INTRODUCTION

TO

STELLARATOR THEORY

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Short history

- **Stellarator concept older than tokamak concept:**
  L. Spitzer (1951): Figure–8 stellarator

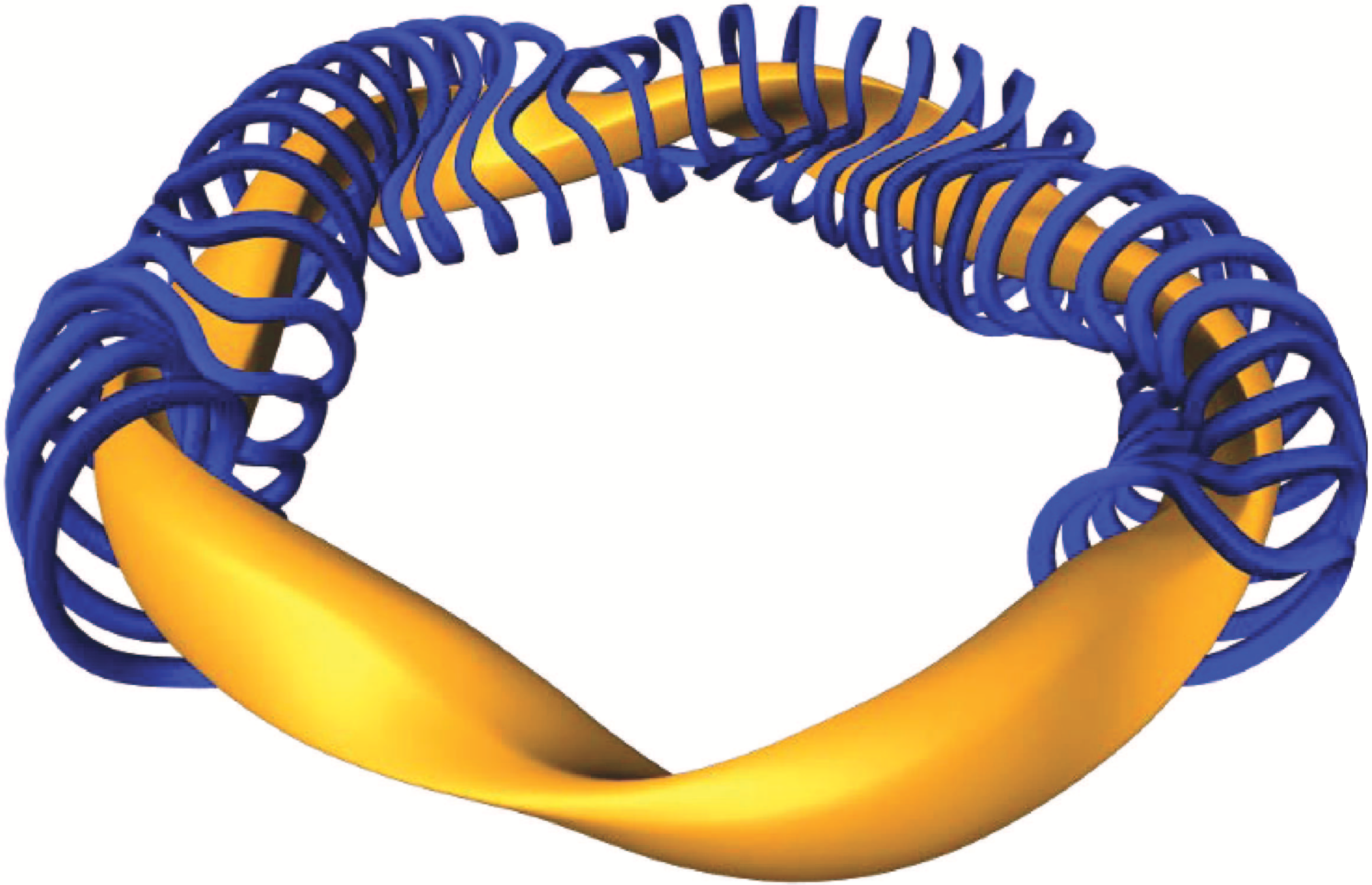
- **One of the first stellarators (Princeton C stellarator) was not successful:**
  $\Rightarrow$ Did not confine plasma very well
  Also theoretical objections against stellarators where raised

- **Same time (1968): High electron temperature (T_e =1keV) achieved in the Russian T3 tokamak**

$$\Downarrow$$

Tokamaks became main line of fusion research

- **Further development of stellarator line mainly at IPP Garching (Germany) and NIFS (Japan)**



Overview

Confinement in stellarators requires some points to be treated in the right way

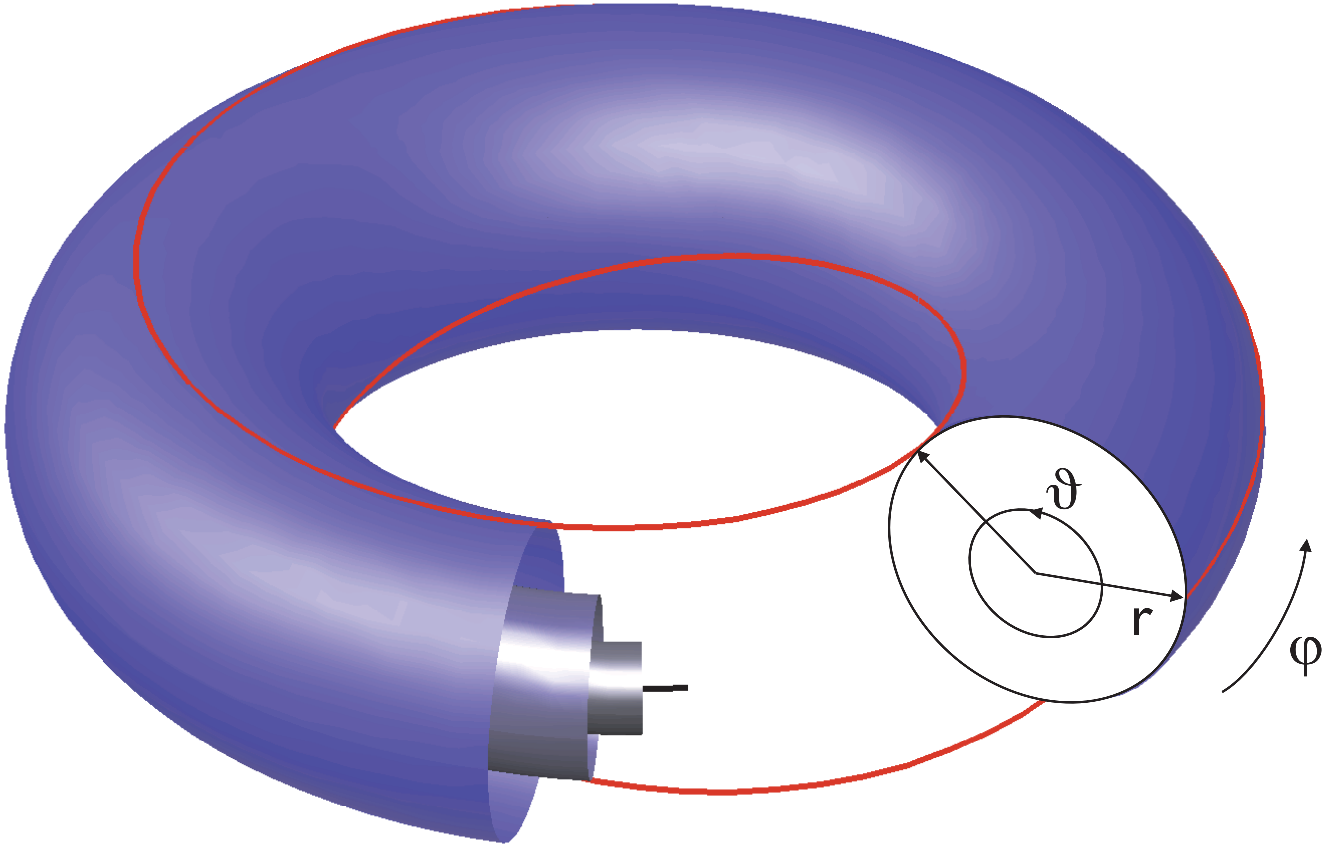
- **Equilibrium**
  Magnetic surfaces
  Pfirsch–Schlüter currents

