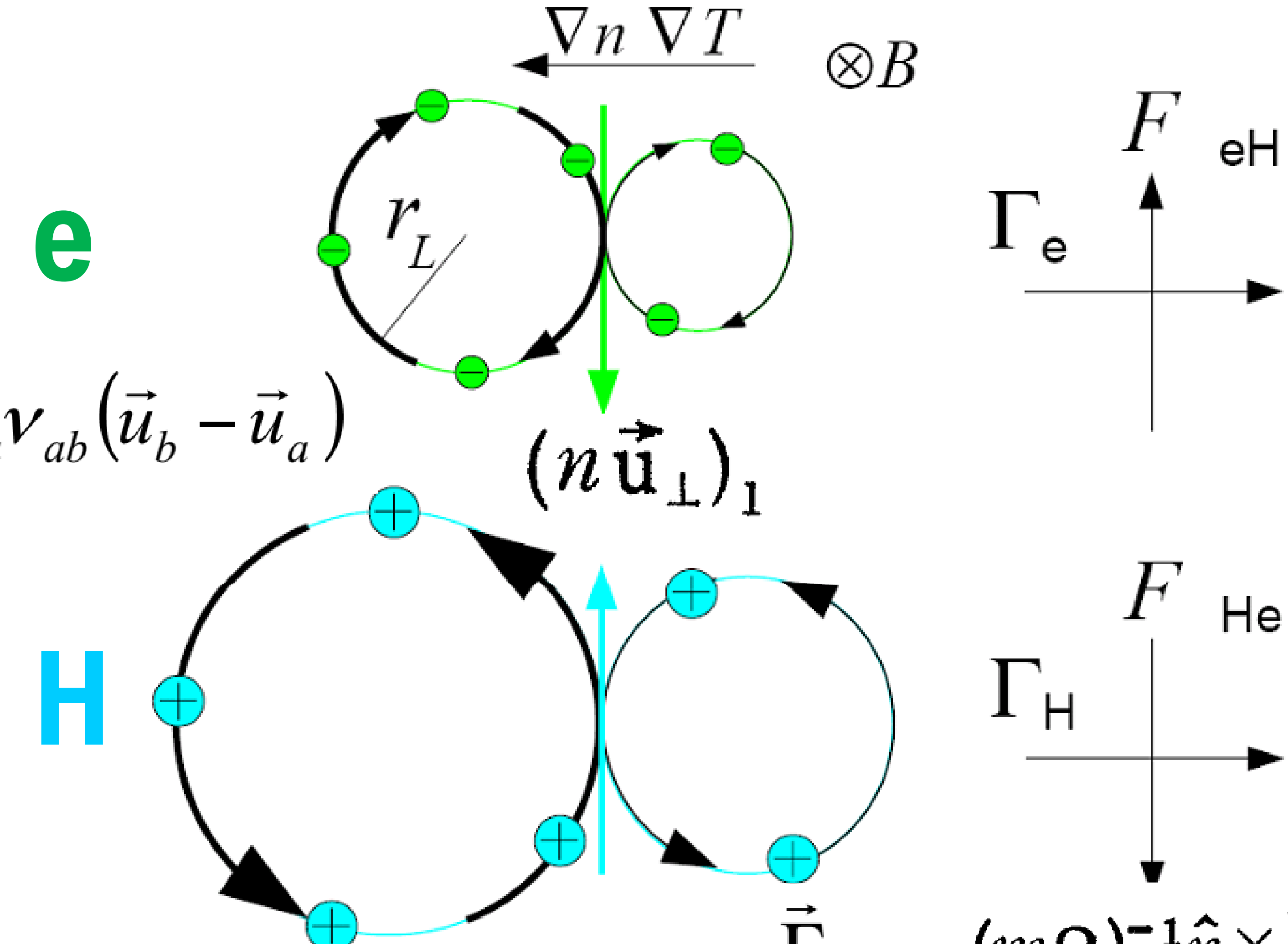
- Classical transport is produced by collisions in a magnetized plasma in a homogeneous magnetic field with straight field lines
- In a fluid picture, it due to the friction between the fluid flows which are produced by Larmor gyration in the presence of density and temperature ( pressure ) gradients ( perp to B )

$$(n\vec{\mathrm{u}}_{\perp})_1 = (m\Omega)^{-1}\,\hat{n}\times\vec{\nabla}\bar{p}$$

- These flows are called diamagnetic because the resulting electric current usually acts to reduce the strength of the magnetic field
- The resulting friction force $\vec{F}_{ab} = m_a n_a \nu_{ab}\left(\vec{u}_b - \vec{u}_a\right)$ moves particles in direction perp to both B and the diamagnetic flow

$$\mathbf{v}_d = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2} \qquad \vec{\Gamma}_{CL} = -(m\Omega)^{-1}\hat{n}\times\vec{\mathrm{F}}$$

# Simple picture of diamagnetic flows and of classical transport

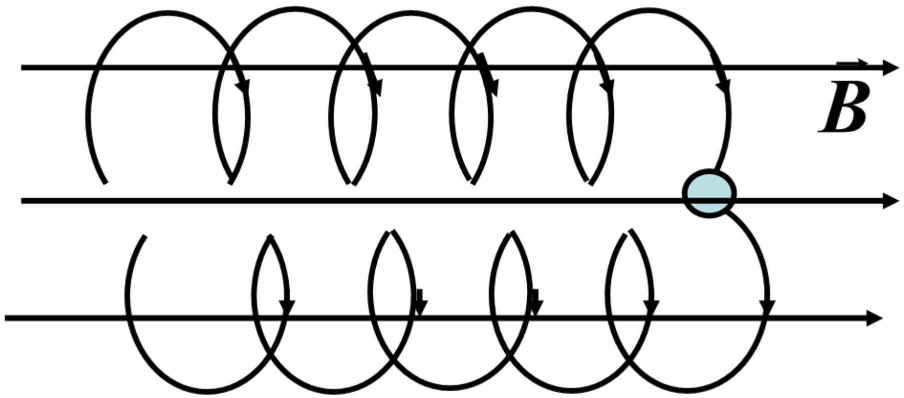


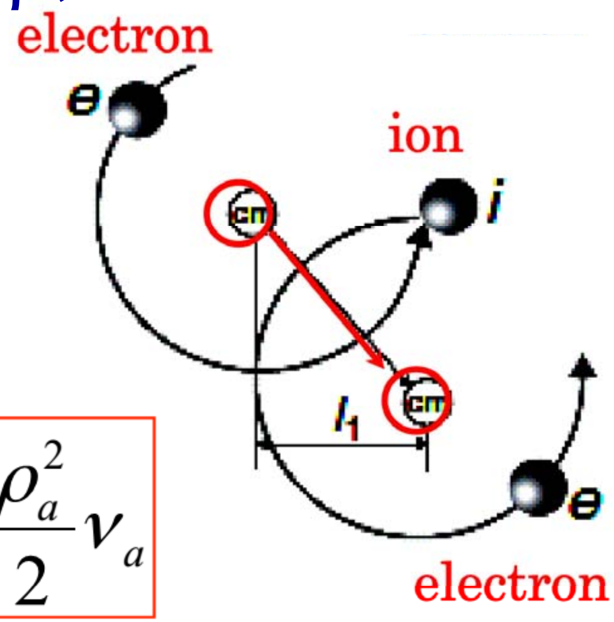

$$\vec{F}_{ab} = m_a n_a \nu_{ab} \left( \vec{u}_b - \vec{u}_a \right)$$

$$\vec{\Gamma}_{CL} = -(m\Omega)^{-1} \hat{n} \times \vec{\mathbf{F}}$$

# Classical transport as random walk

- **Random walk: step size is Larmor radius ρ, characteristic frequency is collision frequency**




$$D_{CL}^{a} = \frac{m_a k_B T}{e_a^2 B^2} \sum_{b \neq a} \nu_{ab}$$

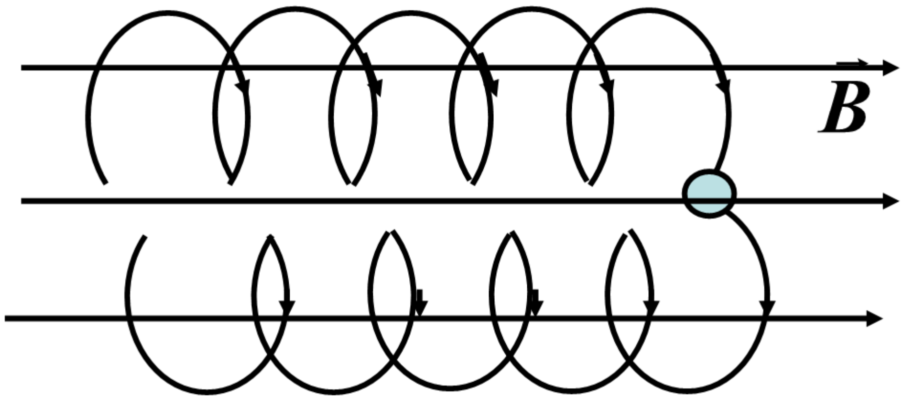
$$\boxed{D_{CL}^{a} = \frac{\rho_a^2}{2} \nu_a}$$

- **Complete derivation shows that it includes diffusion and convection**

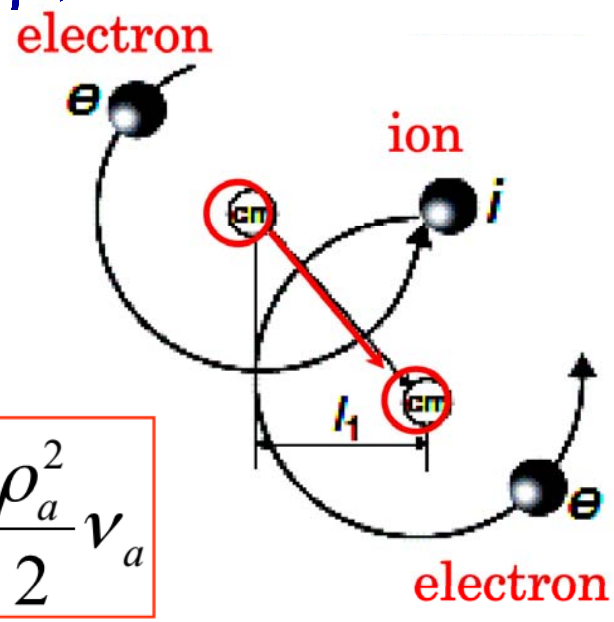
$$\boxed{\vec{\Gamma}_{CL}^{a} = -\nabla n_a \sum_{b \neq a} D_{CL}^{ab} + n_a \sum_{b \neq a} D_{CL}^{ab} \frac{e_a}{e_b} \left( \frac{\nabla n_b}{n_b} - \frac{\nabla T}{T} \left[ \frac{3 m_{ab}}{2 m_b} - 1 - \frac{e_b}{e_a} \left( \frac{3 m_{ab}}{2 m_a} - 1 \right) \right] \right) \qquad D_{CL}^{ab} = \frac{\rho_a^2}{2} \nu_{ab}}$$

# Classical transport as random walk

- Random walk: step size is Larmor radius ρ, characteristic frequency is collision frequency




$$D_{CL}^{a} = \frac{\rho_a^2}{2} \nu_a$$




- Heavy impurities: convection proportional to impurity charge, inward (pinch) with centrally peaked density, outward with peaked temperatures

$$\vec{\Gamma}_{CL}^{Z} = \frac{\rho_Z^2 \nu_{ZH}}{2} \left\{ -\nabla n_Z + n_Z Z \left( \boxed{\frac{\nabla n_H}{n_H}} \boxed{-\frac{1}{2}\frac{\nabla T}{T}} \right) \right\}$$

Temperature screening, outward with centrally peaked profiles

# From Classical to Neoclassical transport

- **SUMMARY:**
- **We have seen that classical (collisional) transport is caused by perpendicular friction between diamagnetic flows (which are perpendicular to both B and the (radial) direction of the temperature and density gradients ).**
- **Diamagnetic flows are fluid flows produced by Larmor gyration in the presence of density and temperature (pressure) gradients**
- **In the simple "particle" picture, classical diffusion can be thought as a random walk process with characteristic step size the Larmor radius and characteristic time the inverse of the collision frequency**

- **Neoclassical transport is the analogous of classical transport, but is produced by parallel friction rather than by perp friction**
- **Since it is connected to parallel motion, it is strongly determined by the specific geometry of the field lines, that is of the magnetic confinement system of the fusion device (B is inhomogeneous with curved field lines)**

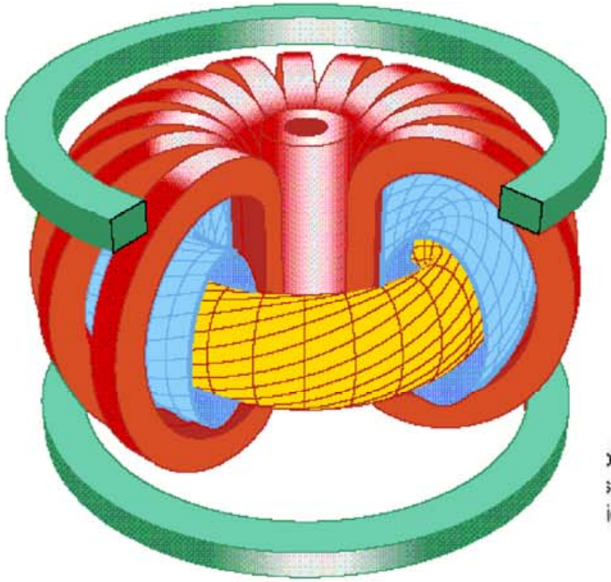
# First order parallel flows

- Collisional transport : time variations are order $O(\delta^2\omega)$
- Then $\partial n/\partial t + \vec{\nabla}\cdot(n\vec{u}) = 0$ implies $\vec{\nabla}\cdot(n\vec{u})_1 = 0$
- (Perpendicular) diamagnetic flows (from momentum conservation)

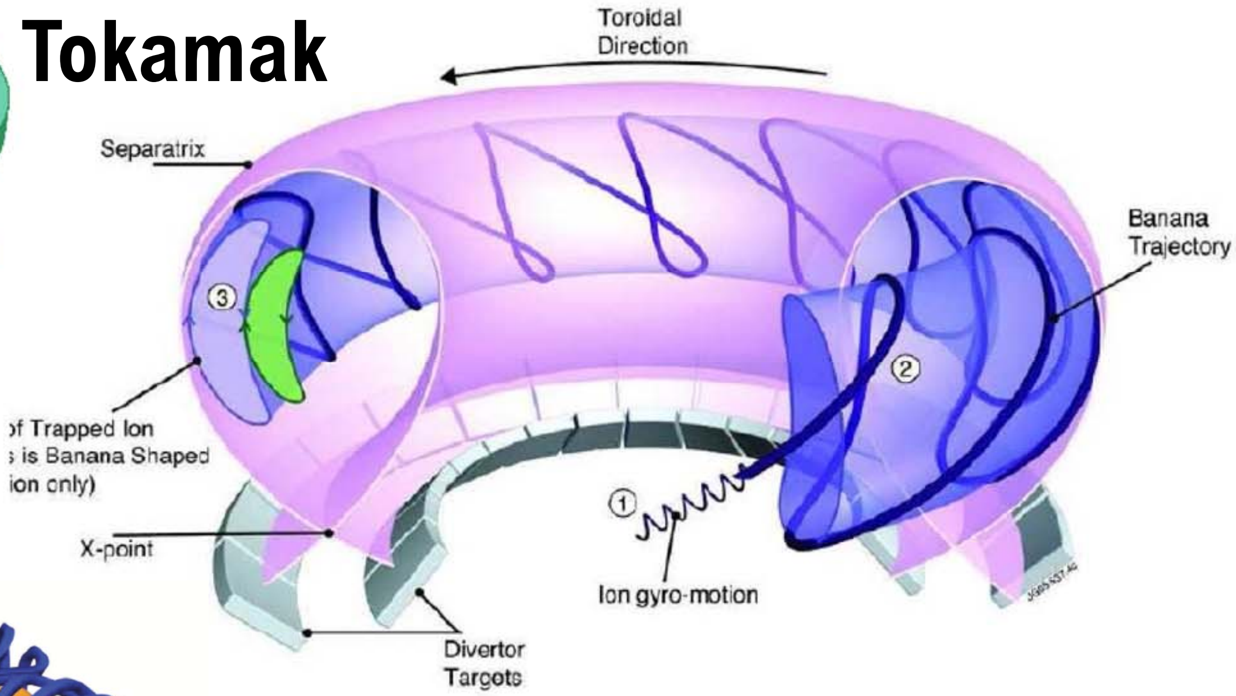
$$(n\vec{u}_\perp)_1 = (m\Omega)^{-1}\,\hat{n}\times(\vec{\nabla}\bar{p} + e\bar{n}\vec{\nabla}\Phi)$$

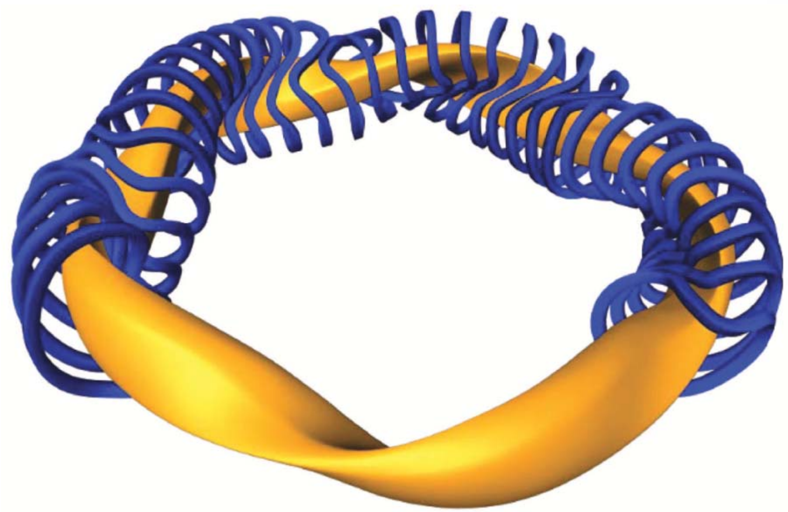
- We conclude that $(n\vec{u})_1$ also comprises a parallel component ("return" flow) which ensures that total divergence is zero
- Friction between parallel flows produces the neoclassical transport
- Calculation of radial ($2^{nd}$ order) neoclassical transport requires the solution of $1^{st}$ order parallel transport (parallel flows connected to radial gradients of density and temperature by toroidal geometry)

# Geometry of magnetic confinement plays an essential role



Tokamak



Stellarator

# Regimes of low and high collisionality

- At this point it is useful to compare the collision frequency with the inverse of the characteristic orbit times ( the bounce frequency )

$$\nu_a^* = \frac{\nu_{a,eff}}{\omega_b} = \frac{\nu_a}{\varepsilon}\frac{qR_0}{\sqrt{\varepsilon}v_{Ta}} = \frac{\nu_a qR_0}{v_{Ta}}\varepsilon^{-3/2}$$