- **Particle motion**
  Neoclassical transport

- **Optimisation of confining magnetic field**

Introduction

Toroidal geometry and rotational transform



Rotational transform, flux surface

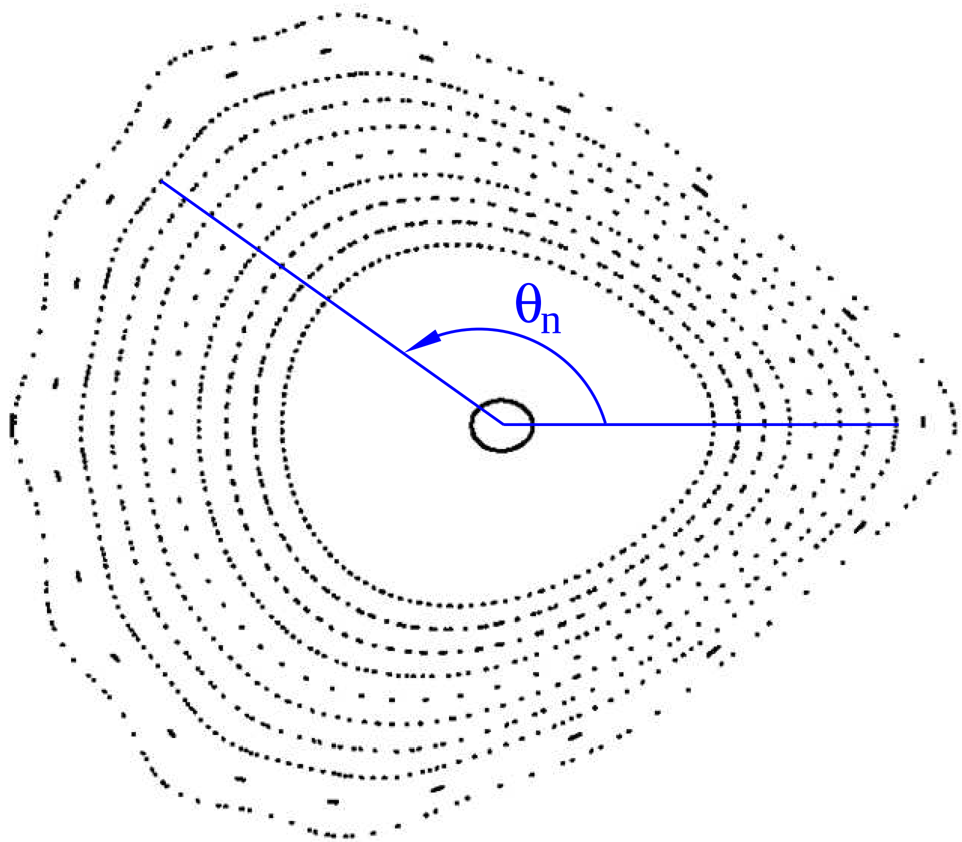
Definition of rotational transform ι:

Pick field line on flux surface, follow it for n toroidal turns

$$\iota := \lim_{n\to\infty} \frac{\theta_n}{2\pi n}$$

- **Rational ι: closed field line**
- **Irrational ι: field line covers surface ergodically**

Poincaré plot



Definition of flux surface:

$$\exists \Psi(\vec{r}) : \vec{B} \cdot \nabla \Psi = 0$$

Tokamak $\Longleftrightarrow$ Stellarator

Two important classes of fusion devices produce rotational transform

Tokamak

- ι produced by plasma current
- ι decreases outwards
- Axisymmetric

plasma current

$\Downarrow$

$*$ Two-dimensional configuration
$*$ Instabilities (disruptions)
$*$ Non steady–state operation

Stellarator

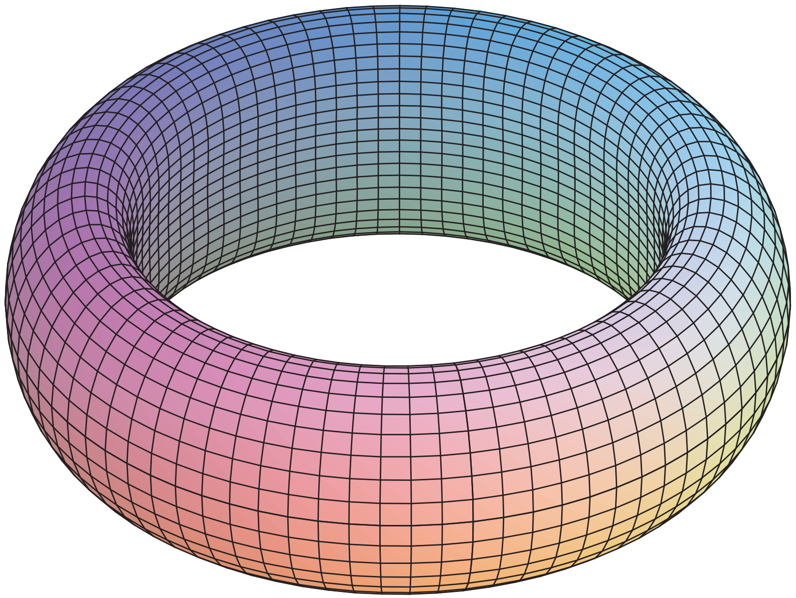
- ι produced only by coils
- ι increases outwards

no externally driven
total toroidal plasma current
(definition of a stellarator)

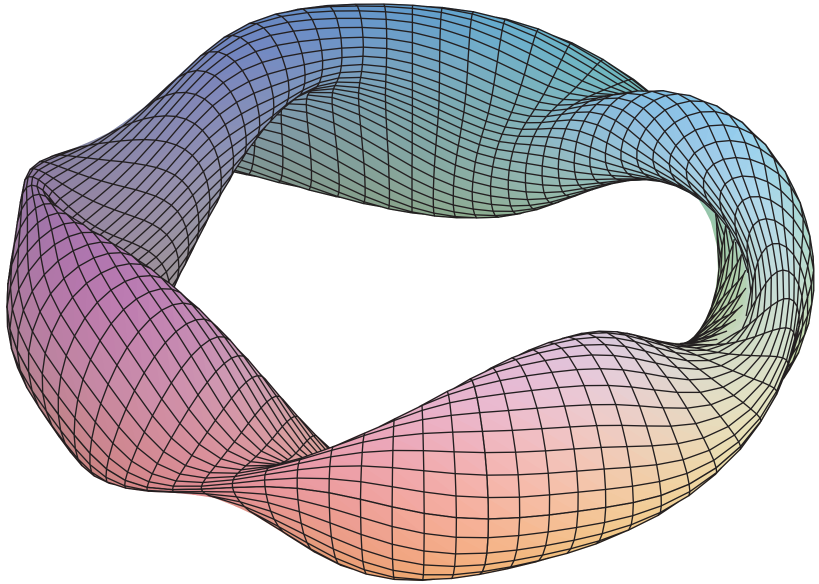
$\Downarrow$

$*$ No disruptions
$*$ Steady state operation
$*$ Toroidal configuration *must* be three-dimensional
$*$ Geometrically and computationally complex

Tokamak



Stellarator



Symmetry (mandatory):

- **Axisymmetry**

Symmetries (imposed):

- **Periodicity** $\varphi \rightarrow \varphi + \frac{2\pi}{P}$
  P: Number of field periods
- **Stellarator symmetry (flipping symmetry):** $(\vartheta, \varphi) \rightarrow (-\vartheta, -\varphi)$ **around certain lines**

Equilibrium

Equilibrium

Equations for MHD equilibrium:

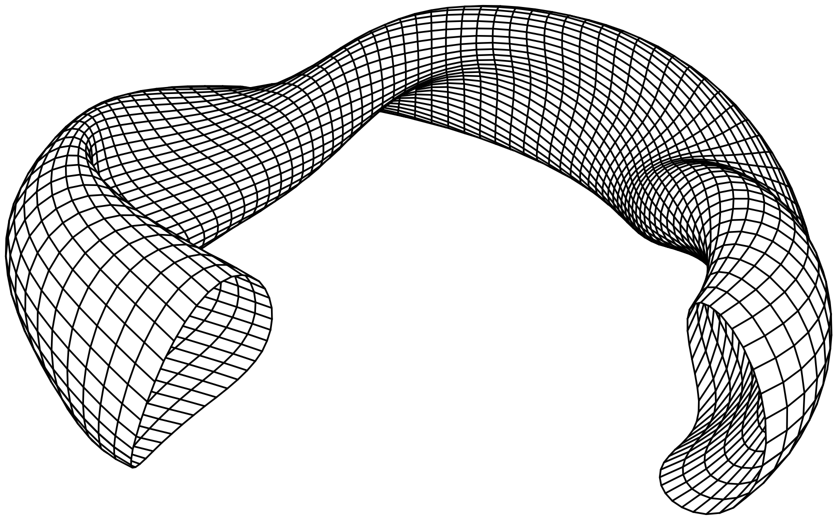
$$\vec{j} \times \vec{B} = \nabla p$$
$$\nabla \times \vec{B} = \mu_0 \vec{j}$$
$$\nabla \cdot \vec{B} = 0$$

Equilibrium completely determined by:

- **Radial profiles (e.g. pressure and total toroidal current $J = 0$)**
- **Geometry of outer flux surface $\mathcal{S}$**

  Boundary parameterisation:
  Cylindrical coordinates (R, Z, ϕ), parameters $U, V \in [0:2\pi]$

$$R = \sum_{m,n} R_{m,n} \cos(m\,U - n\,V)$$
$$Z = \sum_{m,n} Z_{m,n} \sin(m\,U - n\,V)$$
$$\phi = V$$



- **Boundary conditions: $\vec{B}$ tangential to $\mathcal{S}$**

$\Rightarrow$ **Solution of MHD equilibrium equations inside $\mathcal{S}$**

Equilibrium

- **Tokamak:**
  - Two-dimensional problem
  - Existence of flux surfaces $\Psi(\vec{r}) = const.$ guaranteed

$\Rightarrow$ **Grad–Shafranov–Equation**

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right) + \frac{\partial^2}{\partial z^2}\Psi + \mu_0 r^2 \frac{\partial p(\Psi)}{\partial\Psi} + \frac{\mu_0}{8\pi^2}\frac{\partial I^2(\Psi)}{\partial\Psi} = 0$$

- **Stellarator:**
  - Three-dimensional problem
  - Existence of flux surfaces *not* guaranteed
    Structure of solution can only be found a posteriori

$\Rightarrow$ **Difficult to solve (numerical solution necessary)**

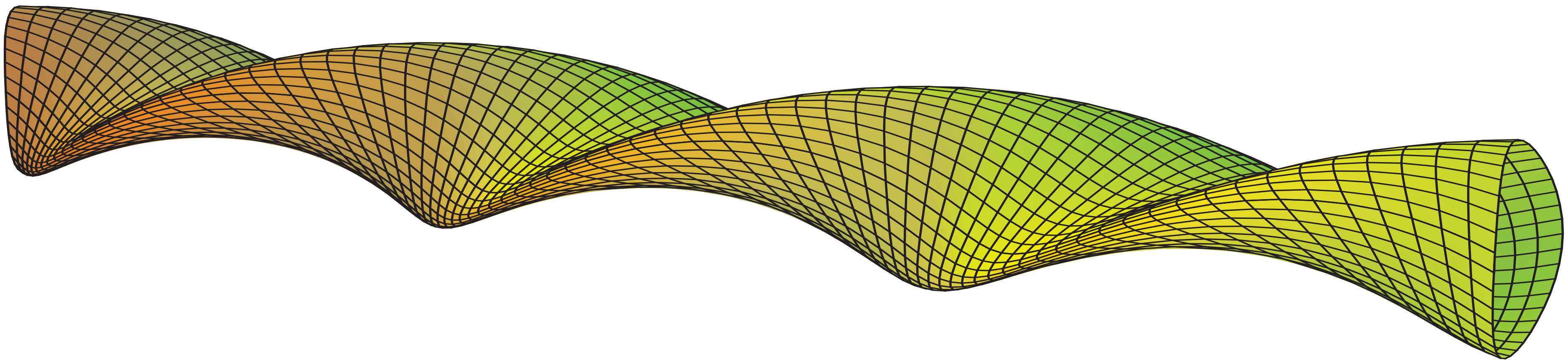
Straight stellarator

Helical symmetry (the only analytical stellarator equilibria):

assume $\vec{B} = \vec{B}(r, \vartheta - kz)$

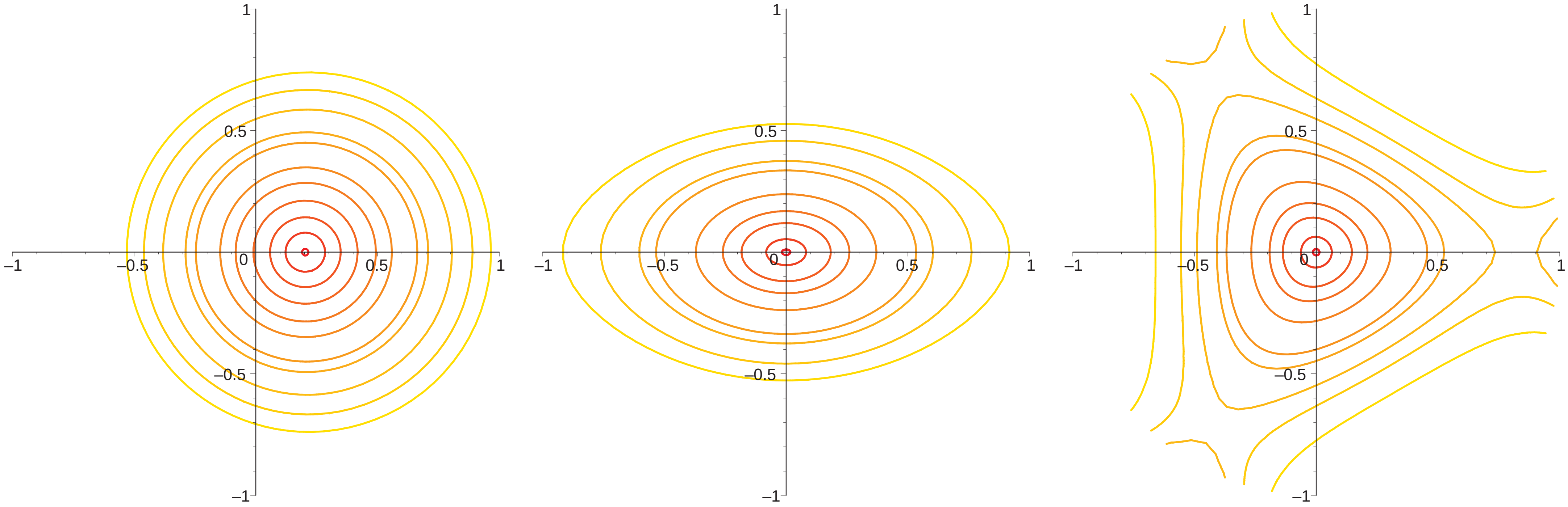
Vacuum field ($p = 0$): $\quad \Delta\Phi = 0 \quad \Rightarrow \quad \Phi = B_0 z + \frac{1}{k}\sum_{\ell=1}^{\infty} b_\ell \, I_\ell(\ell k r)\, \sin \ell(\vartheta - kz)$

Flux surfaces: $\quad \Psi = B_0 \frac{kr^2}{2} - r \sum_{\ell=1}^{\infty} b_\ell \, I'_\ell(\ell k r)\, \cos \ell(\vartheta - kz) = const.$