**Effective ν : collision frequency at which trapped particle is scattered into an untrapped orbit**

$$\nu(\Delta\Theta) = \nu_c/(\Delta\Theta)^2$$

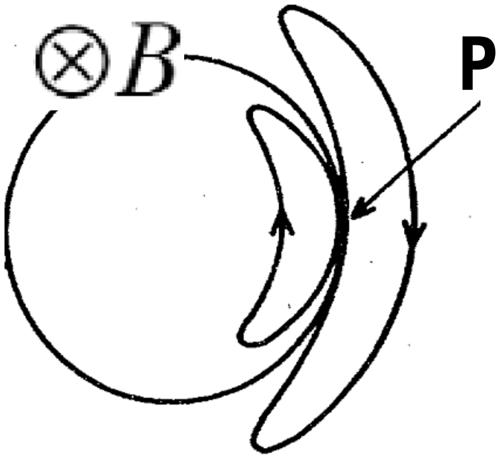
- $\nu_a^*$ (ratio of effective collision frequency to bounce frequency) allows us to identify 3 regimes of collisionality

$$\begin{array}{ll} \nu_a^* < 1 & \text{low : banana} \\ 1 < \nu_a^* < \varepsilon^{-3/2} & \text{medium : plateau} \\ \nu_a^* > \varepsilon^{-3/2} & \text{high : Pfirsch - Schlüter} \end{array}$$

- We shall give a simple picture of neoclassical transport in the low collisionality (banana) regime as random walk process

# Collisionless regime: banana regime

- If a collision in point P reverses sign of $v_{\parallel}$ , center of orbit shifts radially by a banana width




- Random walk : step size is banana width (largest radial excursion of g.c. orbit), $\tau$ is time to reverse $v_{\parallel}$

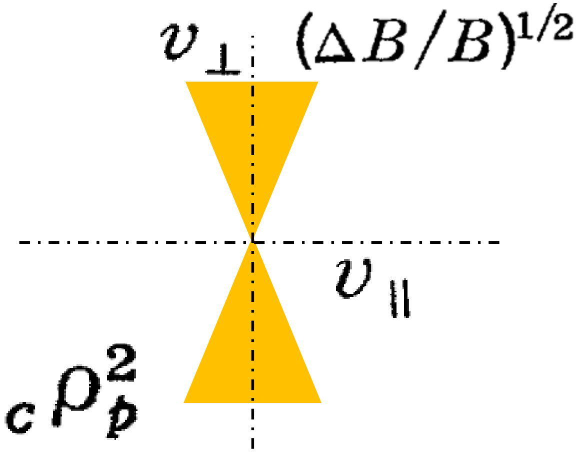
- Trapped particle reverses sign of v|| with pitch-angle diffusion in v-space of $(\Delta B/B)^{1/2}$

- Collision frequency $\nu_c$ refers to 90° diffusion, thereby $\nu_t \simeq (B/\Delta B)\,\nu_c$

- Random walk: step size = banana width

$$\Delta_b = \rho_p (a/R)^{1/2}$$

- … and collision frequency $\nu_t$

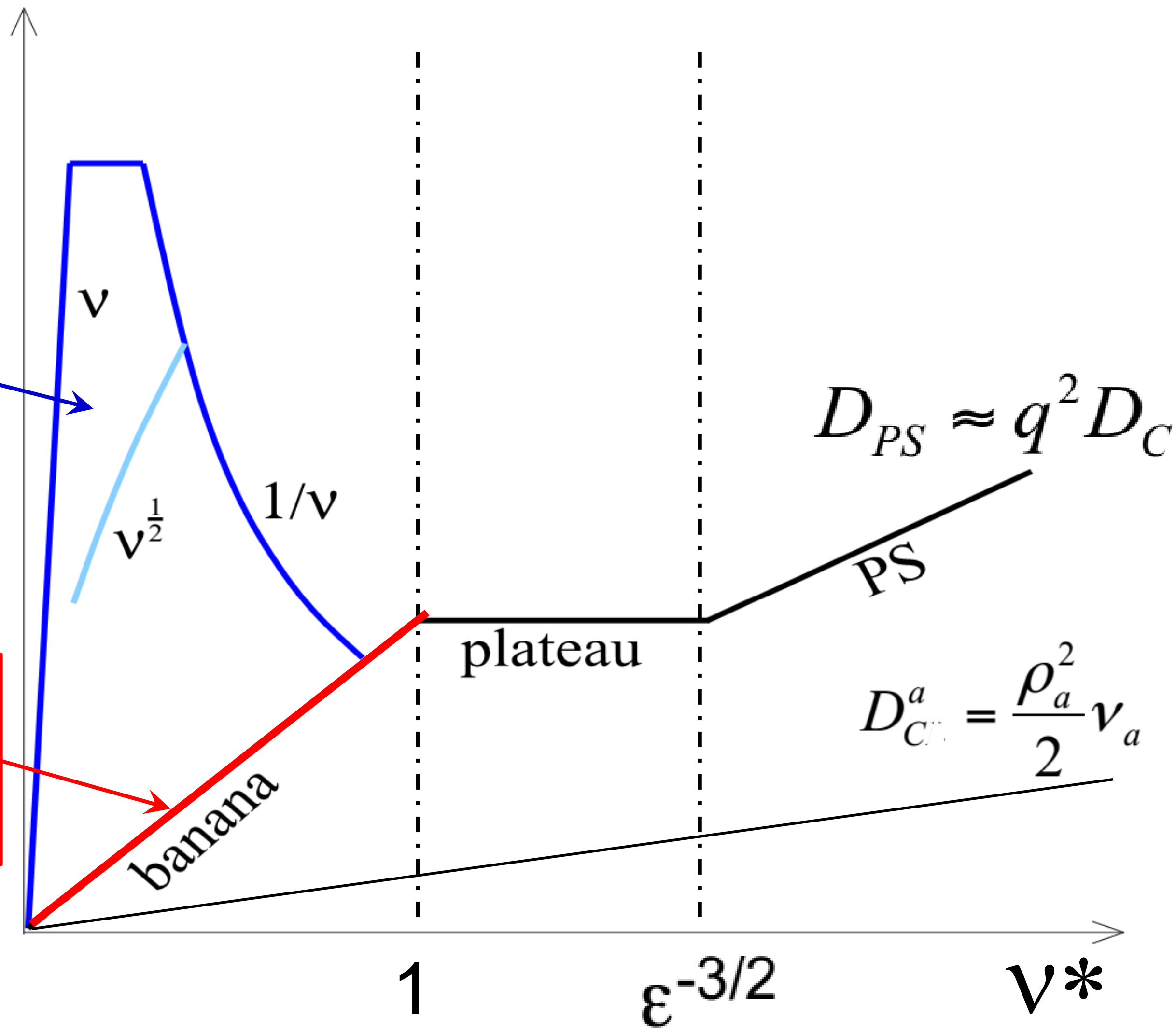



$$D_b \simeq (a/R)^{1/2} \nu_t \Delta_b^2 \simeq (a/R)^{1/2} \nu_c \rho_p^2$$

# Neoclassical transport regimes, overall dependence on collisionality

**More in Stellarator Lectures on Thursday**

$$D_B \approx \frac{q^2}{\varepsilon^{3/2}} D_{CL}$$



# Neoclassical transport matrix

- Like in classical transport, neoclassical transport comprises both diagonal diffusive and off-diagonal (convective) contributions
- Neoclassical transport produces a full matrix which relates thermo-dynamic forces to thermodynamic fluxes (Onsager symmetry)

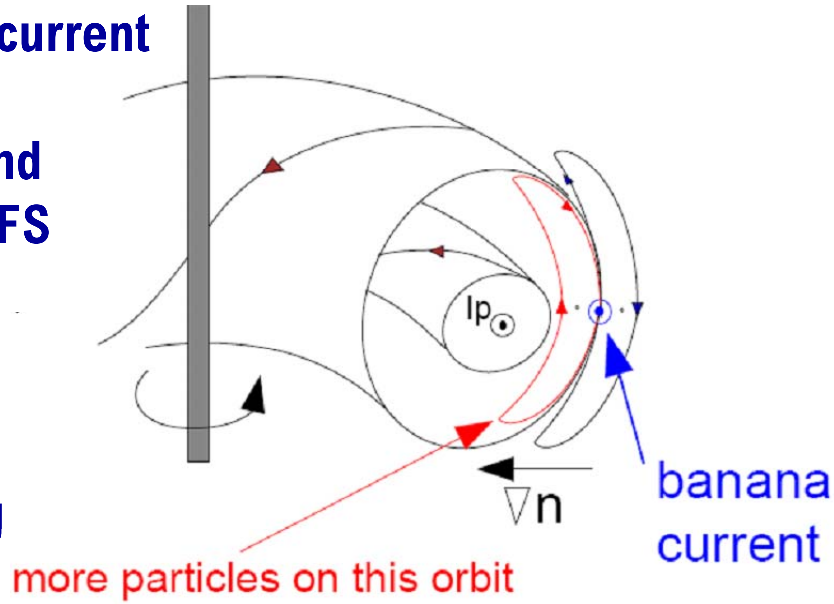
$$\begin{pmatrix} \Gamma_e \dfrac{d\psi}{d\rho} \\[2ex] \dfrac{Q_e}{T_e}\dfrac{d\psi}{d\rho} \\[2ex] \left\langle \dfrac{j_\parallel B}{T_e} \right\rangle - \left\langle \dfrac{j_{\parallel S} B}{T_e} \right\rangle \\[2ex] -\dfrac{I(\psi)\langle E_* B\rangle n_e}{\langle B^2\rangle} \end{pmatrix} = \begin{bmatrix} \mathcal{L}^e_{11} & \mathcal{L}^e_{12} & \mathcal{L}^e_{13} & \mathcal{L}^e_{14} \\ \mathcal{L}^e_{21} & \mathcal{L}^e_{22} & \mathcal{L}^e_{23} & \mathcal{L}^e_{24} \\ \mathcal{L}^e_{31} & \mathcal{L}^e_{32} & \mathcal{L}^e_{33} & \mathcal{L}^e_{34} \\ \mathcal{L}^e_{41} & \mathcal{L}^e_{42} & \mathcal{L}^e_{43} & \mathcal{L}^e_{44} \end{bmatrix} \begin{pmatrix} \dfrac{1}{p_e}\dfrac{\partial p_e}{\partial \psi} + \dfrac{1}{p_e}\dfrac{\partial p_i}{\partial \psi} \\[2ex] \dfrac{1}{T_e}\dfrac{\partial T_e}{\partial \psi} \\[2ex] \dfrac{\langle E_\parallel B\rangle}{\langle B^2\rangle} \\[2ex] \dfrac{q_e}{n_i T_e}\dfrac{K_i(\psi)}{I(\psi)}\langle B^2\rangle \end{pmatrix}$$

# Some interesting off-diagonal effects: the bootstrap current

- Radial gradients of density (pressure) and temperature sustain a parallel current
- This should not be surprising as we have already underlined that the toroidal geometry strongly couples parallel motion to radial gradients

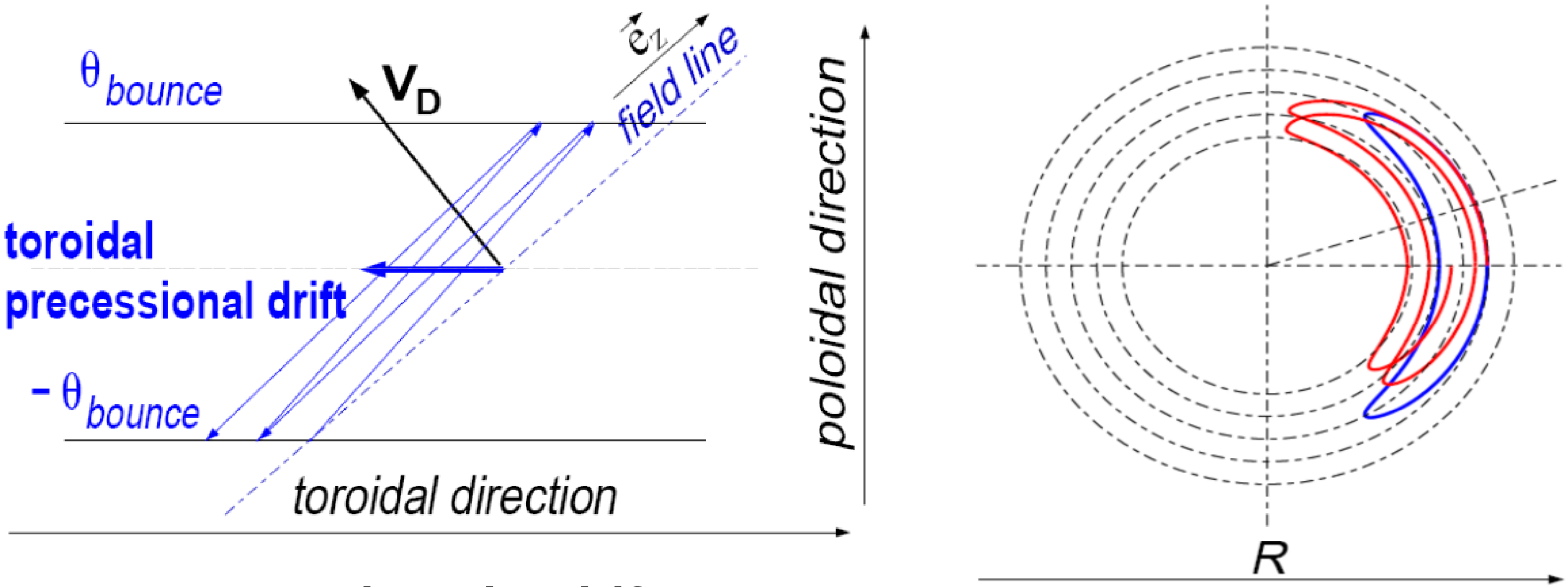
$$\langle j_{\|}B\rangle = \sigma_{\text{neo}}\langle E_{\|}B\rangle - I(\psi)p_e\left[\mathcal{L}_{31}\frac{p}{p_e}\frac{\partial \ln p}{\partial \psi} + \mathcal{L}_{32}\frac{\partial \ln Te}{\partial \psi} + \mathcal{L}_{34}\alpha\frac{\partial \ln Ti}{\partial \psi}\right],$$

- Neoclassical conductivity reduced from classical (Spitzer) as trapped particles do not carry current
- Banana orbits in the presence of density and temperature gradients produce a (small) LFS parallel current (similar principle of the diamagnetic current)
- Banana current acts as a seed for a much bigger current, which is carried by passing particles (bootstrap current)



# Onsager symmetric process of the bootstrap current: the Ware pinch

➤ Mirror Force ( B ∝ 1/R ) ⇒ particle trapping (Banana orbits)




➤Curvature and grad B drift

$$\mathbf{v}_C + \mathbf{v}_{\nabla B} = \frac{v_\parallel^2}{\Omega_c}\mathbf{B} \times \mathcal{K}_B + \frac{v_\perp^2}{2\Omega_c}\mathbf{B} \times \frac{\nabla B}{B}$$

➤Parallel electric field ⇒ orbit displacement ⇒ Ware pinch ⇒ $\propto \dfrac{1}{q\,T^{5/2}}$

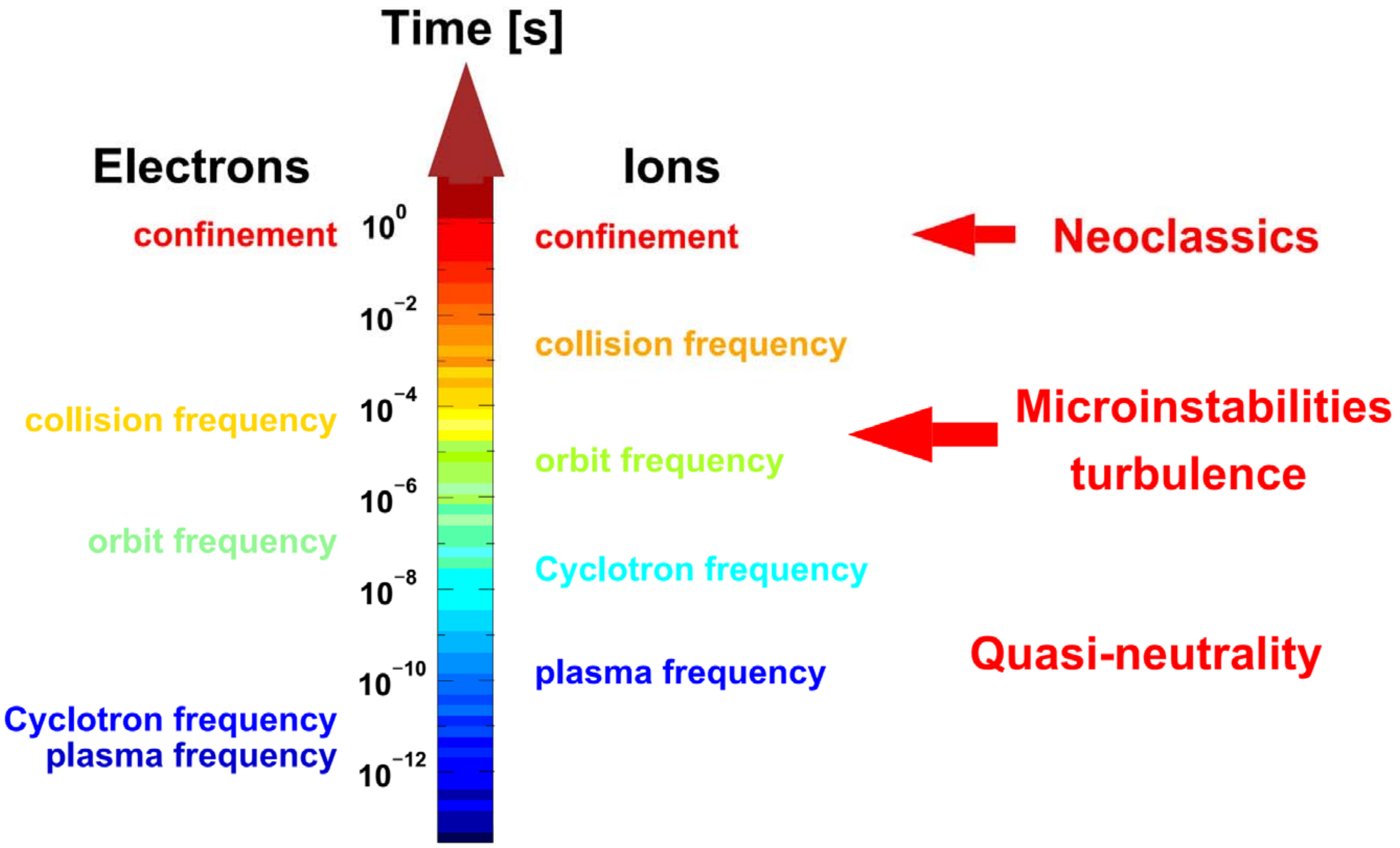
# Neoclassical (collisional) transport, experimental relevance

- Neoclassical (collisional) transport provides an unavoidable level of transport
- In the radial direction :
- In stellarators it can be large enough to carry a big part of the heat flux produced by plasma heating → stellarators require optimization against neocl transport
- In tokamaks, it is small (wrt exp. values, by 1 order of magnitude for ions, 2 orders for electrons), thereby it provides transport levels which are insufficient to carry the heat flux provided by plasma heating. Temperature gradients increase up to the point to destabilize micro-instabilities and turbulence (from next slide)
- Neoclassical transport can still be important in tokamaks in improved confinement regimes (e.g. transport barriers)
- Since collisionality is proportional to Z^2, it remains important for highly charged impurities and it is a reason of major concern because it can produce accumulation of highly radiating impurities

- In the parallel direction predicts parallel electric conductivity and parallel bootstrap currents which are regularly found to be consistent with measurements

# From collisions
# to
# turbulence

# Time scales relevant to transport



# We shall adopt a simple fluid approach: moments of the distribution function

## Density

$$n = \int d^3\vec{v}\, f$$

## Particle flux

$$n\vec{u} = \int d^3\vec{v}\, \vec{v}\, f$$

## Stress tensor

$$\mathsf{P} = \int d^3\vec{v}\, m\vec{v}\,\vec{v}\, f$$

## Energy flux

$$\vec{Q} = \int d^3\vec{v}\,(m v^2/2)\,\vec{v}\, f$$

## Scalar pressure

$$p = nT = \mathrm{Tr}\{\mathsf{P}\}/3$$

## Energy weighted stress tensor

$$\mathsf{R} = \int d^3\vec{v}\,(m v^2/2)\,\vec{v}\,\vec{v}\, f$$

# Fluid equations (conservation laws)

- **Even moments (particle and energy conservation)**

$$\partial n/\partial t + \vec{\nabla}\cdot(n\vec{u}) = 0\ ,$$

$$(\partial/\partial t)\,3p/2 + \vec{\nabla}\cdot\vec{Q} = Q + \vec{u}\cdot(\vec{F} + en\vec{E})\ .$$

- **Odd moments (momentum and energy flux conservation)**

$$(\partial/\partial t)mn\vec{u} + \vec{\nabla}\cdot\mathsf{P} - en(\vec{E} + c^{-1}\vec{u}\times\vec{B}) = \vec{F}\ ,$$

$$(\partial/\partial t)\vec{Q} + \vec{\nabla}\cdot\mathsf{R} - (3/2)(e/m)\vec{E}p - (e/m)\vec{E}\cdot\mathsf{P} - (e/mc)\vec{Q}\times\vec{B} = \vec{G}$$

- **Equation evolving moment *n* involves moment *n+1* $\longrightarrow$ infinite set of equations $\rightarrow$ a fluid closure is required**

# We consider first order collisionless

- **Particle conservation**

$$\frac{\partial n_q}{\partial t} + \nabla \cdot (n_q \mathbf{v}_q) = 0,$$

- **Momentum balance, perpendicular direction**

$$\mathbf{v}_i = \mathbf{v}_E + \mathbf{v}_{*i}$$

$$\mathbf{v}_{*\sigma} = \frac{\hat{\mathbf{e}}_{||} \times \nabla(n_\sigma T_\sigma)}{Z_\sigma e n_\sigma B}$$

- **Momentum balance, parallel direction**

$$m_i n_i \frac{\partial v_{||i}}{\partial t} = -e Z_i n_i \nabla_{||} \phi - \nabla_{||}(n_i T_i)$$

- **Energy conservation**

$$\frac{3}{2} n_q \left( \frac{\partial}{\partial t} + \mathbf{v}_q \cdot \nabla \right) T_q + n_q T_q \nabla \cdot \mathbf{v}_q = -\nabla \cdot \mathbf{q}_q$$

# Ordering: time derivatives of order 1 (and not order 2 as previously for collisional transport )

- **We describe the dynamical evolution of first order perturbations of all fluid moments and of the electrostatic potential, which can vary in time, in Fourier space**

$$u = u_0(x) \; + \; \delta u \, \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$$

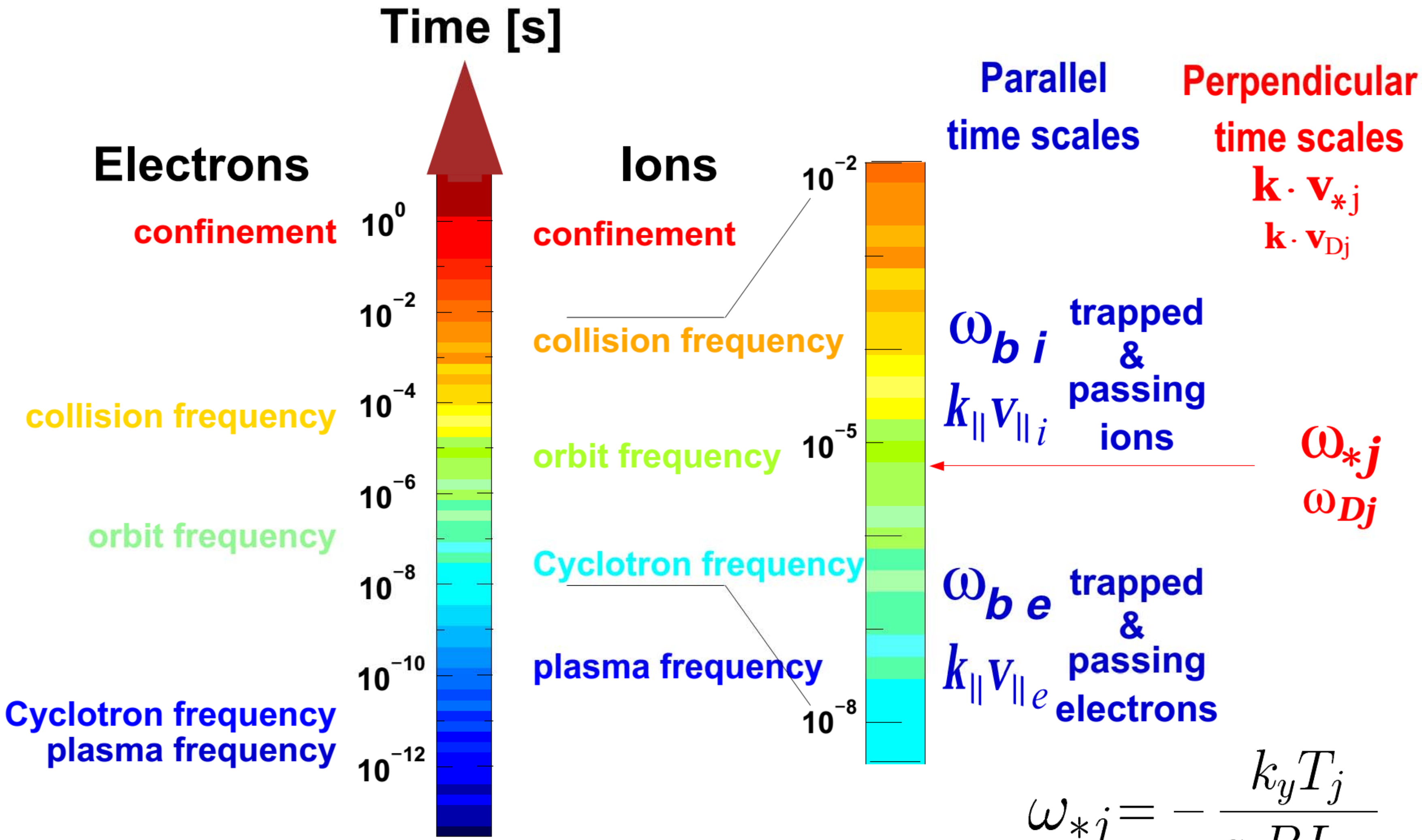
$$\frac{\partial n_i}{\partial t} + n_i \nabla\cdot\mathbf{v}_E + \mathbf{v}_E\cdot\nabla n_i + \nabla\cdot(n_i\,\mathbf{v}_{*i}) = 0$$

$$-\omega\frac{\delta n_i}{n_i} + 2\omega_{Di}\frac{Z_i T_e}{T_i}\frac{e\,\delta\phi}{T_e} - \omega_{*i}\frac{Z_i T_e}{T_i}\frac{e\,\delta\phi}{T_e} + 2\omega_{Di}\frac{\delta n_i}{n_i} + 2\omega_{Di}\frac{\delta T_i}{T_i} = 0$$

**Drift frequency** **Diamagnetic frequency**

$$\mathbf{k}\cdot\mathbf{v}_{D\sigma} = 2\omega_{D\sigma} = \frac{-2k_y T_\sigma}{Z_\sigma e B R}\;; \quad \omega_{*\sigma} = \frac{-k_y T_\sigma}{Z_\sigma e B L_{n\sigma}}\;; \quad \frac{1}{L_X} = -\frac{\nabla X}{X}$$

# Parallel and perpendicular time scales




$$\omega_{*j} = -\frac{k_y T_j}{q_j B L_{nj}}$$

# Normalizations, parallel and perpendicular time scales

- **We are considering harmonic fluctuations** $\propto$ *exp( **ik** · **x** - iω t )*
- **Perpendicular motion characterisitc frequency is the drift frequency**

$$\omega_D = -\frac{k_y T}{eBR} \quad \Rightarrow \quad \hat{\omega} = \omega \,/\, \omega_D$$

- **Parallel motion :** $k_{||} v_{\rm th} = k_{||}\sqrt{2T/m} \Rightarrow \hat{k}_{||} = k_{||} v_{\rm th} / 4\,\omega_D$
- **Ions and electrons:**
  **same mobility perpendicular to the field line,**
  **very different mobility along the field line**
- **Parallel velocity moment ( parallel force balance )**

$$\hat{\omega}\tilde{u} + 4\tilde{u} + 2\hat{k}_{||}(\tilde{n} + \tilde{T}) = -2\,\hat{k}_{||}\,\hat{\phi}$$

- **Ions : inertia is dominant** $\Rightarrow$ **zero order, negligible parallel motion**
- **Electrons : parallel streaming is dominant** $\Rightarrow$ **adiabatic response**

# Algebraic equations for the Fourier amplitudes of the fluctuations (normalizations introduced)

➤ **Continuity (zero moment)**

$$\hat{\omega}\tilde{n} \; + \; 2\,(\tilde{n} + \tilde{T}) \; + \; 4\,\hat{k}_{||}\,\tilde{u} \; = \; \left[\frac{R}{L_n} - 2\right]\hat{\phi}$$

➤ **Parallel velocity moment**

$$\hat{\omega}\tilde{u} \; + \; 4\tilde{u} \; + \; 2\hat{k}_{||}(\tilde{n} + \tilde{T}) \; = \; -2\,\hat{k}_{||}\,\hat{\phi}$$

➤ **Energy balance ( second order moment )**

$$\hat{\omega}\tilde{T} \; + \; \frac{4}{3}\tilde{n} \; + \; \frac{14}{3}\tilde{T} \; + \; \frac{8}{3}\hat{k}_{||}\tilde{u} \; = \; \left[\frac{R}{L_T} - \frac{4}{3}\right]\hat{\phi}$$

- **We are adopting a simplified geometry, we are neglecting actual derivatives along the field line (formally parallel wave vector), and we neglect finite Larmor radius effects (we should not, see later …)**
- **These fluid equations can be also derived as moments of the drift-kinetic and gyro-kinetic equations (this goes beyond the scope of this lecture)**

# Parallel and perpendicular time scales, passing and trapped particles

- **Motion $\perp$ to B : Drift frequency** $\omega_D = -\dfrac{k_y T}{eBR}$

- **Motion || to B :**

  **Passing particles** $k_\| v_{\| i} << \omega \sim \omega_D << k_\| v_{\| e}$

  **Trapped particles (Mirror force due to B $\propto$ 1/R)**

$$\omega_{bi} \simeq \sqrt{\epsilon}\,\frac{v_{Ti}}{qR} << \omega_D \quad \& \quad \omega_{be} \simeq \sqrt{\epsilon}\,\frac{v_{Te}}{qR} >> \omega_D$$

- **Ions: bounce time longer than the characteristic time zero order $\Rightarrow$ no difference between passing and trapped**

  **Electrons : many bounces in a characteristic time different behaviour between passing and trapped electrons**

# Passing electrons: adiabatic response

- Fast unconstrained motion parallel to B

- Parallel force balance for passing electrons

$$\hat{\omega}\tilde{u} \; + \; 4\tilde{u} \; + \; 2\hat{k}_{||}(\tilde{n} + \tilde{T}) \; = \; 2\,\hat{k}_{||}\,\hat{\phi}$$

- Balance of dominant terms $\tilde{n} \; = \; \hat{\phi}$

or in non-normalized form $$\frac{\delta n_{ep}}{n_{ep}} = \frac{e\,\phi}{T_e}$$

# Passing and trapped ions: continuity and energy balance

- Neglect parallel motion (strong inertia) $\tilde{u} = 0$
- Continuity and energy balance

$$\hat{\omega}\tilde{n} + 2(\tilde{n} + \tilde{T}) = \left[\frac{R}{L_n} - 2\right]\hat{\phi}$$

$$\hat{\omega}\tilde{T} + \frac{4}{3}\tilde{n} + \frac{14}{3}\tilde{T} = \left[\frac{R}{L_T} - \frac{4}{3}\right]\hat{\phi}$$

- Quasineutrality condition $\tilde{n}_e = \tilde{n}_i$
  - Assuming (for the moment) that all electrons are passing ( more precisely adiabatic ), quasi-neutrality becomes $\tilde{n} = \hat{\phi}$
    we obtain a homogeneous system of three equations in three unknowns

# The mechanism leading to an instability

➤ **We look for the eigenvalues of this homogeneous system**

$$\hat{\omega}\tilde{n} \;+\; 2\left(\tilde{n}+\tilde{T}\right) \;=\; \left[\frac{R}{L_n}-2\right]\hat{\phi}$$

$$\hat{\omega}\tilde{T} \;+\; \frac{4}{3}\tilde{n} \;+\; \frac{14}{3}\tilde{T} \;=\; \left[\frac{R}{L_T}-\frac{4}{3}\right]\hat{\phi}$$

$$\tilde{n} \;=\; \hat{\phi}$$

➤ **Can be solved analytically (trivial)**
**here we focus on the basic coupling mechanism leading to an instability**

# The mechanism leading to an instability

➤ **We look for the eigenvalues of this homogeneous system**

$$\boxed{\hat{\omega}\tilde{n}} + 2\left(\tilde{n} + \boxed{\tilde{T}}\right) = \left[\frac{R}{L_n} - 2\right]\hat{\phi}$$

$$\boxed{\hat{\omega}\tilde{T}} + \frac{4}{3}\tilde{n} + \frac{14}{3}\tilde{T} = \left[\boxed{\frac{R}{L_T}} - \frac{4}{3}\right]\hat{\phi}$$

$$\boxed{\tilde{n} = \hat{\phi}}$$

➤ $\hat{\omega}^2\,\hat{\phi} = -2\,\dfrac{R}{L_T}\,\hat{\phi}$ $\Rightarrow$ **imaginary roots**

$\Rightarrow$ **pure growing mode** $\hat{\gamma} = \sqrt{2\,\dfrac{R}{L_T}}$

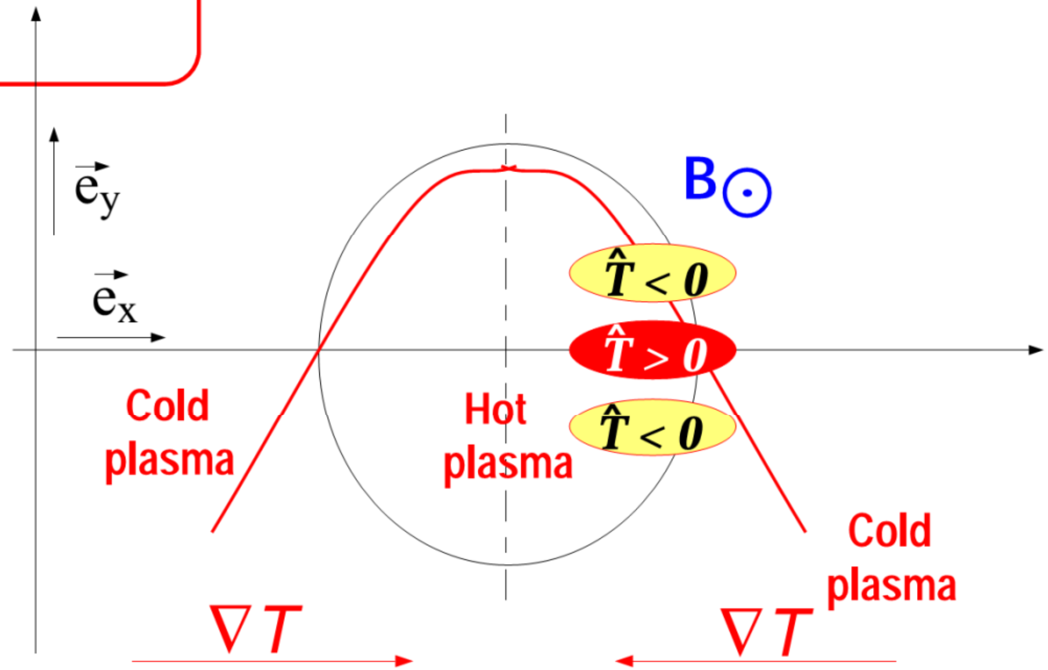
# The Ion Temperature Gradient (ITG) mode

$$\hat{\omega}\tilde{n} + 2\tilde{T} = 0$$

$$\hat{\omega}\tilde{T} + \frac{R}{L_T}\hat{\phi} = 0$$

$$\tilde{n} = \hat{\phi}$$

**Initial Temperature perturbation**



# The Ion Temperature Gradient (ITG) mode

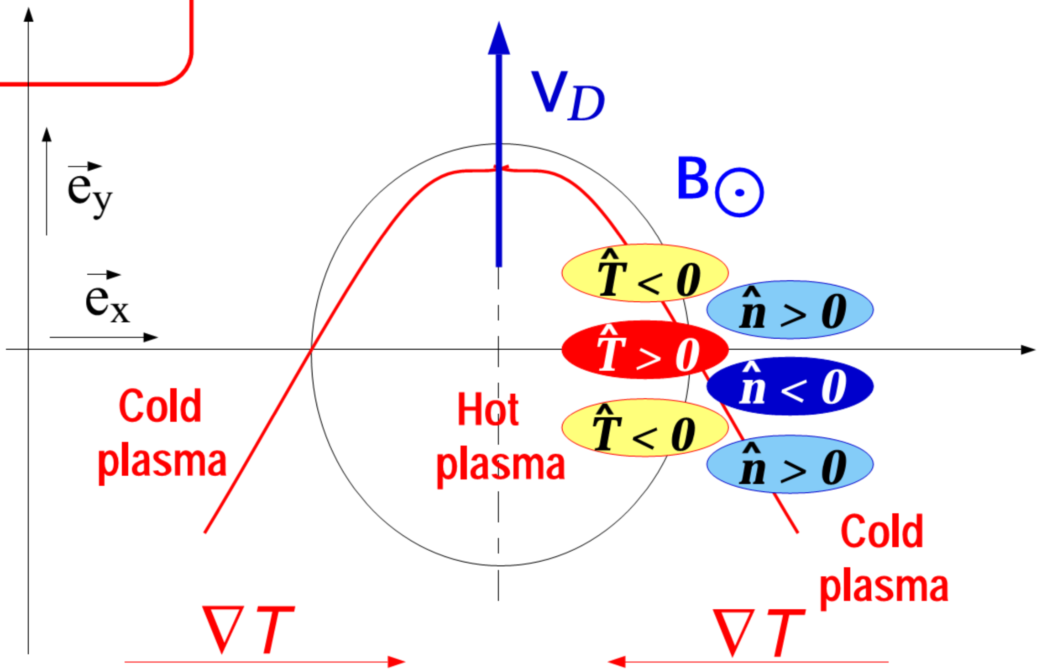
$$\hat{\omega}\tilde{n} + \underbrace{2\tilde{T}}_{\text{curvature and } \nabla B \text{ drift}} = 0$$