Straight stellarator

$\ell = 1$ $\qquad$ $\ell = 2$ $\qquad$ $\ell = 3$



Toroidal stellarator: Island formation

Nested flux surfaces necessary for confinement

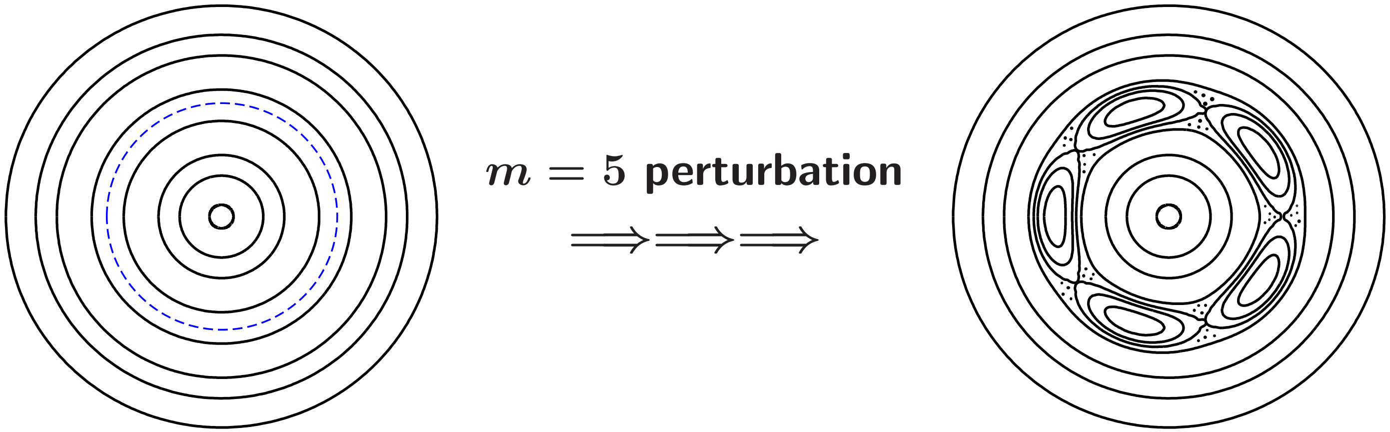
Axi-/helical symmetric: Flux surfaces always exist (Grad-Shafranov equation)

Non symmetric: Straight stellarator (with flux surfaces) + perturbation $\mathcal{P} = \frac{B_{\mathrm{mn}}}{B_0}$
(e.g. bending straight stellarator to a torus)

- Islands form where $\iota = \frac{n}{m}$ (resonances)
- Ergodic regions near x–points

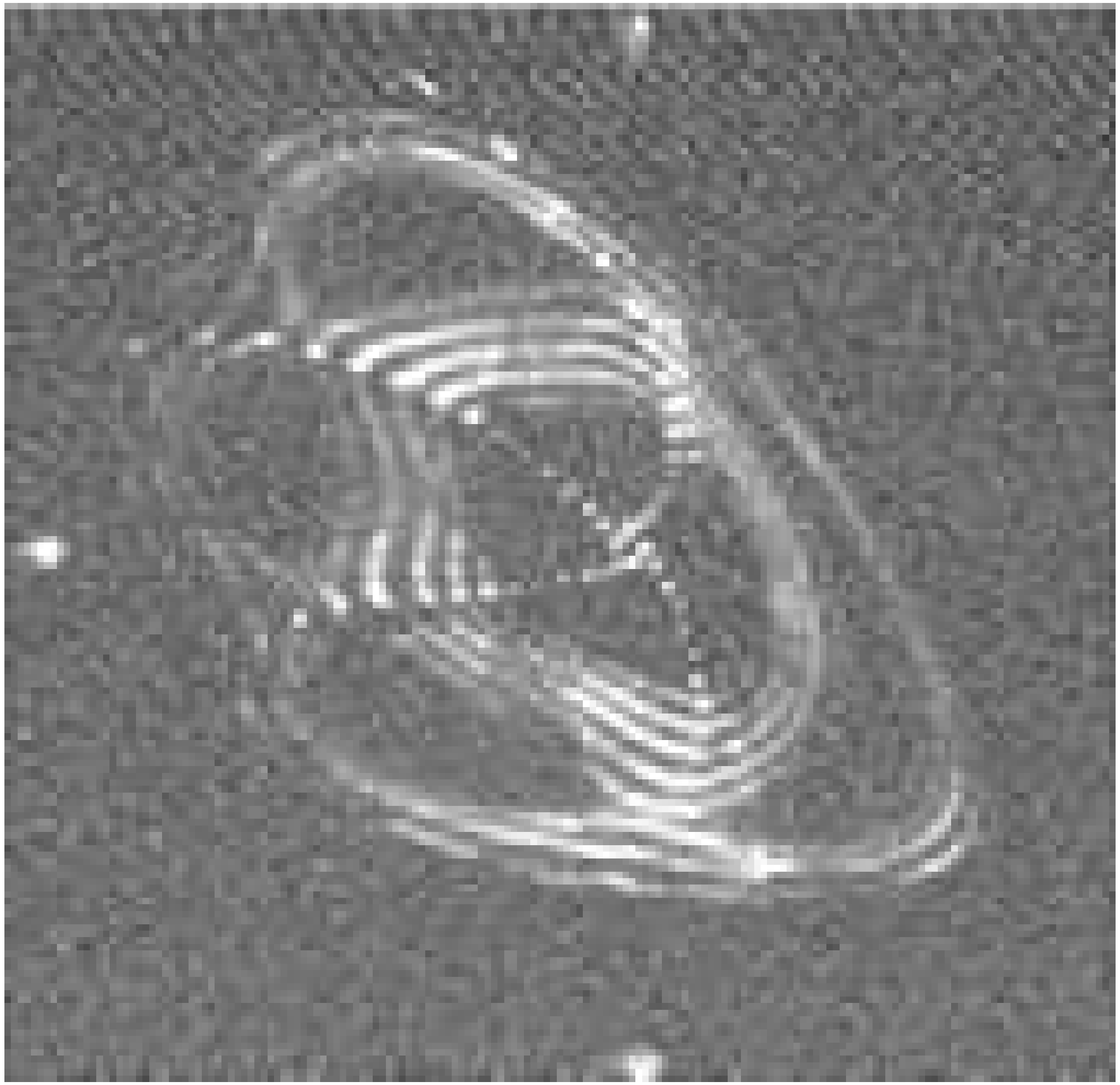
(Analogy: Integrable system + perturbation $\Rightarrow$ KAM theorem)

Island width: $w = 4\sqrt{\frac{R}{m\iota'}\mathcal{P}}$ (e.g. W7–X: $\mathcal{P} \sim 10^{-4} \Rightarrow w/a \sim 0.2$):