$$\hat{\omega}\tilde{T} + \frac{R}{L_T}\hat{\phi} = 0$$

$$\tilde{n} = \hat{\phi}$$

**Initial Temperature perturbation**

**Generates a density perturbation**



# The Ion Temperature Gradient (ITG) mode

curvature and ∇B drift

$$\hat{\omega}\tilde{n} + 2\tilde{T} = 0$$

ExB flow advection

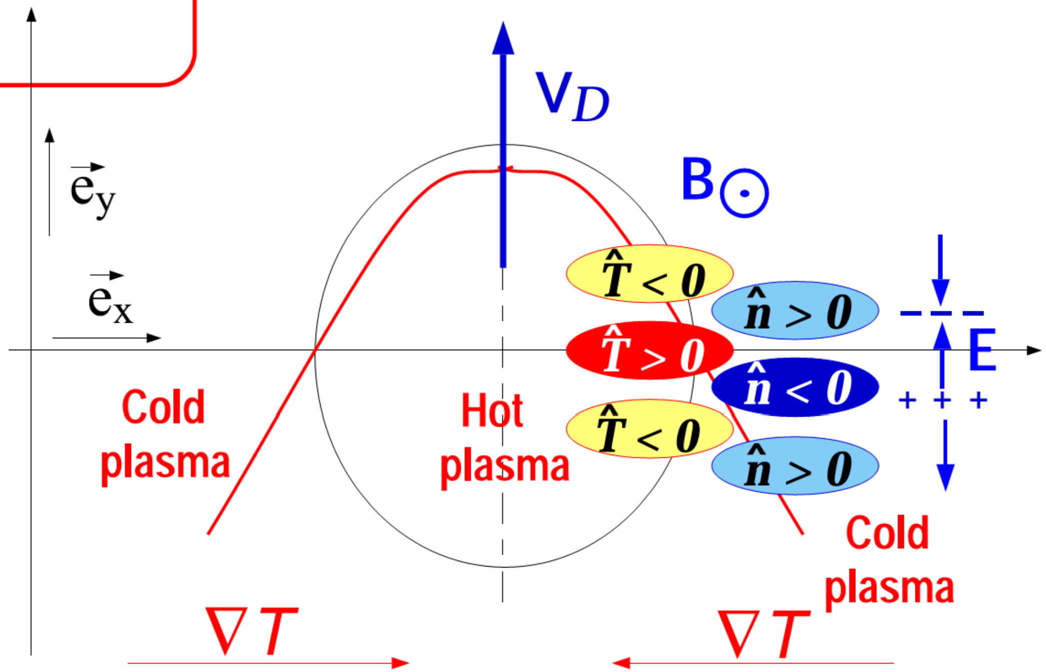
$$\hat{\omega}\tilde{T} + \frac{R}{L_T}\hat{\phi} = 0$$

$$\tilde{n} = \hat{\phi}$$

**Initial Temperature perturbation**

**Generates a density perturbation**

**Passing electrons neutralise the charge separation**



# The Ion Temperature Gradient (ITG) mode

curvature and ∇B drift

$$\hat{\omega}\tilde{n} + 2\tilde{T} = 0$$

ExB flow advection

$$\hat{\omega}\tilde{T} + \frac{R}{L_T}\hat{\phi} = 0$$

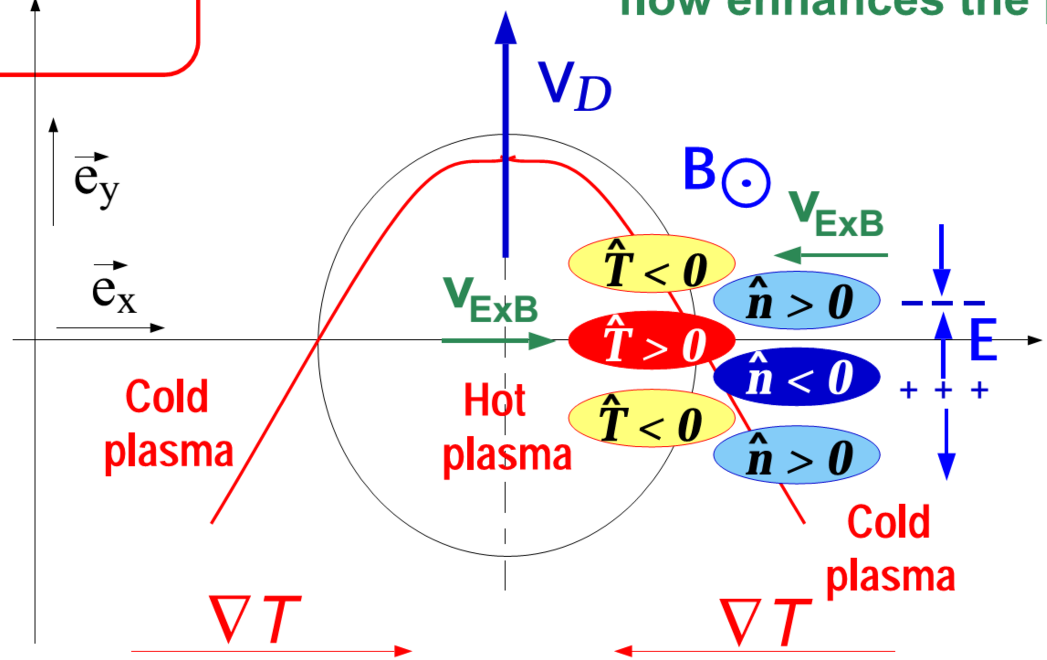
$$\tilde{n} = \hat{\phi}$$

**Initial Temperature perturbation**

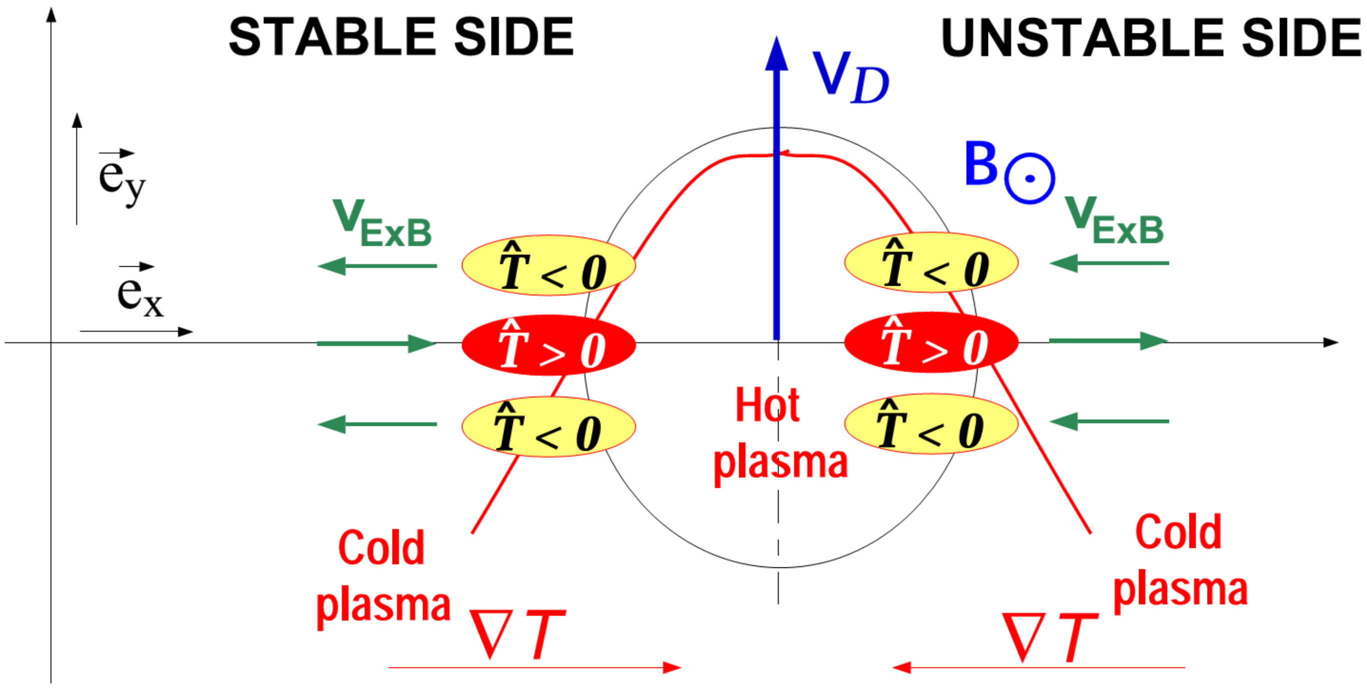
**Generates a density perturbation**

**Passing electrons neutralise the charge separation**

**Parallel force balance implies an electrostatic potential, the ExB flow enhances the perturbation**



# ITG mode, the "bad curvature" region




- **Low field side ⇒ unstable ( bad curvature ) region : warm plasma moves in the warm regions**
- **High field side ⇒ stable region (reversed temperature gradient) warm plasma moves in the cold regions**

# ITG mode, what did we neglect so far ?

- **Back to our simple fluid model**

$$\boxed{\hat{\omega}\tilde{n}} + 2\,(\tilde{n} + \boxed{\tilde{T}}) = \left[\frac{R}{L_n} - 2\right]\hat{\phi}$$

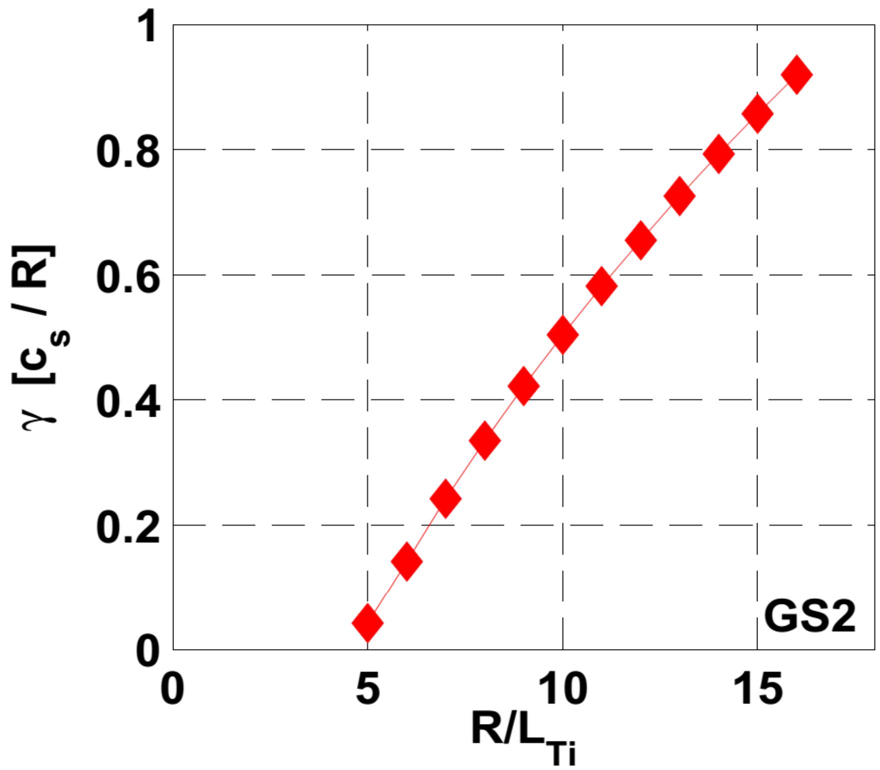
$$\boxed{\hat{\omega}\tilde{T}} + \frac{4}{3}\tilde{n} + \frac{14}{3}\tilde{T} = \left[\boxed{\frac{R}{L_T}} - \frac{4}{3}\right]\hat{\phi}$$

$$\boxed{\tilde{n} = \hat{\phi}}$$

- **By keeping all the terms, we would have found that the eigenvalues are not just imaginary, but complex numbers in which an imaginary part (an unstable mode) occurs provided that the normalized logarithmic temperature gradient R/LT exceeds a certain value (threshold) (otherwise only real roots)**

# ITG mode : the threshold

- The mode is stable for values of R/LT smaller than the threshold value (R/LT critical )




- The threshold value is not a universal number, but depends itself on plasma parameters

- In particular it increases with increasing $T_i$ / $T_e$ and for adiabatic electrons increases with increasing R/Ln ( $\eta_i$ mode , $\eta_i = L_n / L_T$ )

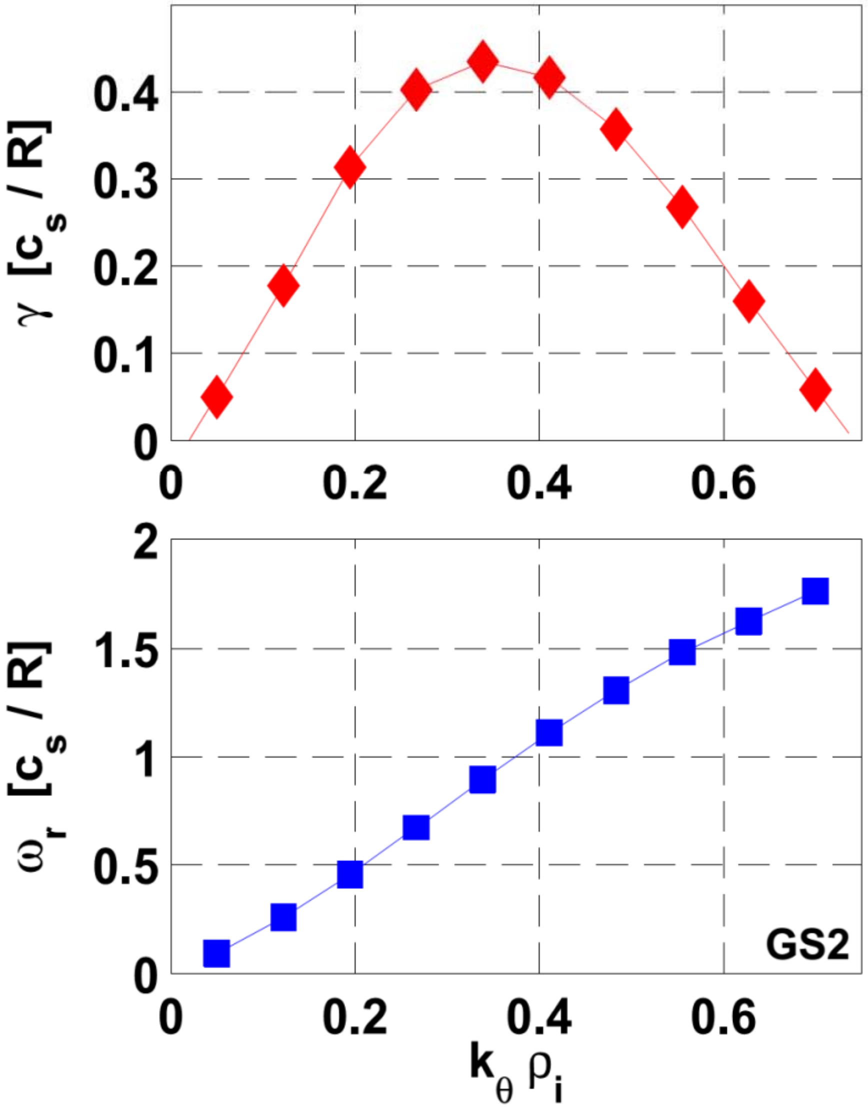
# ITG mode : the spectrum

- **Simple (reduced) fluid model**

$$\gamma = \omega_D \sqrt{2\frac{R}{L_T}}$$

$$\gamma = k_\theta \, \rho_s \frac{c_s}{R} \sqrt{2\frac{R}{L_T}}$$

**$\gamma$ is an increasing linear function of the wave number**




**This formula also shows the generic scaling of the growth rate of the ITG instability:**
**Inversely proportional to the sqrt of the major radius and the sqrt of the Ti gradient length LT**

# ITG mode : the spectrum

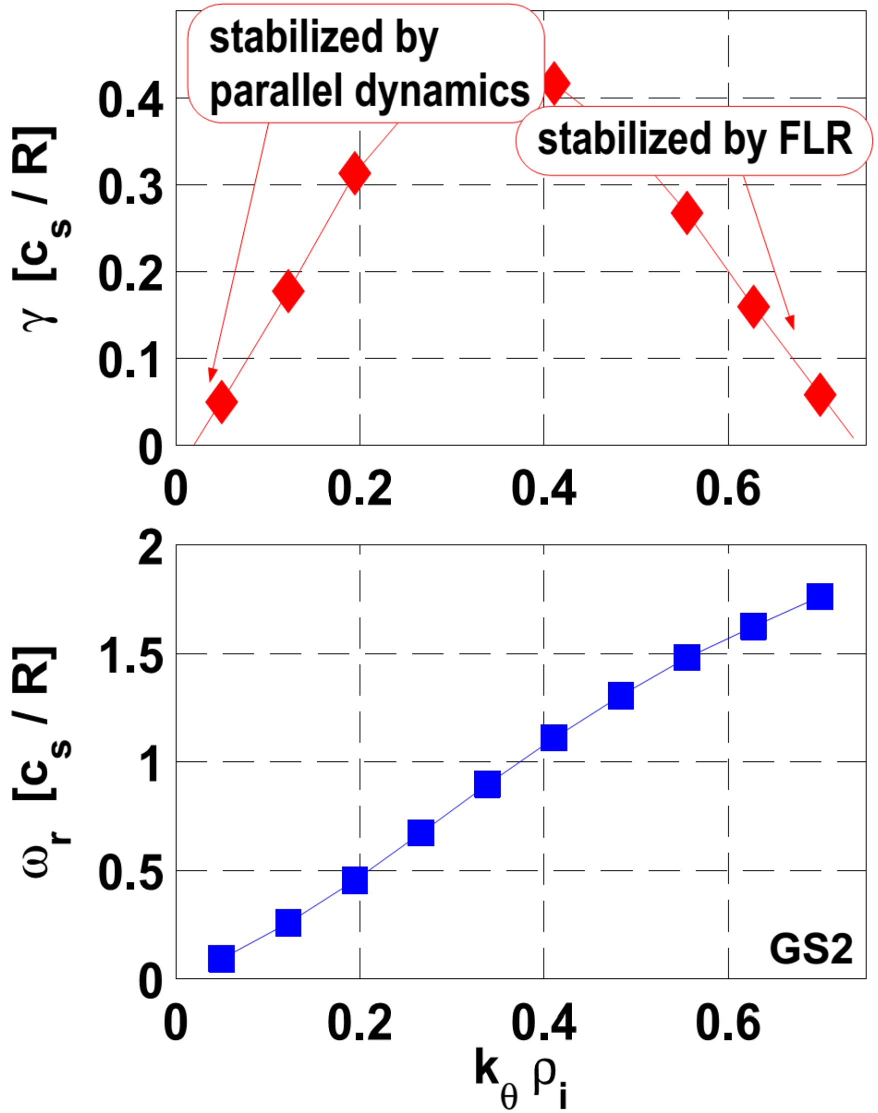
➤ **Simple (reduced) fluid model**

$$\gamma = \omega_D \sqrt{2\,\frac{R}{L_T}}$$

$$\gamma = k_\theta\, \rho_s\, \frac{c_s}{R}\, \sqrt{2\,\frac{R}{L_T}}$$

$\gamma$ **is an increasing linear function of the wave number**

➤ **Parallel dynamics and finite Larmor radius effects, which were neglected in the simple fluid model, modify significantly the spectrum**



# Kinetic electrons

- So far we have explored the (electrostatic) ITG mode with adiabatic electrons
- The inclusion of the electron dynamics leads to several other modes, in particular the trapped electron mode (TEM) and the electron temperature gradient (ETG) mode
- These are mainly electrostatic modes. Simple pictures can be built for these modes, analogous to that we have presented for the ITG, but this time assuming an adiabatic response of the ions
- In addition, at sufficiently high (electron) plasma pressure, fluctuating currents produce also fluctuations of the magnetic field (and magnetic potential), opening the possibility to instabilities of electromagnetic type, in particular the kinetic ballooning modes (KBM) and the micro-tearing modes (MTM)
- All of these modes can occur concurrently, at different scales, but also at the same scales (e.g. ITG and TEM or ITG, KBM and MTM), and have different dependencies on plasma parameters

# TEM and ETG

- These are instabilities connected with the electron dynamics, and are driven by an electron temperature gradient above a critical threshold

- Presence of trapped electrons also implies an instability driven by the density gradient

- The TEM instability can be considered as the analogous of the ITG instability, still at the ion Larmor radius scale, but where the slow (average) motion along the field line of electrons is caused by trapping (while it is caused by inertia for the ions )

- The ETG mode is the analogous of the ITG mode at the electron Larmor radius scale (practically where the role of ions and electrons are swapped)

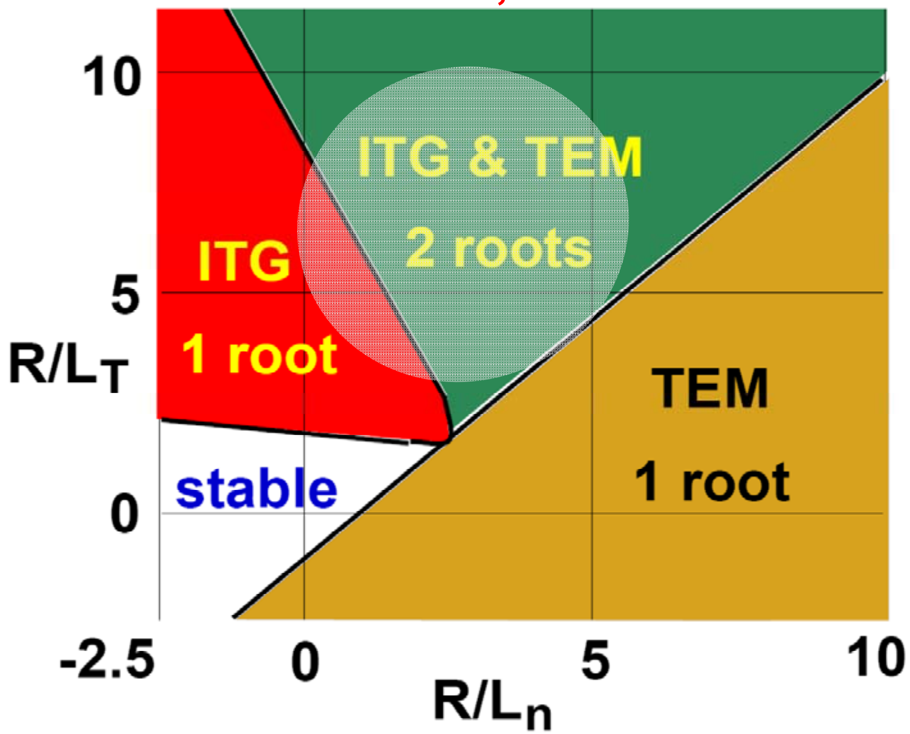
# Summary on turbulent transport so far: Example of ITG and TEM instability diagram

- Curvature and grad B drifts depend on particle energy, and electric charge
- This implies coupling between density and temperature fluctuations $\Rightarrow$ leads to instabilities ( and turbulence )
- Trapped electrons $\Rightarrow$ slow motion in parallel direction, can lead to instability like ions (slow motion by inertia)
- Presence of trapped electrons implies an additional density gradient driven instability (TEM type, next lecture)
- Instabilities occur when

$$\frac{R}{L_T} = -\frac{R}{T}\frac{dT}{dr} > \frac{R}{L_{T\,crit}}$$

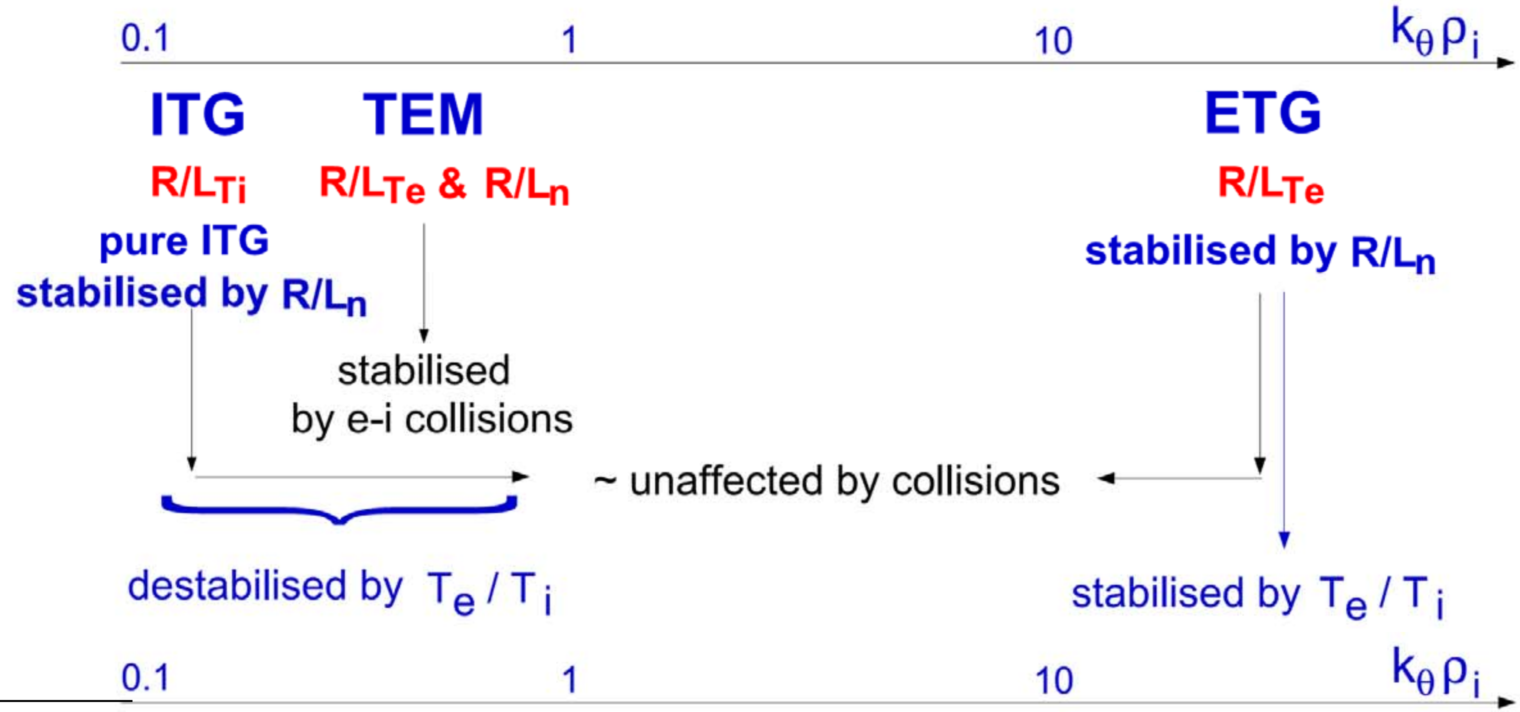
$$\frac{R}{L_n} = -\frac{R}{n}\frac{dn}{dr} > \frac{R}{L_{n\,crit}}$$

C. Bourdelle et al, Nucl. Fusion 2002



# Summary of instabilities

- instability
- drive



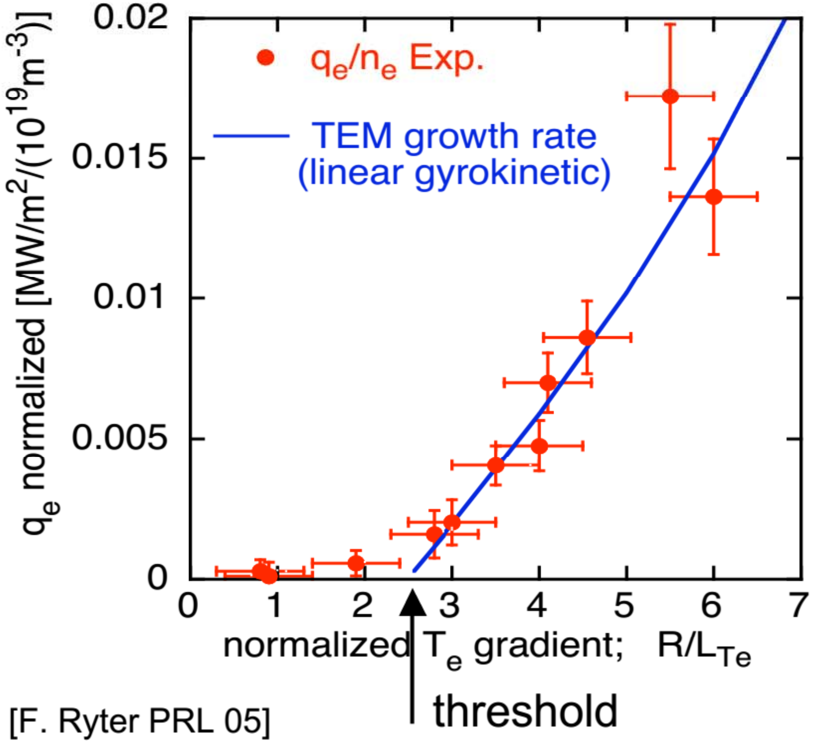

- e.m. instability **MTM , KBM**
- drive $R/L_{Te}$, $\beta_e$    $q^2 \beta_e$, $\beta'$ ( $R/L_T$, $R/L_n$ )

**MTM = microtearing mode**
**KBM = kinetic ballooning mode**

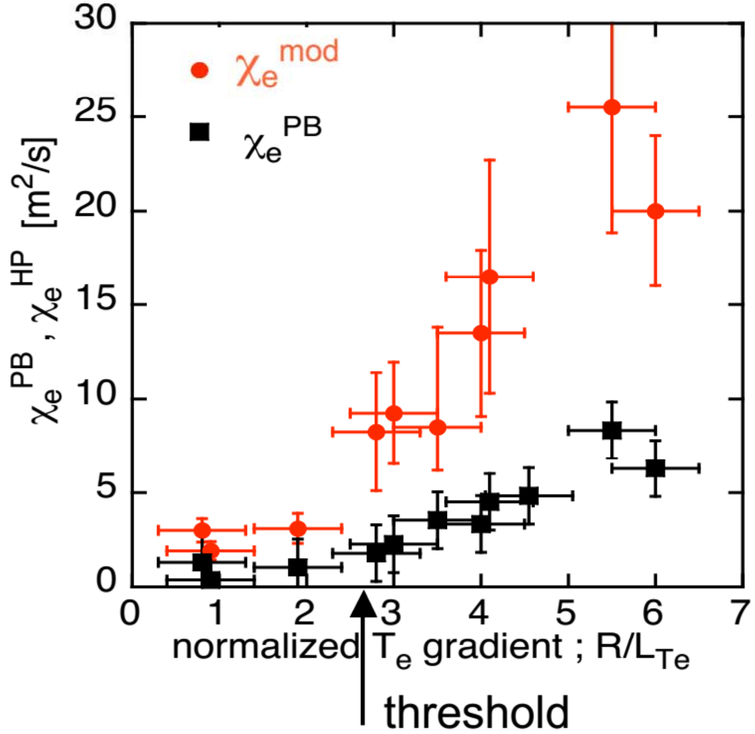
# Experimental evidence: existence of threshold

- Existence of threshold can be experimentally investigated by combined stationary and transient (modulation of heating power) experiments
- Experimental evidences have been obtained, which have been identified with thresholds predicted for TEM and ITG turbulence

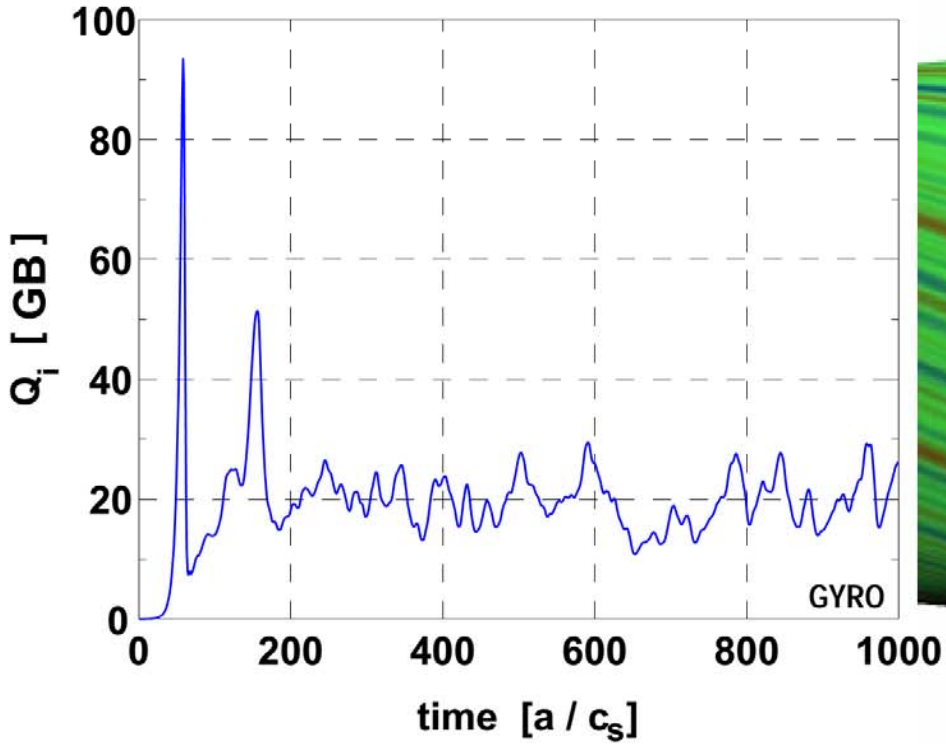
[e.g. Ryter PRL 05 (TEM), Mantica PRL 09 (ITG)]

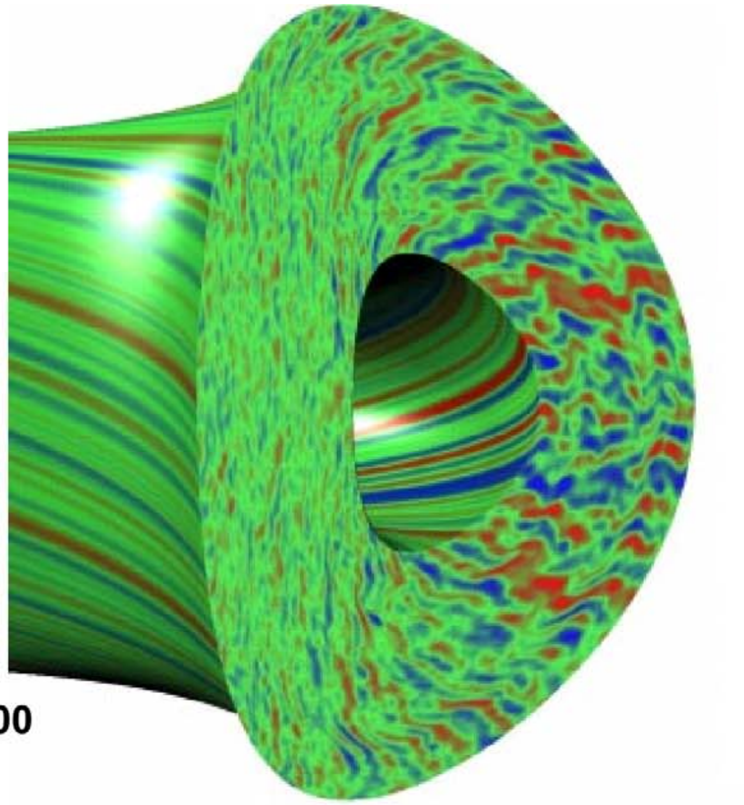



[F. Ryter PRL 05]



# From instabilities to turbulence



- **Time evolution of the ion heat flux in a gradient driven nonlinear gyrokinetic simulation (code GYRO [Candy and Waltz, General Atomics])**