Islands (small perturbations) are very bad for confinement

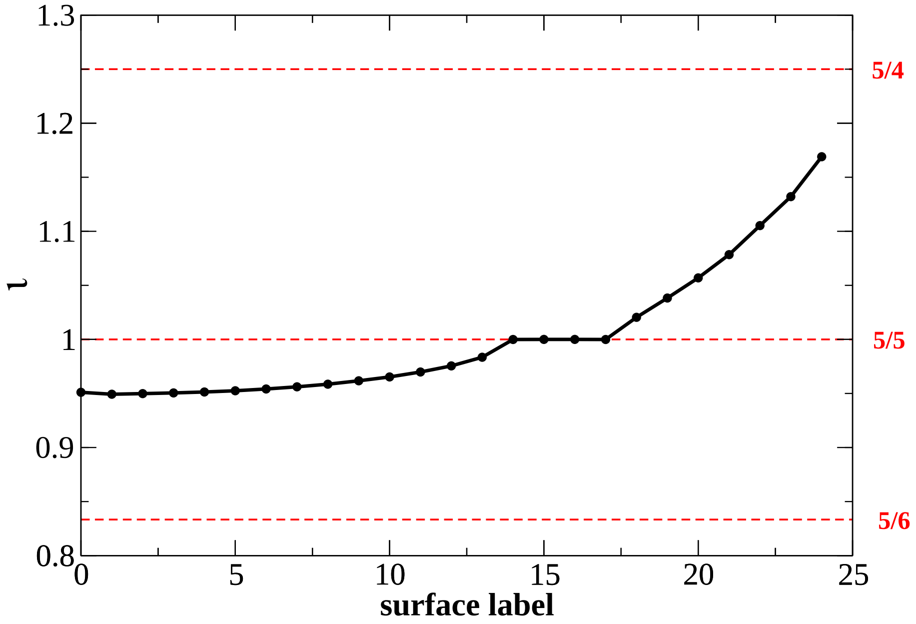
WEGA Stellarator

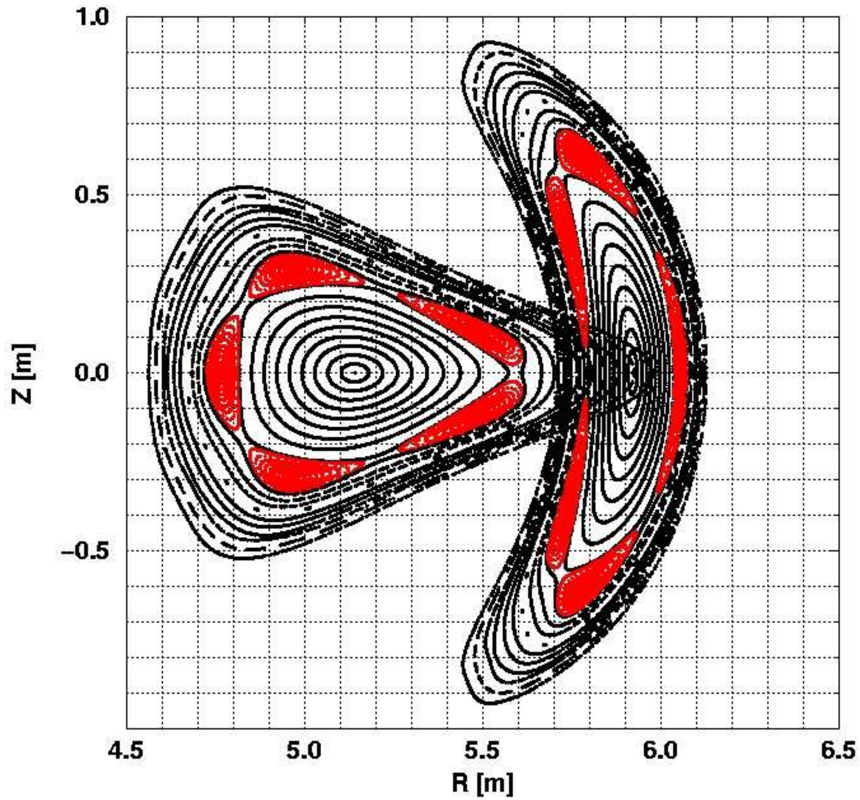


M.Otte

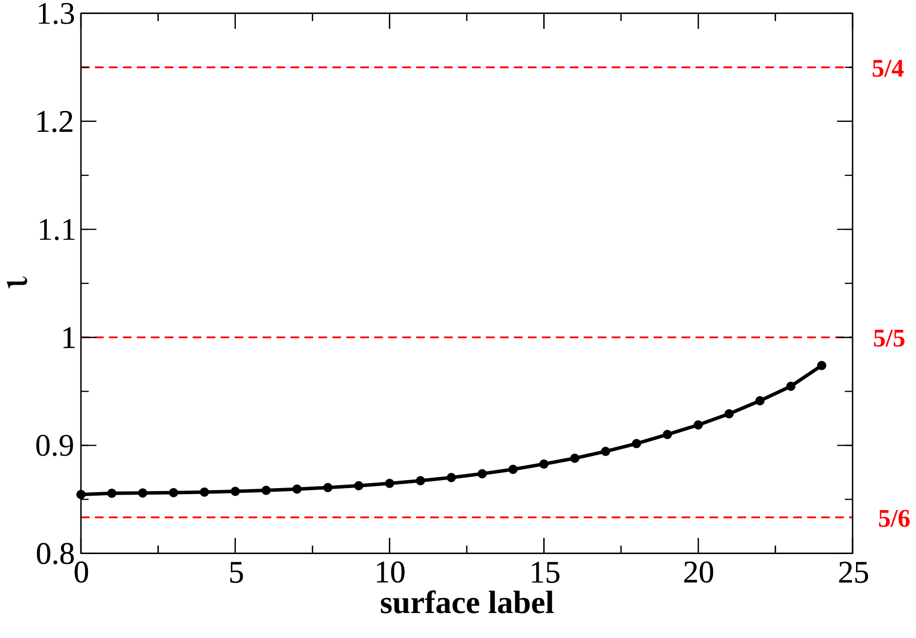
Example: Formation of islands at resonance

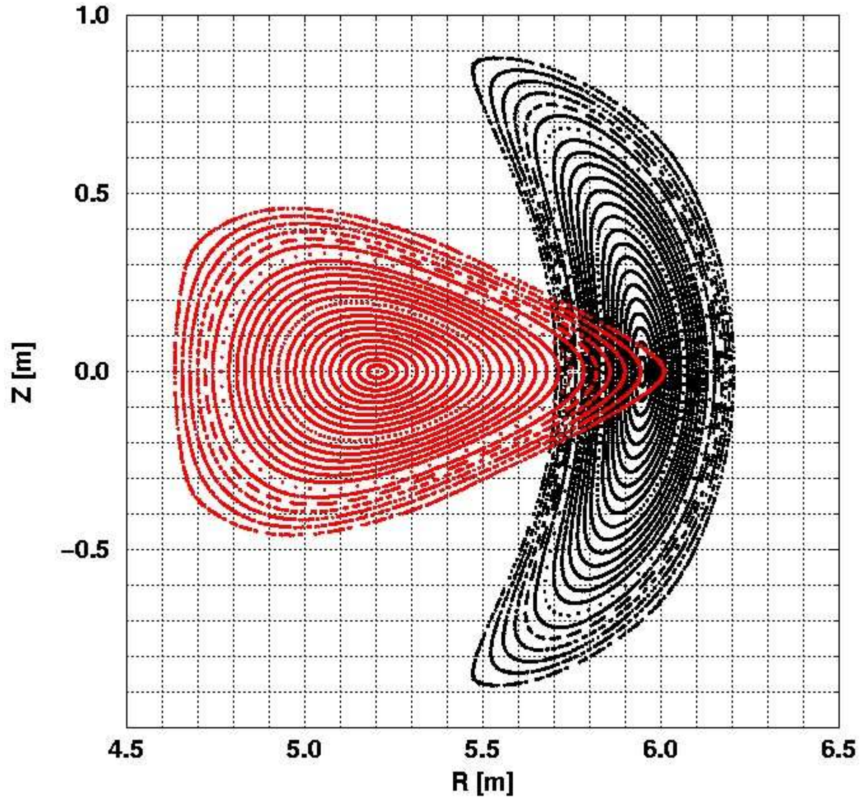
Stellarator with 5 field periods $\Rightarrow$ resonances possible at $\iota = \frac{5n}{m}$