# From instabilities to turbulence

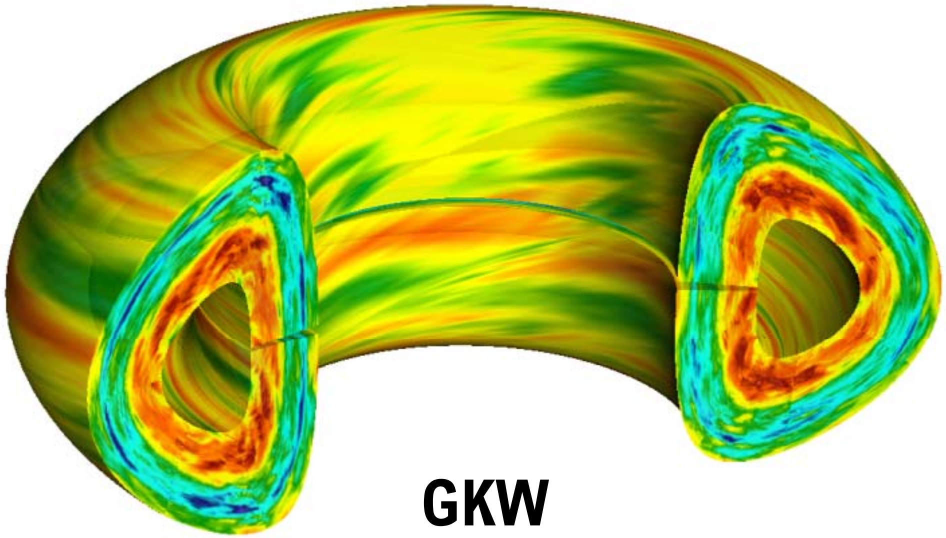
- So far we considered a linearized model. Did you notice this ?
- Consider the continuity equation $$\frac{\partial n_q}{\partial t} + \nabla \cdot (n_q \, \mathbf{v}_q) = 0,$$

$$\frac{\partial n_i}{\partial t} + n_i \nabla \cdot \mathbf{v}_E + \mathbf{v}_E \cdot \nabla n_i + \nabla \cdot (n_i \, \mathbf{v}_{*i}) = 0$$

- Terms with the fluctuating ExB drift have been only included with the background density, terms with fluctuating ExB drift times fluctuating density were ignored
- These terms imply a non-linearity
- In addition, these terms introduce toroidal mode coupling: all toroidal modes are involved in these terms (these are de-coupled in the linearized equation)
- Their inclusion leads to the description of a turbulent state in the plasma. Numerical codes (solving nonlinear gyro-fluid and gyro-kinetic systems) have been developed to compute this state and the consequent transport

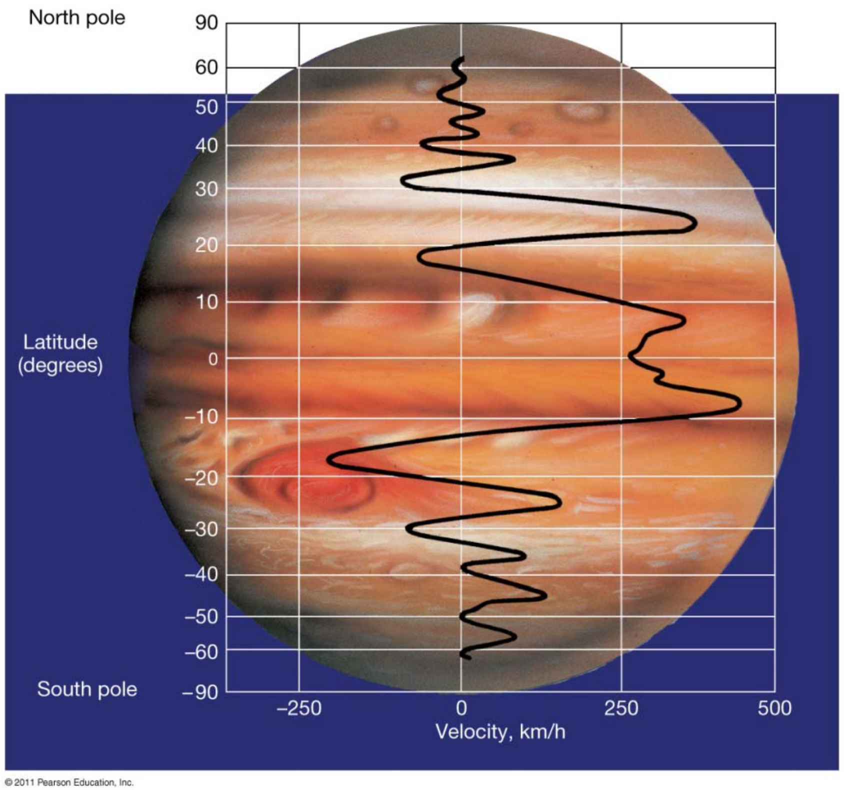
# Turbulence saturation and zonal flows

- **Zonal flows (stationary, poloidally and toroidally homogeneous (n=0,m= 0), radially varying ExB flows) are produced by turbulence (self-organization) and act as saturation and self-regulation mechanism of the turbulence**



**GKW**

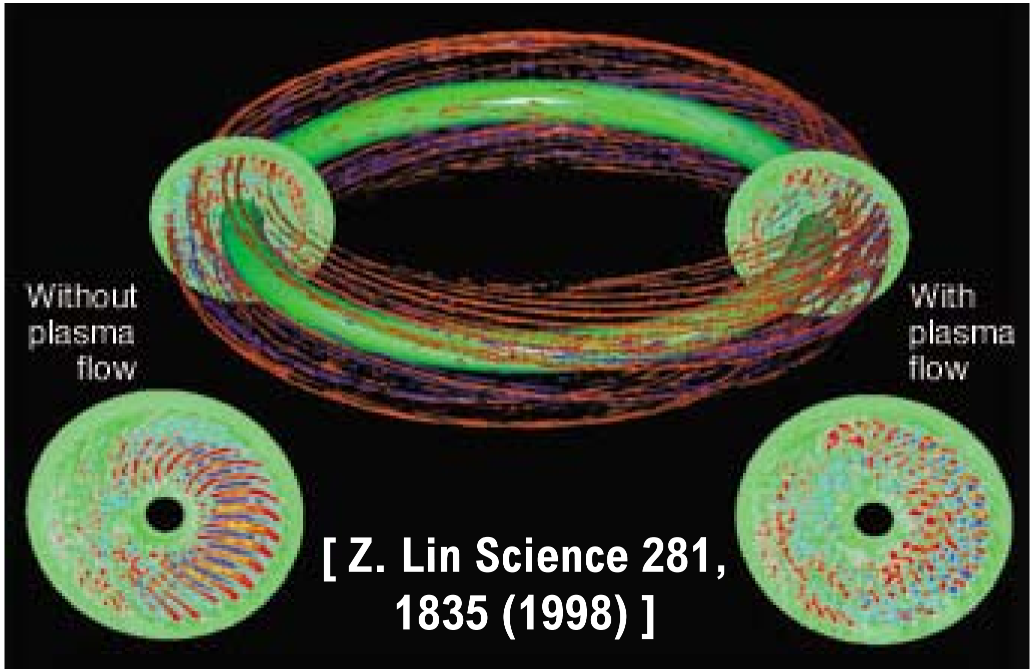
**https://code.google.com/p/gkw/**



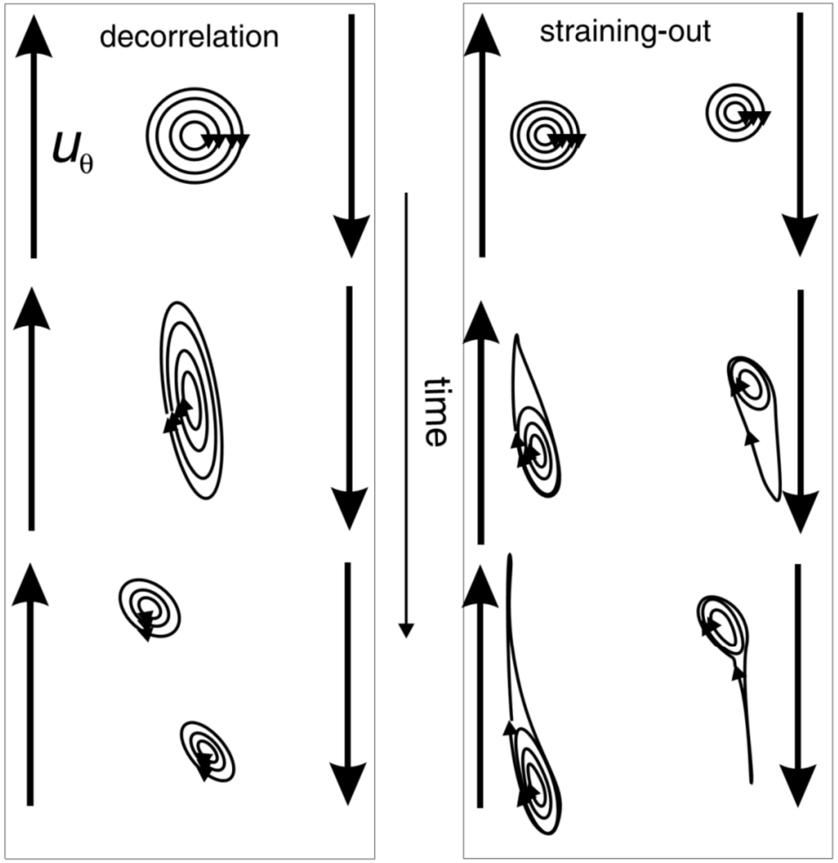

**Diamond et al, "Zonal flows in plasma-a review“, *Plasma Phys. Control. Fusion* 47, R35 (2005)**

# Stabilizing mechanisms, sheared flows

- More in general, any sheared flow can produce a reduction of the turbulence and of the associated transport
- Strong rotational shear (gradient in rotation velocity) can even quench the turbulence




[ Z. Lin Science 281, 1835 (1998) ]




[P. Manz PRL 09, U. Stroth PPCF 11]

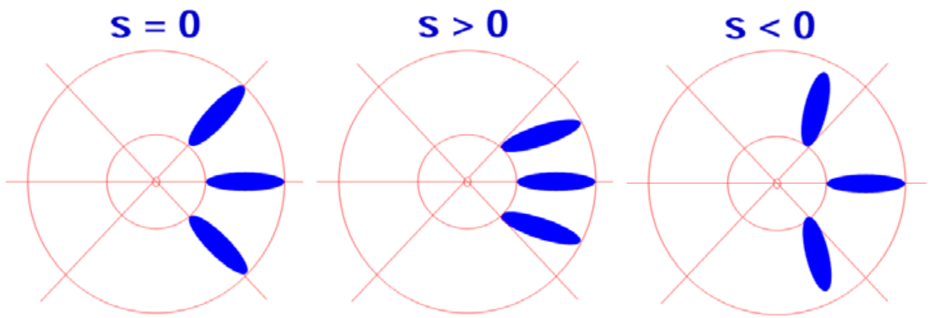
# Stabilizing mechanisms, sheared flows

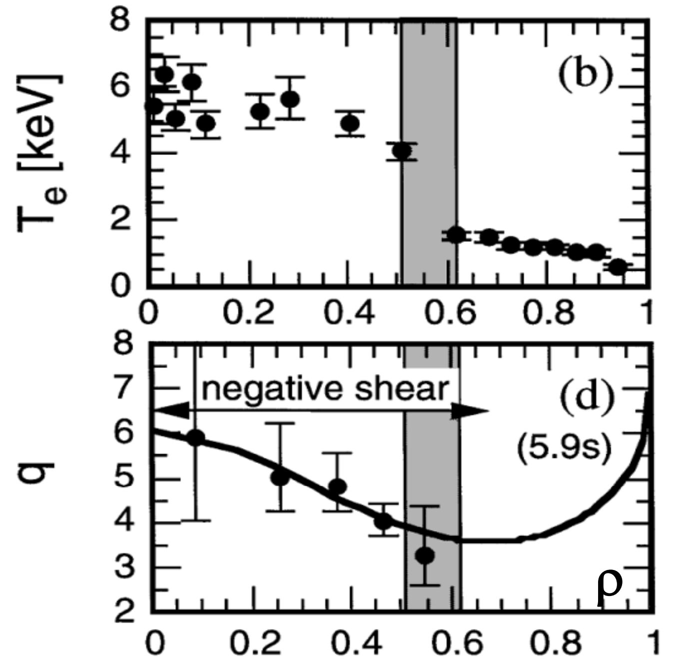
**Code:** **GYRO**

**Authors:** Jeff Candy and Ron Waltz

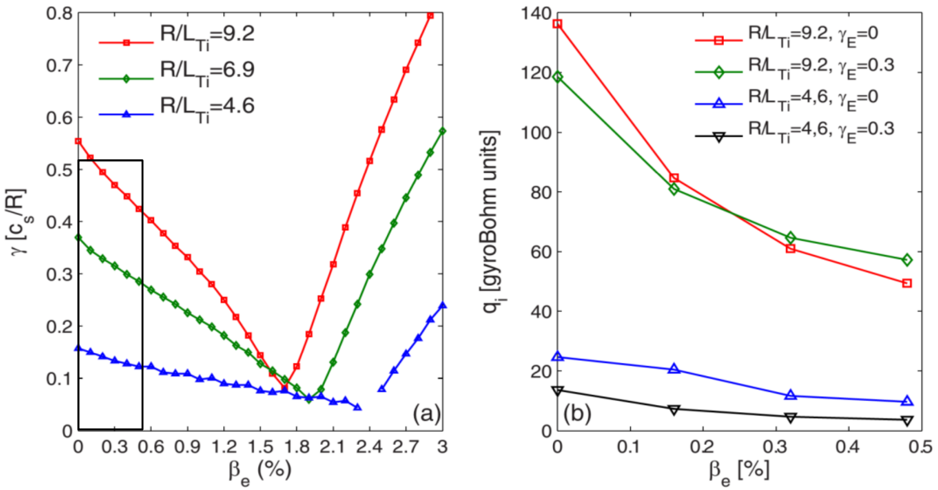
# Additional stabilizing mechanisms

➢ **Magnetic shear**




JT-60U [ Fujita PRL 78, 2377 (1997) ]




[ J. Citrin Nuclear Fusion 54, 023008 (2014) ]

➢ **Fluctuations of the magnetic field (finite beta effects)**

➢ **Ion to electron temperature ratio (for ITG modes). Dilution due to impurities**

# The local scaling of turbulent transport

- The natural scaling of turbulent transport can be obtained by considering a displacement of one Larmor radius in the time of one inverse growth rate

$$\rho_s = \frac{c_s}{\Omega_c} \quad \text{and} \quad \gamma \sim \frac{c_s}{R} \quad \Rightarrow D \propto \rho_s^2 \, \gamma \quad \Rightarrow D \propto \frac{c_s^3}{\Omega_c^2 R}$$

- It is called gyroBohm, and it is the natural scaling of local transport, derived from the dimensionless form of the equations

- The name gyroBohm shows that it is given by Bohm diffusion reduced by one normalized gyroradius factor

- Bohm diffusion is given by a displacement of one Larmor radius in a time of the order of one gyroperiod

$$D_B = \rho_s^2 \, \Omega_c = \frac{T}{e\,B} \Rightarrow D_{GB} = D_B \frac{\rho_s}{a} = \frac{m^{1/2} \, T^{3/2}}{e^2 \, B^2} \frac{1}{a}$$

# Profile stiffness

- **We look now at the scaling of the heat flux**

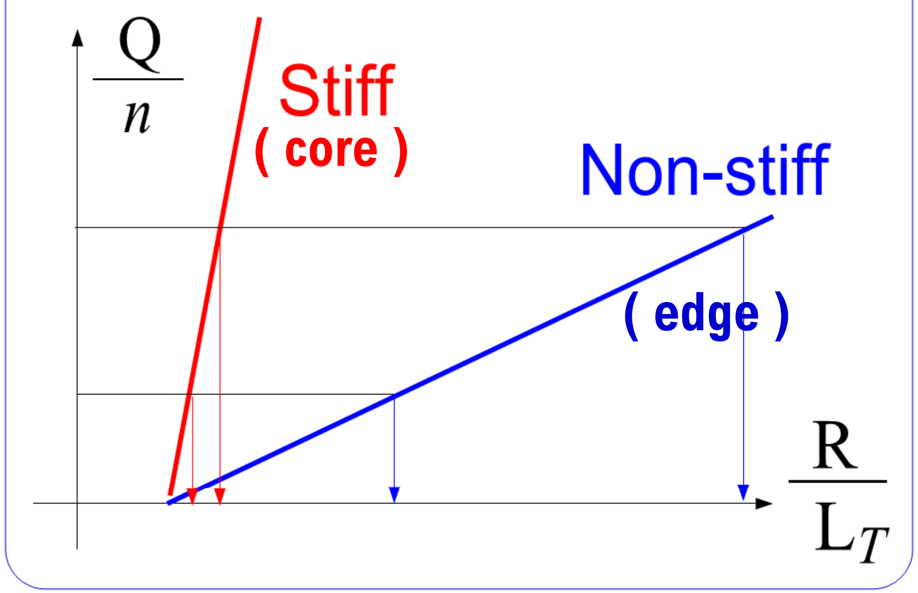
$$\frac{Q}{n} = -T\chi\frac{\nabla T}{T}$$

$$\chi_{GB} = D_B\frac{\rho_s}{a} = \frac{m^{1/2}T^{3/2}}{e^2B^2}\frac{1}{a}$$

$$\frac{Q}{n} = \hat{\chi}_s T\frac{\chi_{GB}}{R}\left(\frac{R}{L_T} - \frac{R}{L_{Tcrit}}\right) = \hat{\chi}_s\frac{m^{1/2}T^{5/2}}{e^2B^2R^2}\left(\frac{R}{L_T} - \frac{R}{L_{Tcrit}}\right)$$

- **Strong scaling of the heat flux with temperature**
- **Core, high temperature, is stiff (little change of gradient in response to large change of heat flux)**
- **Edge, low temperature, is non stiff**

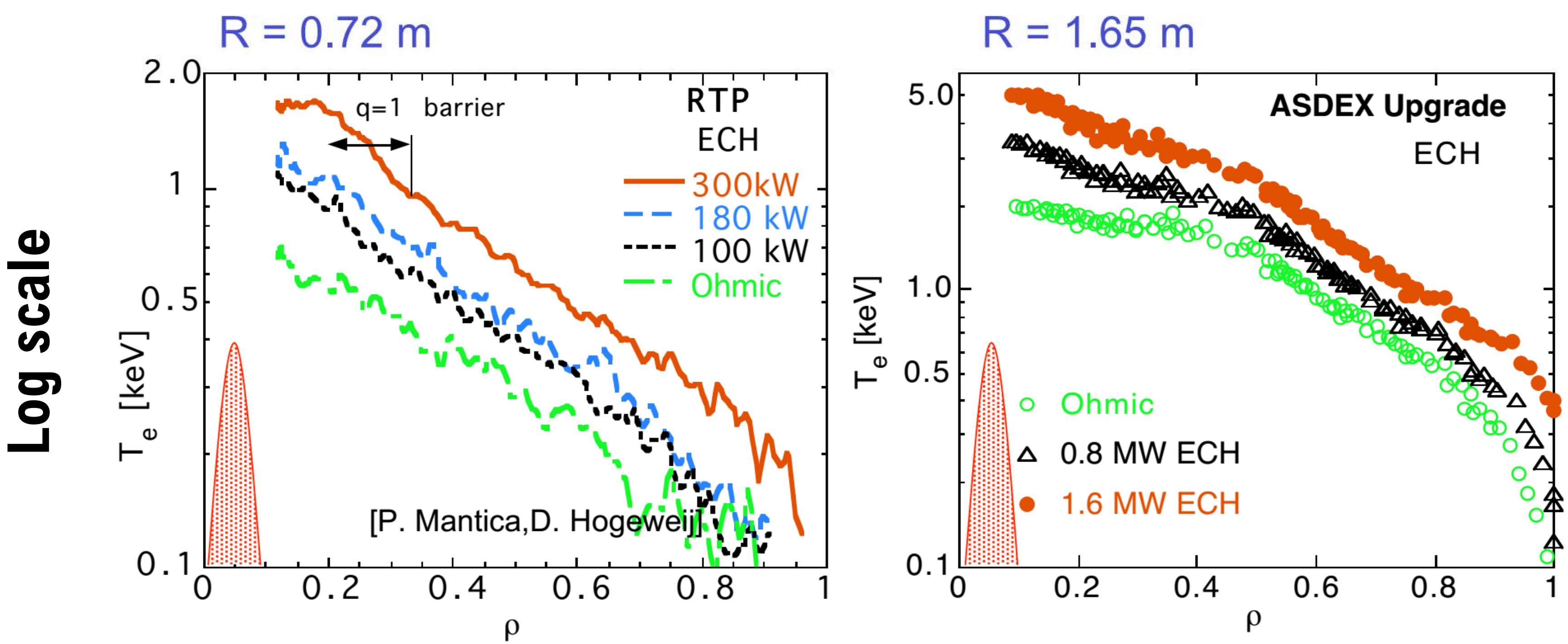
- **With reactor high temperatures, core profiles will be close to the threshold**