Example: Avoid mayor resonances to avoid islands





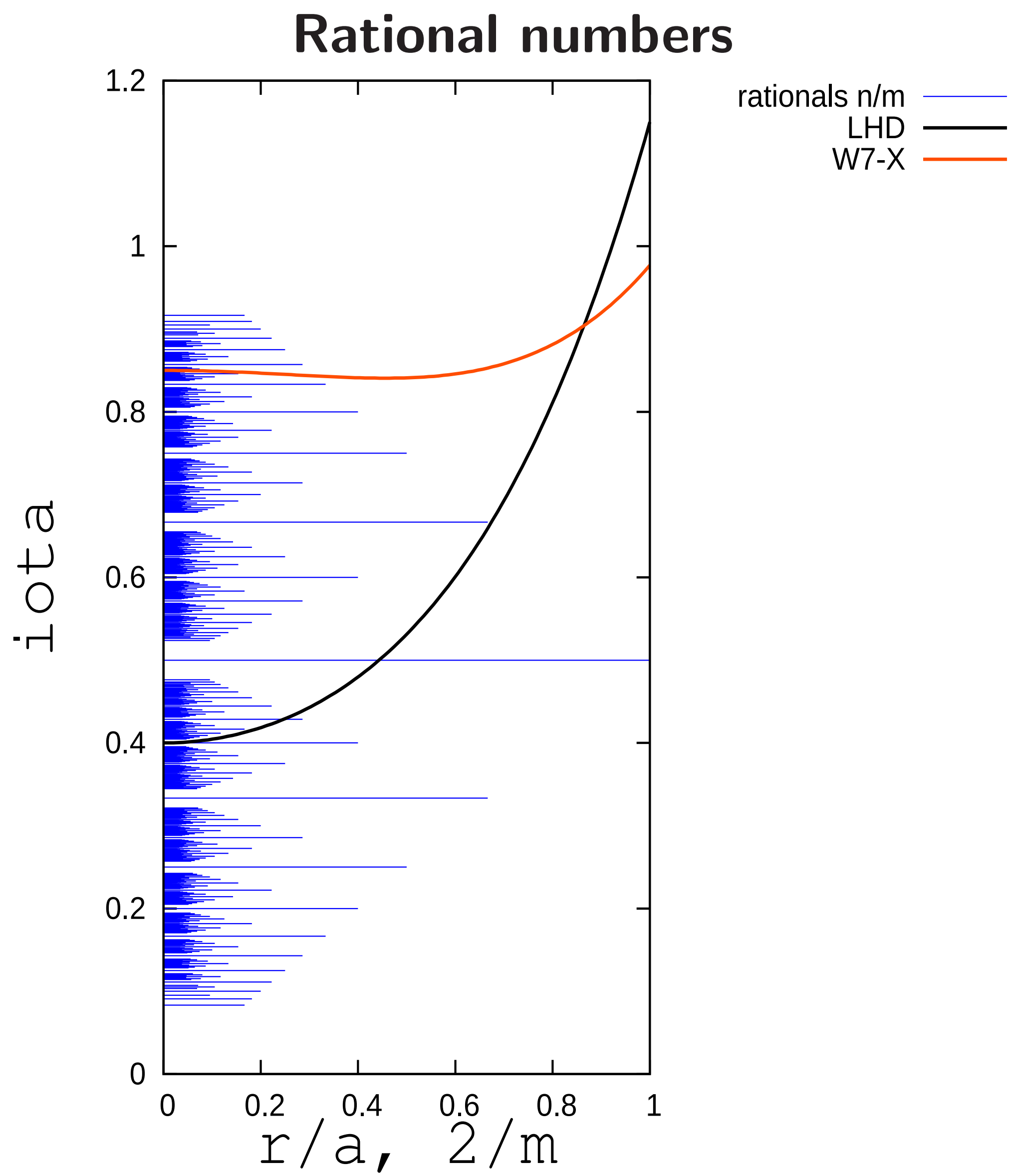
How to avoid islands

Large islands are very bad for confinement
$\Rightarrow$ Equilibrium must have nested flux surfaces

Island size: $w \sim \dfrac{1}{\sqrt{m\,\iota'}}$

Strategies to avoid islands:

- **Do not avoid resonances but use high shear to cross them fast (e.g. LHD)**
- **Avoid low order (small m) rationals and use low shear (e.g. W7-X)**



Equilibrium calculation: VMEC code

Main tool for stellarator equilibrium calculations

Energy principle (equivalent to MHD equilibrium equations)

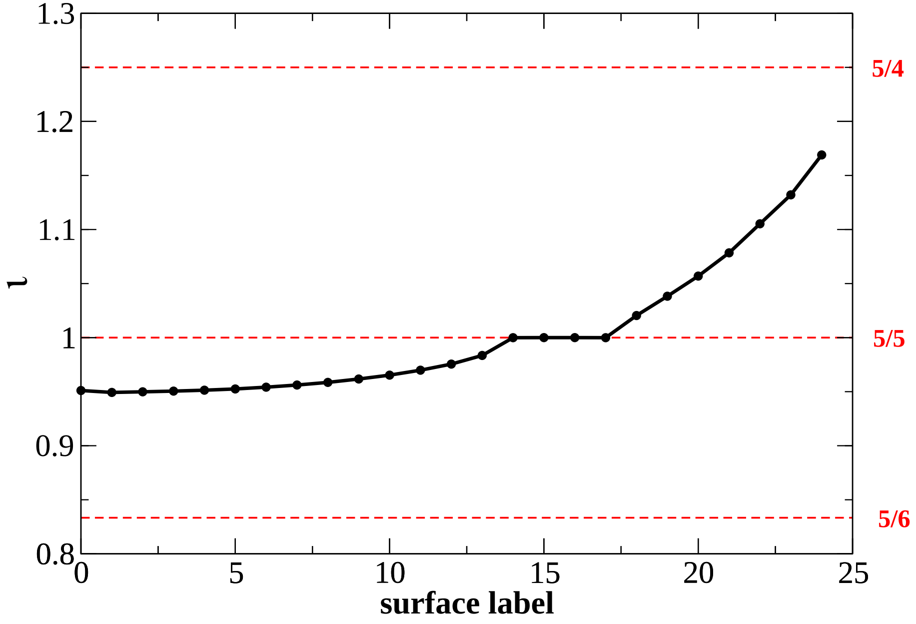
$$W = \int \left(\frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1} \right) dV$$

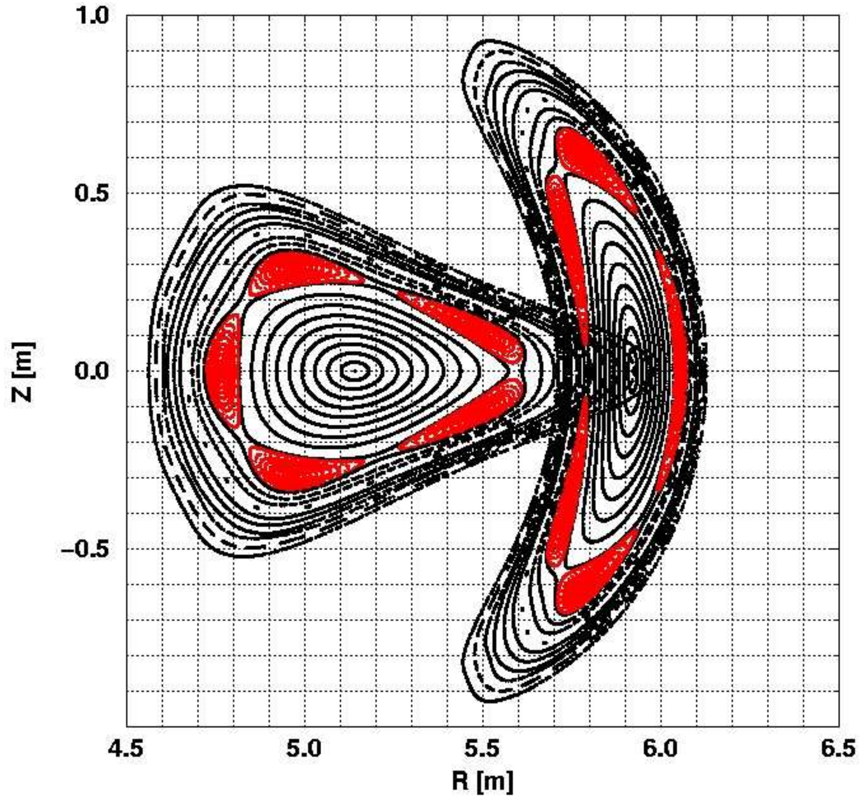
Minimisation of W under constraints:

* No total toroidal current
* Conserve mass enclosed by flux surface
* Assume existence of flux surfaces