# Profile stiffness, experimental evidences

➢ **With central heating, temperature increases maintaining self-similar profile shape**

Comparable $B_T$ and $n_e$, central ECH, different size [F. Ryter ppcf 2001]




$T_e$ profiles: const $\nabla T_e/T_e$ shifted according to edge $T_e$

"Profile resilience" observed since 1980 in tokamaks
(TFR, TFTR, T10, ASDEX, DIII-D, ASDEX Upgrade, Tore Supra, ... JET )

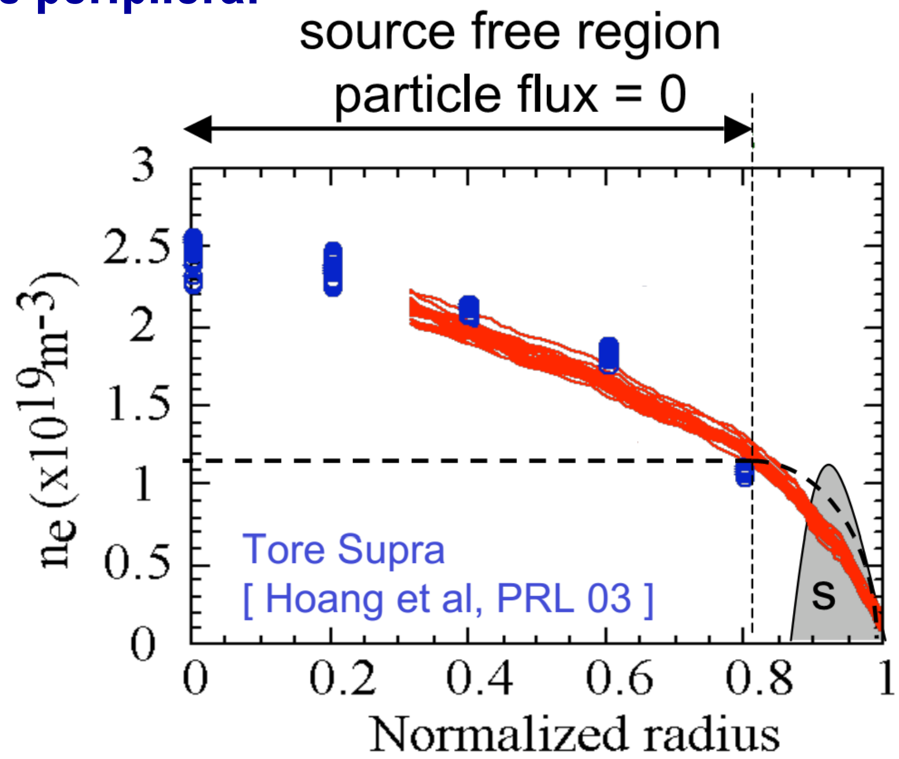
# Turbulence and particle transport (tokamaks): Density profiles are peaked in the centre

- **In most conditions, electron density profiles in tokamaks are centrally peaked, even when the particle source is peripheral**
- **Existence of inward convection (pinch) balancing outward diffusion**

$$\Gamma = -D\nabla n_e + n_e V$$

$$\Downarrow$$

$$-\frac{\nabla n_e}{n_e} = -\frac{V}{D} + \frac{\Gamma}{n_e D}$$




- **Theoretical studies have shown that turbulence can produce inward convection which sustains centrally peaked density profiles**
- **Predictions are found to qualitatively (quantitatively) reproduce observations**

# Turbulence and particle transport

- In fusion devices, there are conditions in which profiles are observed to be peaked in the absence of particle source

- Pinch (inward convection) must be present which sustains a finite density gradient in the absence of a source

- Theory of turbulent transport predicts the existence of this inward convection.
- Its properties are found to be consistent with several experimental observations

- Like neoclassical transport, also turbulent transport cannot be described by simple Fick's laws (diffusion only), but also comprises off-diagonal terms

$$\frac{\Gamma}{n} = -D\frac{\nabla n}{n} - \underbrace{D_T\frac{\nabla T}{T}}_{\text{Thermo-diffusion}} + \underbrace{V_p}_{\text{Pure convection}}$$

# Magnetic field inhomogeneity and curvature at the origin of turbulent convection

- **Trapped electron density $n$ and fluctuating electrostatic potential $\tilde{\phi}$**

$$\frac{\partial n}{\partial t} + \nabla \cdot ( n \, \tilde{\mathbf{v}}_{E\times B} ) = 0$$

$$\frac{\partial n}{\partial t} + \nabla n \cdot \tilde{\mathbf{v}}_{E\times B} + n \nabla \cdot \tilde{\mathbf{v}}_{E\times B} = 0$$

# Magnetic field inhomogeneity and curvature at the origin of turbulent convection

- **Trapped electron density** $n$ **and fluctuating electrostatic potential** $\tilde{\phi}$

$$\frac{\partial n}{\partial t} + \nabla \cdot ( n \, \tilde{\mathbf{v}}_{E\times B} ) = 0$$

$$\frac{\partial n}{\partial t} + \nabla n \cdot \tilde{\mathbf{v}}_{E\times B} = 0$$

- **Constant B with straight field lines :** $\nabla \cdot \tilde{\mathbf{v}}_{E\times B} = 0$ $\Rightarrow$ **Diffusion only**

# Magnetic field inhomogeneity and curvature at the origin of turbulent convection

- Trapped electron density $n$ and fluctuating electrostatic potential $\tilde{\phi}$

$$\frac{\partial n}{\partial t} + \nabla \cdot \left( n \, \tilde{\mathbf{v}}_{E\times B} \right) = 0$$

$$\frac{\partial n}{\partial t} + \nabla n \cdot \tilde{\mathbf{v}}_{E\times B} + n \nabla \cdot \tilde{\mathbf{v}}_{E\times B} = 0$$

- Constant B with straight field lines : $\nabla \cdot \tilde{\mathbf{v}}_{E\times B} = 0$

- Magnetic field in tokamak geometry : $\nabla \cdot \tilde{\mathbf{v}}_{E\times B} = -\tilde{\mathbf{v}}_{E\times B} \cdot \left( \frac{\nabla B}{B} + \mathcal{K}_B \right)$

**Curvature and grad B yield a compression term, produces a “curvature” pinch, trapped electrons are transported inward**

# Momentum transport and toroidal rotation

- Tokamaks : toroidal symmetry, from conservation of angular momentum → neoclassical transport is intrinsically ambipolar, no radial current, Er and toroidal velocity are unconstrained (plasma is free to rotate in toroidal direction)
- Stellarators: Neoclassical transport is not intrinsically ambipolar, ambipolarity sets the radial electric field, which also determines the plasma rotation
- Tokamaks: turbulent transport also produces a toroidal viscosity which plays a critical role in determining radial profile of toroidal rotation in the presence of ext. torques
- However, tokamak plasmas can rotate in the toroidal direction even in the absence of any external torque (called intrinsic rotation)
- Turbulence also produces a momentum pinch (as it produces a particle pinch)
- Sources of intrinsic toroidal rotation are one of the topics of current research (external torque will be limited in a reactor)

# Summary

- From the plasma physics standpoint, in fusion devices plasma confinement plays the most critical role towards the achievement of practical fusion energy
- In this lecture we have given a quick overview of the general concepts and the main (collisional and turbulent) processes which are involved
- Turbulence implies that the topic becomes extremely involved and quantitative prediction are very difficult to obtain
- However, at least in the plasma core, quasi-linear theory often gives results which are in acceptable agreement with the nonlinear simulations. Several (not all yet) experimental observations are qualitatively (quantitatively) reproduced
- The description of the plasma edge is more involved, and requires a fully nonlinear treatment in most conditions
- The topic of transport and confinement in fusion devices is continuously evolving. I hope this lecture will allow you to build the background to be able to increase your knowledge by reading published reviews and more recent papers

# If you want to learn more about specific topics, please ask ! Here some general references :

- **Neoclassical transport:**

**F. L. Hinton and R.D. Hazeltine, *Theory of plasma transport in toroidal confinement systems,* Rev. Mod. Phys. 48, 239 (1976); http://journals.aps.org/rmp/abstract/10.1103/RevModPhys.48.239**

**S.P. Hirshman and D.J. Sigmar, *Neoclassical transport of impurities in tokamak plasmas,* Nucl. Fusion 21, 1079 (1981); http://iopscience.iop.org/0029-5515/21/9/003**

**P. Helander and D.J. Sigmar, *Collisional Transport in Magnetized Plasmas*, Cambridge University Press (2001) ISBN 0521807980**

- **Turbulent transport:**

**J. Weiland, *Collective modes in Inhomogeneous Plasmas*, Inst. of Phys. Pub. (1999) ISBN-13: 978-0-7503-0589-1**

**W. Horton, *Drift Waves and Transport*, Rev. Mod. Phys. 71, 735 (1999); http://journals.aps.org/rmp/abstract/10.1103/RevModPhys.71.735**

**P. W. Terry, *Suppression of turbulence and transport by sheared flow* , Rev. Mod. Phys. 72, 109 (2000); http://journals.aps.org/rmp/abstract/10.1103/RevModPhys.72.109**

# THE END

- Additional slides on collisional transport ( classical, Pfirsch-Schlüter flows and neoclassical )

# Fluid approach requires definition of moments of the distribution function

**Density**

$$n = \int d^3\vec{v}\, f$$

**Particle flux**

$$n\vec{u} = \int d^3\vec{v}\, \vec{v}\, f$$

**Stress tensor**

$$\mathsf{P} = \int d^3\vec{v}\, m\, \vec{v}\, \vec{v}\, f$$

**Energy flux**

$$\vec{Q} = \int d^3\vec{v}\, (m v^2/2)\, \vec{v}\, f$$

**Scalar pressure**

$$p = nT = \mathrm{Tr}\{\mathsf{P}\}/3$$

**Energy weighted stress tensor**

$$\mathsf{R} = \int d^3\vec{v}\, (m v^2/2)\, \vec{v}\, \vec{v}\, f$$

# Fluid approach also requires definition of moments of the collision operator

**Friction Force**

$$\vec{\mathbf{F}} = \int d^3\vec{\mathbf{v}}\, m\vec{\mathbf{v}}\, C(f)\,;$$

**Collisional energy exchange**

$$Q = \int d^3\vec{\mathbf{v}}\,(m/2)\,(\vec{\mathbf{v}} - \vec{\mathbf{u}})^2\, C(f)$$

**Conservation laws (particle, momentum, energy)**

$$\int d^3\vec{\mathbf{v}}\, C_{ab} = 0\,,$$

$$\int d^3\vec{\mathbf{v}}\,[m_a\vec{\mathbf{v}}\, C_{ab} + m_b\vec{\mathbf{v}}\, C_{ba}] = 0\,,$$

$$\int d^3\vec{\mathbf{v}}\,[m_a v^2\, C_{ab} + m_b v^2\, C_{ba}] = 0\,.$$

$\Rightarrow$

$$\sum_a \vec{\mathbf{F}}_a = 0\,,$$

$$\sum_a (Q_a + \vec{\mathbf{F}}_a \cdot \vec{\mathbf{u}}_a) = 0\,.$$

# Fluid equations

- **Even moments (particle and energy conservation)**

$$\partial n/\partial t + \vec{\nabla}\cdot(n\vec{u}) = 0 \,,$$

$$(\partial/\partial t)\,3p/2 + \vec{\nabla}\cdot\vec{Q} = Q + \vec{u}\cdot(\vec{F} + en\vec{E}) \,.$$

- **Odd moments (momentum and energy flux conservation)**

$$(\partial/\partial t)mn\vec{u} + \vec{\nabla}\cdot \mathsf{P} - en(\vec{E} + c^{-1}\vec{u}\times\vec{B}) = \vec{F} \,,$$

$$(\partial/\partial t)\vec{Q} + \vec{\nabla}\cdot \mathsf{R} - (3/2)(e/m)\vec{E}p - (e/m)\vec{E}\cdot \mathsf{P} - (e/mc)\vec{Q}\times\vec{B} = \vec{G}$$

- **Equation evolving moment *n* involves moment *n+1* → infinite set of equations → a fluid closure is required**

# Orderings

- Orderings are an essential element of theory of plasma transport
- Depending on the orderings which are assumed, one can obtain very different consequences and compute different transport processes starting from the same equations
- Generic ordering for magnetized plasmas: B is Big ! that is cyclotron frequency is large and Larmor radius is small

$$\delta \equiv \rho / l = \omega / \Omega \ll 1$$

$$\rho \equiv v_{\mathrm{th}} / \Omega = m v_{\mathrm{th}} / (eB) \quad ; \quad \omega \equiv v_{\mathrm{th}} / l \; ; \; l \cong |\nabla \ln p|^{-1}$$

- In addition, collisional transport theory orders small the time derivatives (no rapid fluctuation)

$$\partial \ln p / \partial t = O(\delta^2 \omega)$$

# Zero order in $\delta \equiv \rho / l$ : no variation along B

- It can be shown that with these orderings the distribution function at the lowest order it a local Maxwellian distribution
- Note: this is NOT an assumption, but follows from the orderings
- Orderings applied to odd moments of fluid equations imply :

$$\vec{B} \cdot \vec{\nabla} T = O(\delta) \;, \quad \vec{B} \cdot \vec{\nabla} [n \exp(-e\Phi / T)] = O(\delta)$$

Combined with $\sum_a e_a n_a = 0$

$$\vec{B} \cdot \vec{\nabla} n = O(\delta) \qquad \vec{B} \cdot \vec{\nabla} \Phi = O(\delta)$$

- Note that orderings also imply that :

$$\{ n\vec{u}, \vec{Q}, \vec{F}, (\mathsf{P} - \mathsf{I}p) \} = O(\delta)$$

# First order: ordering applied to momentum conservation

- Momentum conservation can be used as typical example

$$(\partial/\partial t) m n \vec{\mathbf{u}} + \vec{\nabla} \cdot \mathsf{P} - en(\vec{\mathbf{E}} + c^{-1} \vec{\mathbf{u}} \times \vec{\mathbf{B}}) = \vec{\mathbf{F}}$$

$$(\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

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- Never forget : $\left(\vec{u}\times\vec{B}\right)\times\vec{B} = -\vec{u}_{\perp}B^2$

$$\hat{n} \equiv \vec{\mathrm{B}}/B$$

$$n\vec{\mathrm{u}}_{\perp} = (m\Omega)^{-1}\hat{n}\times[\vec{\nabla}\cdot \mathrm{P} - \vec{\mathrm{F}} - en\vec{\mathrm{E}} + m(\partial/\partial t)n\vec{\mathrm{u}}]$$

- We need to evaluate RHS at order $n$ to obtain LHS at order $n+1$
- Collisional transport: ordering also assumes no rapid variation, time derivatives are ordered small (higher order term at the RHS)
- If we drop this assumption, the same equations can lead to unstable conditions → micro-instabilities and turbulence (later)

# 1st and 2nd order perpendicular flows

$$n\vec{u}_\perp = (m\Omega)^{-1}\hat{n}\times[\vec{\nabla}\cdot \mathrm{P} - \vec{F} - en\vec{E} + m(\partial/\partial t)\, n\vec{u}]$$

**Diamagnetic and ExB flows**

- **1st order** $$(n\vec{u}_\perp)_1 = (m\Omega)^{-1}\hat{n}\times(\vec{\nabla}\bar{p} + e\bar{n}\vec{\nabla}\bar{\Phi})$$

**These flows remain on the magnetic flux surface (no radial component)**

**Classical Transport**

- **2nd order** $$(n\vec{u}_\perp)_2 = -(m\Omega)^{-1}\hat{n}\times\vec{F} \; +$$

**Neoclassical Transport**

$$+(m\Omega)^{-1}\hat{n}\times[\vec{\nabla}\cdot(\mathrm{P} - \mathrm{I}\bar{p}) - e(n\vec{E} + \bar{n}\vec{\nabla}\bar{\Phi})]$$

$$F_\parallel + enE_\parallel \simeq \hat{n}\cdot\vec{\nabla}\mathrm{P}_\parallel - (\mathrm{P}_\parallel - \mathrm{P}_\perp)\hat{n}\cdot\vec{\nabla}B/B$$

# Classical transport: a simple example

**Classical Transport** $n\vec{\mathrm{u}}_c \equiv -(m\Omega)^{-1}\hat{n} \times \vec{\mathrm{F}}$

- **We have to compute the friction force** $\vec{\mathrm{F}} = \int d^3\vec{\mathrm{v}}\, m\vec{\mathrm{v}}\, C(f)$
- **We observe that 1st order electron and ion perp flows are in opposite directions (note also that they are perp to B but remain on the flux surface)**
- **Friction force can be computed by shifted Maxwellian which has those first order flows (should also include other corrections to describe first order energy flow: this is obtained consistently from a kinetic approach)** $\vec{F}_{ab} = m_a n_a \nu_{ab}\left(\vec{u}_b - \vec{u}_a\right)$

# Classical transport: the particle flux

**Classical Transport** $$n\vec{\mathrm{u}}_c \equiv -(m\Omega)^{-1}\hat{n}\times\vec{\mathrm{F}}$$

- **where** $$\vec{F}_{ab} = m_a n_a \nu_{ab}\left(\vec{u}_b - \vec{u}_a\right)$$

- **And :** $$(n\vec{\mathrm{u}}_\perp)_1 = (m\Omega)^{-1}\hat{n}\times(\vec{\nabla}\bar{p} + e\bar{n}\vec{\nabla}\Phi)$$

$$n\vec{\mathrm{u}}_c = \frac{m_a}{e_a B^2}\sum_{b\neq a}\nu_{ab}\left(\frac{\nabla p_b}{e_b n_b} - \frac{\nabla p_a}{e_a n_a}\right)$$

- **Classical transport produced by perpendicular friction of diamagnetic flows**

# General expression of classical transport, impurity pinch and temperature screening

- So far, we have computed classical transport considering only fricition of 1st order (in $\delta$) perpendicular (diamagnetic) flows (1st odd fluid moment)

- Higher odd fluid moments are also present, e.g. the diamagnetic heat flux $\vec{q}_{\perp 1} = (5/2)\,(m\Omega)^{-1}\,\overline{p}\,\hat{n}\times\vec{\nabla}\,\overline{T}$

- 1st order (in $\delta$) distribution function is not just a shifted Maxwellian but also has corrections which produce these higher fluid moments (can be heuristically derived by Taylor expanding the Maxwellian on a gyroradius with both density and temperature radial gradients)

$$f_1 = f_0\left(u_{dia} + \left[\frac{v^2}{v_T^2} - \frac{5}{2}\right]\frac{k_B T'}{e_a B}\right)\frac{2v_z}{v_T^2}$$

- Perpendicular friction force is modified by the explicit appearance of temperature gradients ("thermal force") and so is the radial particle flux

$$\vec{\Gamma}_{CL}^{a} = -\nabla n_a \sum_{b\neq a} D_{CL}^{ab} + n_a \sum_{b\neq a} D_{CL}^{ab}\frac{e_a}{e_b}\left(\frac{\nabla n_b}{n_b} - \frac{\nabla T}{T}\left[\frac{3m_{ab}}{2m_b} - 1 - \frac{e_b}{e_a}\left(\frac{3m_{ab}}{2m_a} - 1\right)\right]\right) \qquad D_{CL}^{ab} = \frac{\rho_a^2}{2}\nu_{ab}$$

# General expression of classical transport, impurity pinch and temperature screening

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$$f_1 = f_0\left(u_{dia} + \left[\frac{v^2}{v_T^2} - \frac{5}{2}\right]\frac{k_B T'}{e_a B}\right)\frac{2v_z}{v_T^2}$$

- For heavy impurities: convection proportional to impurity charge, inward (pinch) due to negative (centrally peaked) density, outward due to temperature gradients

$$\vec{\Gamma}_{CL}^{Z} = \frac{\rho_Z^2 \nu_{ZH}}{2}\left\{-\nabla n_Z + n_Z Z\left(\boxed{\frac{\nabla n_H}{n_H}}\boxed{-\frac{1}{2}\frac{\nabla T}{T}}\right)\right\}$$

Temperature screening, outward with centrally peaked profiles

# Divergence of first order flows

- **Perpendicular first order flows from the perpendicular component momentum balance equation**

$$\vec{u}^{(1)}_{\perp,a} = \frac{\vec{E}\times\vec{B}}{B^2} - \frac{\nabla p_a \times \vec{B}}{e_a n_a B^2} = -\frac{\partial\Phi}{\partial\psi}\frac{\nabla\psi\times\vec{B}}{B^2} - \frac{\partial p_a}{\partial\psi}\frac{\nabla\psi\times\vec{B}}{e_a n_a B^2} = \omega_a(\psi)\frac{\nabla\psi\times\vec{B}}{B^2}$$

$$\omega_a(\psi) = -\frac{\partial\Phi}{\partial\psi} - \frac{1}{e_a n_a}\frac{\partial p_a}{\partial\psi}$$

- **Electrostic potential and pressure are constant on magnetic flux surfaces a lowest order (psi is the poloidal flux, radial coordinate)**

$$\nabla\cdot\vec{u}^{(1)}_{\perp,a} = \underbrace{\frac{\nabla\cdot\omega_a(\psi)}{B^2}\cdot\nabla\psi\times\vec{B}}_{=0} + \omega_a(\psi)\nabla\cdot\left(\frac{1}{B^2}\right)\cdot\nabla\psi\times\vec{B} + \underbrace{\frac{\omega_a(\psi)}{B^2}\nabla\cdot\left(\nabla\psi\times\vec{B}\right)}_{=0}$$

$$= -2\omega_a(\psi)\frac{\nabla B}{B^3}\nabla\psi\times\vec{B}$$

$$= -2\vec{u}^{(0)}_{\perp,a}\cdot\frac{\nabla B}{B}$$

$$\nabla\cdot\left(\nabla\psi\times\vec{B}\right) = \vec{B}\cdot\underbrace{\left(\nabla\times\nabla\psi\right)}_{=0} - \nabla\psi\cdot\underbrace{\left(\nabla\times\vec{B}\right)}_{\mu_0\vec{j}} = 0$$

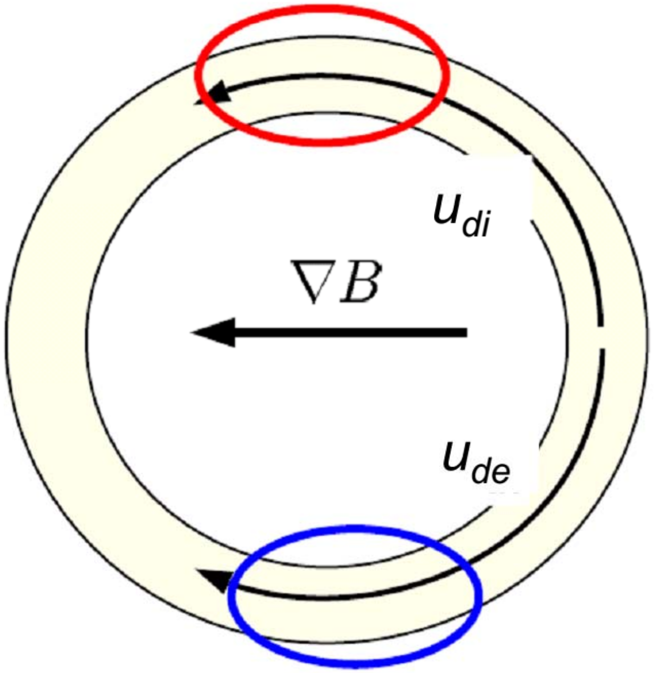
# Divergence of first order flows

- Perpemdicular flows are not divergence free
- The continuity equation then implies:

$$\frac{\partial n_a}{\partial t} = -\nabla\cdot\left(n_a \vec{u}^{(1)}_{\perp,a}\right) = -n_a \nabla\cdot\vec{u}^{(1)}_{\perp,a}$$

$$\nabla\cdot\vec{u}^{(1)}_{\perp,a} = -2\vec{u}^{(1)}_{\perp,a}\cdot\frac{\nabla B}{B}$$

$$\frac{\partial n_a}{\partial t} = 2n_a \vec{u}^{(1)}_{\perp,a}\frac{\nabla B}{B}$$



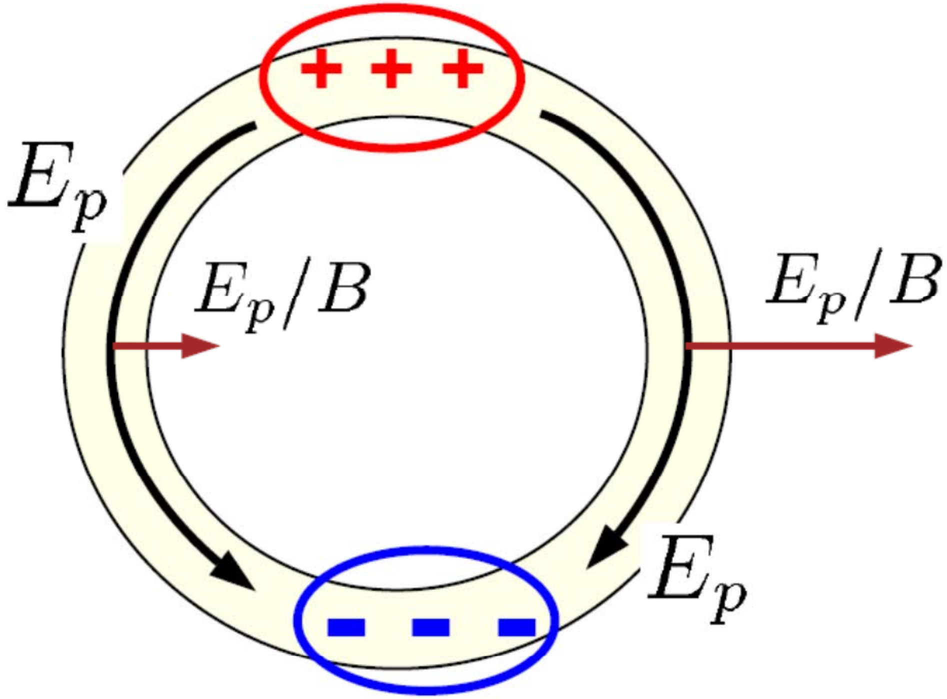

- Densities of ions and electrons pile up on the top and the bottom of the flux surface
- In the particle picture this is found from the curvature and grad B drifts
- We realize therefore that first order flows must be divergence free, that is parallel flows must develop in order to ensure that the divergence of the total first order flows be zero

# Coupling of parallel and perpendicular dynamics

- Ions and electrons piling up on the top and the bottom respectively leads to charge separation and the development of an electric field along the field lines

- A current is driven which prevents further charge separation

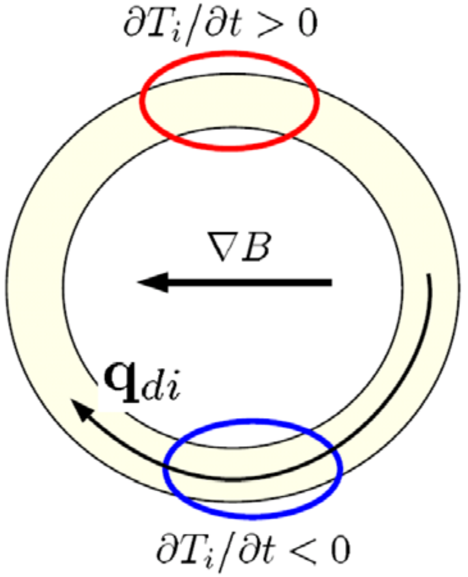
- Parallel electron and ion flows build up to ensure the cancel the up-down asymmetry, and ensure that the total first order flow is divergence free (these are often called return flows)

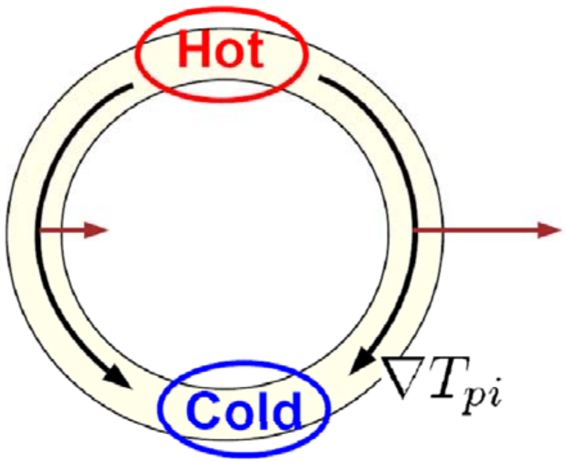



- The residual small charge separation leads to a perpendicular ExB drift in the radial direction (2nd order neoclassical radial flux)

# Coupling of parallel and perpendicular dynamics, heat flows

- So far we have considered ion and electron densities only
- However a similar derivation can be considered also for the perpendicular diamagnetic heat flows and the energy balance equation
- These would cause to up-down asymmetric temperature perturbations, which are balanced by parallel heat flows
- At the 2nd order these lead to radial (neoclassical) energy fluxes



# Pfirsh-Schlüter flow

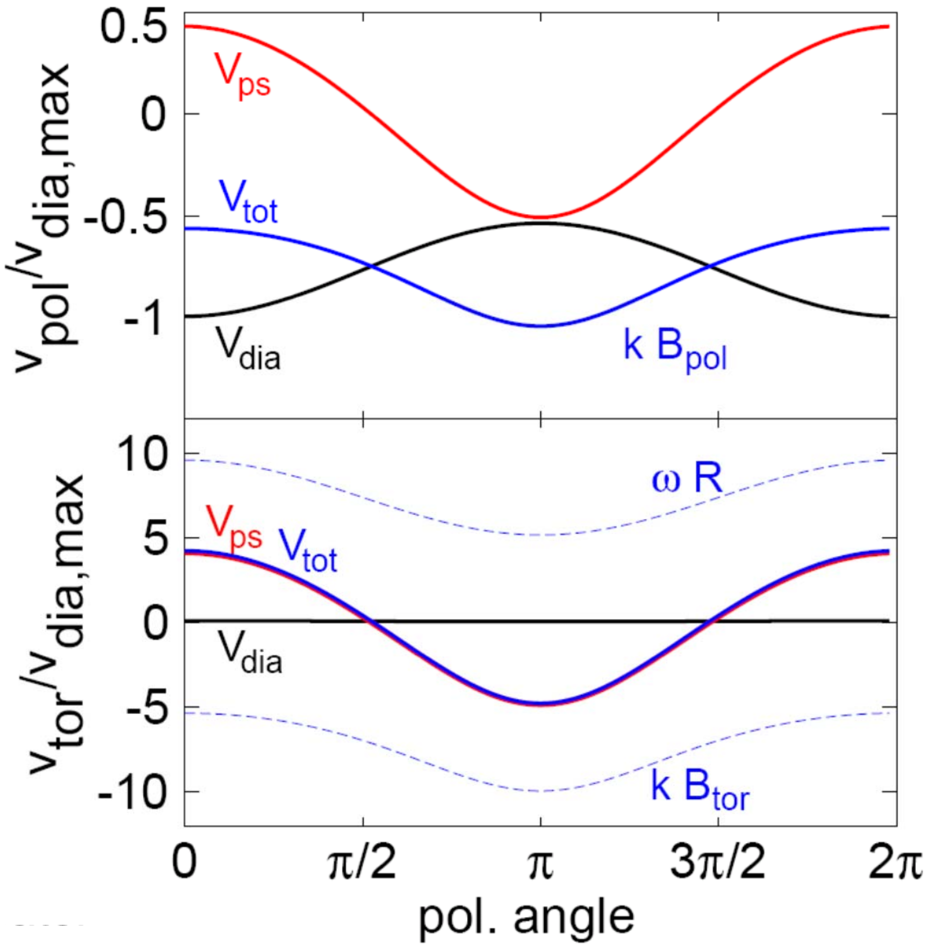
- **Pfirsch-Schlüter parallel velocity**

$$\vec{u}^{(1)}_{\|,a} = \frac{1}{e_a n_a}\frac{\partial p_a}{\partial \psi} R B_t \left[\frac{1}{B} - \frac{B}{\langle B^2 \rangle}\right] \vec{e}_{\|}$$

$$\left\langle \vec{u}^{(1)}_{\|,a} \vec{B} \right\rangle = 0$$

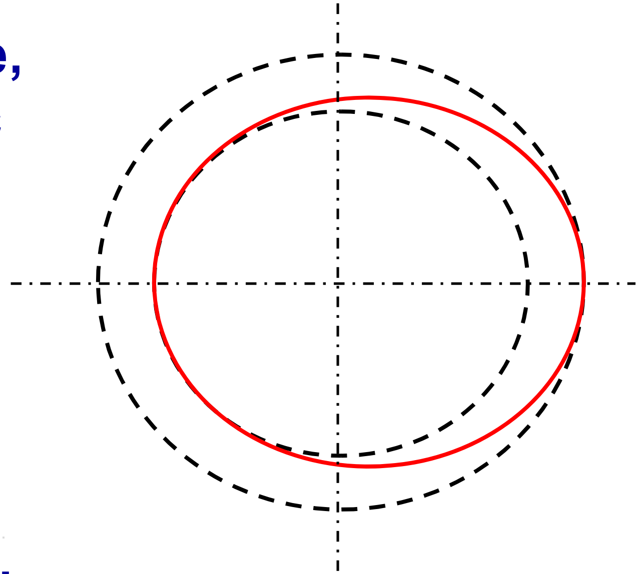
- **Total first order flow obtained by adding perpendicular (diamagnetic and ExB) and parallel (return) Pfirsch-Schüter flows**

$$\vec{u}^{(1)}_a = \omega R \vec{e}_t + K\vec{B}$$




- **Not fully determined,at this stage, since a divergence free velocity $\hat{u}\,\vec{B}/\langle B^2 \rangle$ can be added (this is connected with the banana-plateau transport)**

# Collisional (Pfirsch-Schlüter) regime



- Collision frequency much larger the inverse orbit time, guiding center (g.c.) motion predominantly stochastic
- Parallel transport characterized by random walk process in parallel direction
- Radial drift (ion, V|| > 0) inward in upper half, outward in lower half (positive charge)
- If $\tau$ is time to diffuse along field line half way around, then g.c. reverses radial drift every interval $\tau$ : $D \simeq v_D^2 \tau$
- Parallel random walk is characterized by step size $v_{\mathrm{th}}/\nu_c$
- Then half way around the axis takes $q^2 R^2 = (v_{\mathrm{th}}^2/\nu_c)\tau$
- With $v_D \simeq \rho v_{\mathrm{th}}/R$ we obtain $D_{\mathrm{PS}} = q^2 \rho^2 \nu_c = q^2 D_c$

# Collisionless regimes: particle trapped in local B minima (non axisymmetry, stellarator)

- Same argument applied to particles trapped in local minima of the magnetic fiield ( local variation of the magnetic field is $b$ )
- Random walk : step size is $v_D/\nu_A$
- Effective collision for particle detrapping is $\nu_A = (B/b)\nu_c$
- Considering that only a fraction $(b/B)^{1/2}$ of particles is involved, we obtain

$$D_A = (b/B)^{1/2}\nu_A(v_D/\nu_A)^2 = (b/B)^{3/2}v_D^2/\nu_c$$

- This is the so-called 1/ν regime typical of stellarators
- Magnetic field geometry of stellarators has to be optimized to reduce as much as possible this neoclassical transport component

More in Stellarator Lectures on Thursday

# Intermediate collisionality (plateau) regime

- Undisturbed banana motion removed by collisions, BUT collisionless motion of particles with not too small $v_{\|}$ still present

- Particles with parallel velocity $v_{\|} < v_0$: can remain on this class of orbits with low parallel velocity if the collision frequency is not higher than

$$\nu(v_0) = (v_{\text{th}}/v_0)^2 \nu_c$$

- However, if their parallel velocity is too low, $\nu(v_0) > v_{\|}/qR$ they cannot close one poloidal turn, and their radial drift is not compensated

- Thereby, particles with $v_{\|} < v_0$: , where $v_0/qR = \nu_c (v_{\text{th}}/v_0)^2$ , are fast enough that their motion is (almost) undisturbed, but too slow to close an entire poloidal turn

- Their uncompensated radial step size is $v_D/\nu(v_0)$

$$D_p = (v_0/v_{\text{th}})\nu(v_0)[v_D/\nu(v_0)]^2 = v_D^2/\omega_T$$