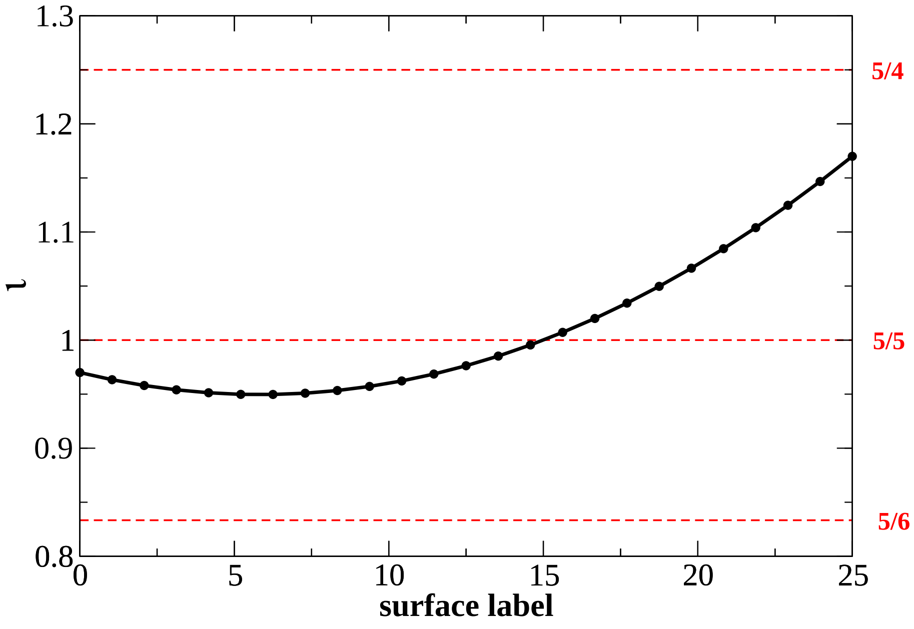
- Solution is only approximation to equilibrium
- No islands (divergent parallel currents instead of islands)
- Application of VMEC not useful for equilibria with large islands
- Fast calculation of equilibria ($O(1)$ CPUh)

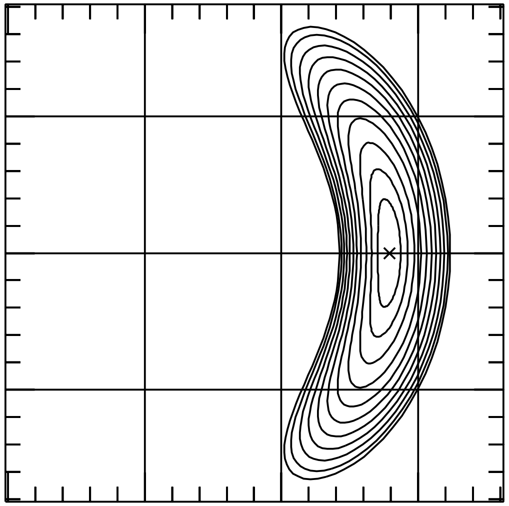
Formation of island at resonance

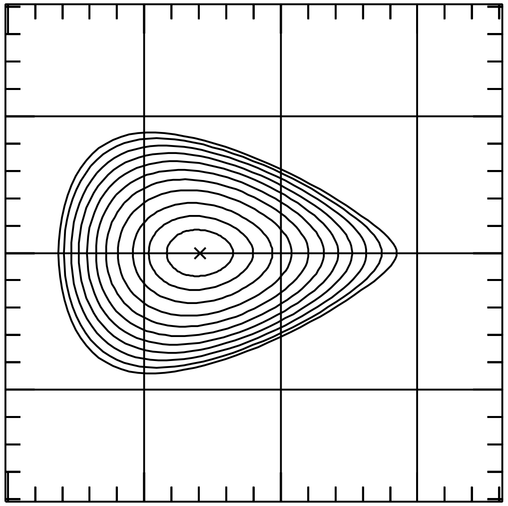




VMEC calculation







Equilibrium calculation: PIES code

Allows for islands

Solve MHD equilibrium equations with Picard iteration:

$$\begin{aligned} \nabla \times \vec{B}^{n+1} &= \mu_0 \vec{j}(\vec{B}^n) \\ \nabla \cdot \vec{B}^{n+1} &= 0 \end{aligned}$$

$\vec{j}(\vec{B}^n)$ **found from**

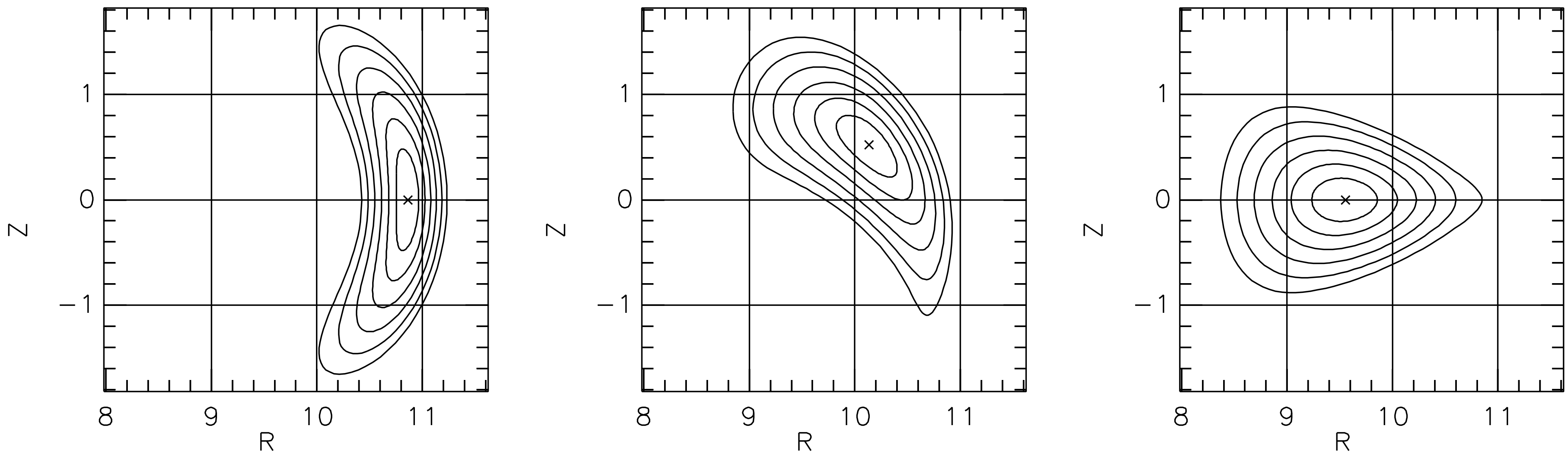
$$\begin{aligned} \vec{j} \times \vec{B}^n &= \nabla p \\ \nabla \cdot \vec{j} &= 0 \end{aligned}$$

Extremely slow convergence $\Rightarrow$ Long computing time ($O(100)$ CPUh)

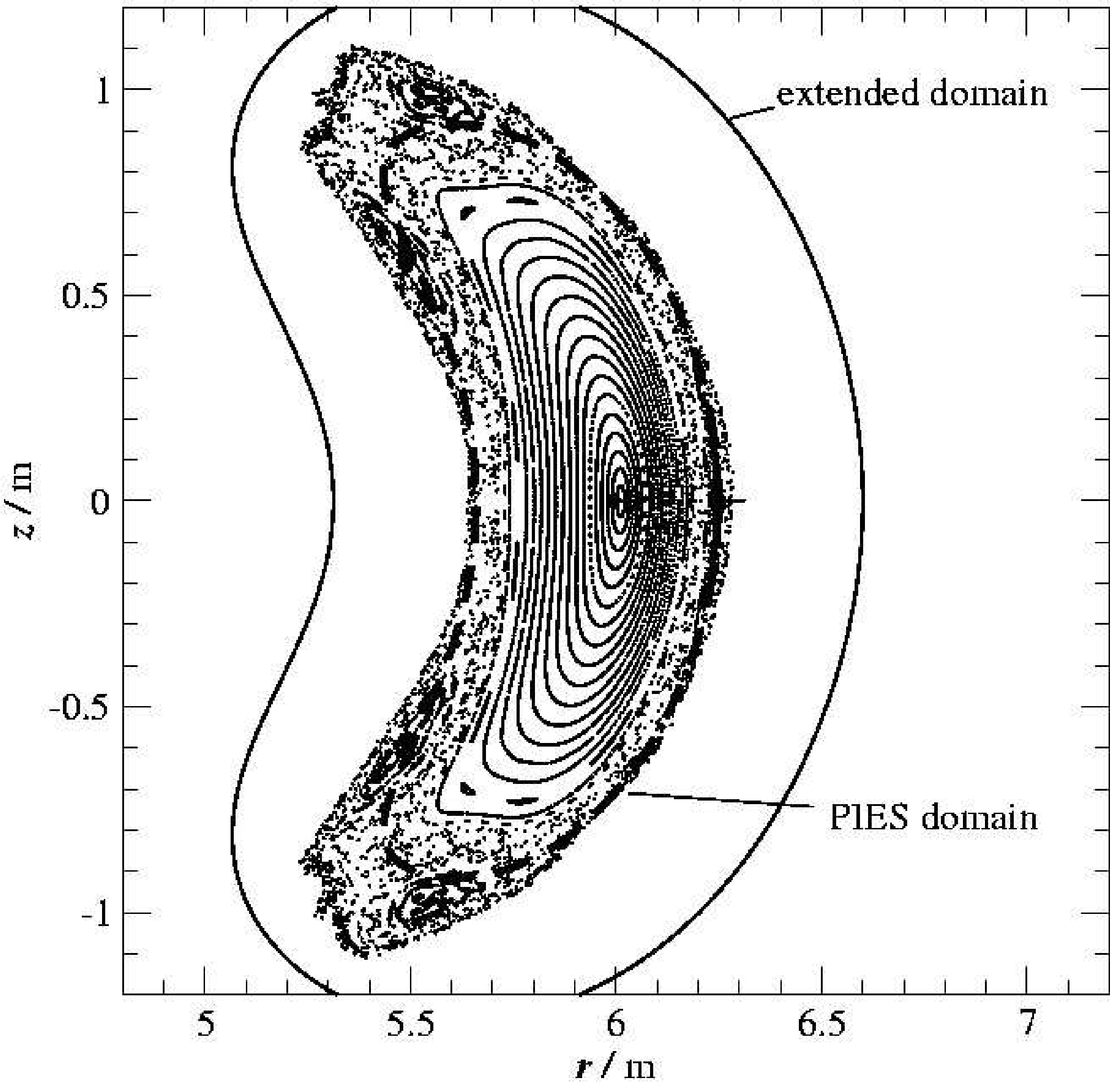
Justification of VMEC solution a posteriori

Flux surfaces for W7–X

VMEC calculation

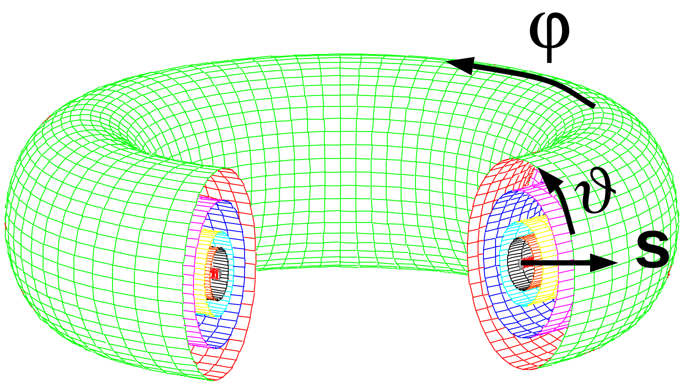


W7–X, $\beta = 4\%$ (PIES calculation)



Magnetic coordinates (straight field line coordinates)

- Because of B field, plasma is highly anisotropic ($L_\perp \sim \mathrm{cm}, L_\| \sim \mathrm{km}$)
- Simple field line description is advantageous
- Introduction of an appropriate curvilinear coordinate system (s, ϑ, φ):
  -Flux surface label (tor. flux): $s \in [0:1]$
  -Angle like coordinates: $\vartheta, \varphi \in [0:2\pi]$



- $\vec{B} = \frac{1}{\sqrt{g}}\left(F_\mathrm{P}'(s)\,\vec{r}_\vartheta + F_\mathrm{T}'(s)\,\vec{r}_\varphi\right)$
  $\vec{B} = I(s)\nabla\vartheta + J(s)\nabla\varphi + \tilde{\beta}(s,\vartheta,\varphi)\nabla s$

  $\Rightarrow$ Field lines are straight
  $\vartheta = \iota\,\varphi \qquad \iota = F_\mathrm{P}'/F_\mathrm{T}'$
- Geometric information contained in the metric tensor $g_{ss}, g_{s\theta}, \ldots$
- Magnetic coordinates are not unique: Boozer, Hamada, ... coordinates

Indispensable formalism to treat stellarators
(same formalism as in general theory of relativity)

Pfirsch-Schlüter current

Shafranov shift

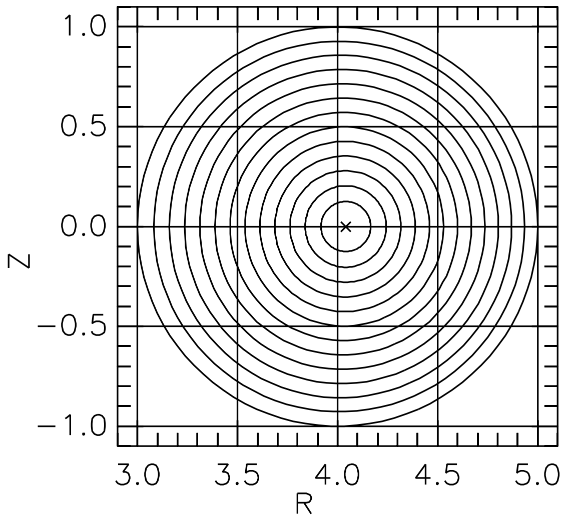
Fusion reactor needs high β value $\beta := \frac{2\mu_0 p}{B^2}$

Measure of cost effectiveness (B field is expensive)

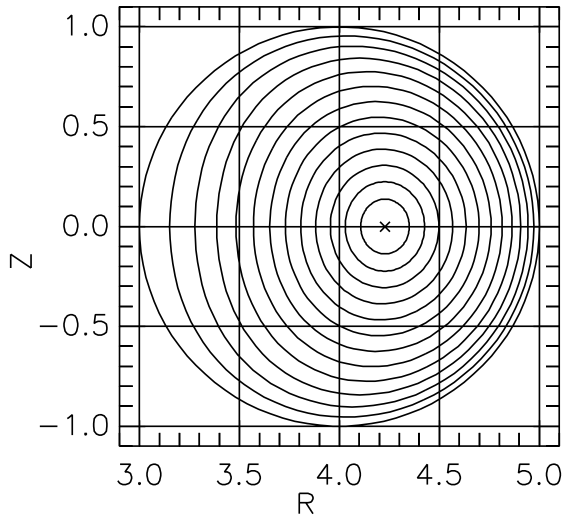
Typical value needed (for a reactor): $\beta \approx 5\%$

- Plasma pressure leads to Shafranov shift Δ

$\beta = 0$



$\beta = 10\%$



- Maximal shift given by $\Delta \approx a \Rightarrow$ equilibrium β limit: β_{eq}

Cause of the Shafranav shift: Pfirsch–Schlüter current

Shafranov shift caused by current parallel to B (Pfirsch–Schlüter current)

- **Equilibrium:** $\vec{j} \times \vec{B} = \nabla p \quad \Rightarrow \quad \vec{j}_\perp = \frac{\vec{B}\times\nabla p}{B^2}$

- **Quasineutrality:** $\nabla \cdot \vec{j} = 0 \quad \Rightarrow \quad \nabla \cdot \vec{j}_\| = -\nabla \cdot \vec{j}_\perp$

$$\Rightarrow \qquad \nabla \cdot \vec{j}_\| = -\frac{2\,p'}{B^3}\underbrace{\left(\nabla r \times \vec{B}\right)\cdot\nabla|\vec{B}|}_{\text{acts only in a flux surface}} = -\frac{2\,p'|\nabla r|}{B}\,\kappa_{\mathrm{g}}$$

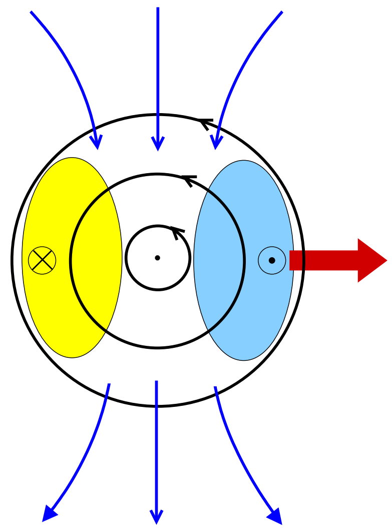
- **Simple toroidal configuration** $(B = \frac{B_0}{1+\frac{r}{R}\cos\vartheta})$:

$$j_\| = -\frac{2p'}{\iota B_0}\cos\vartheta$$

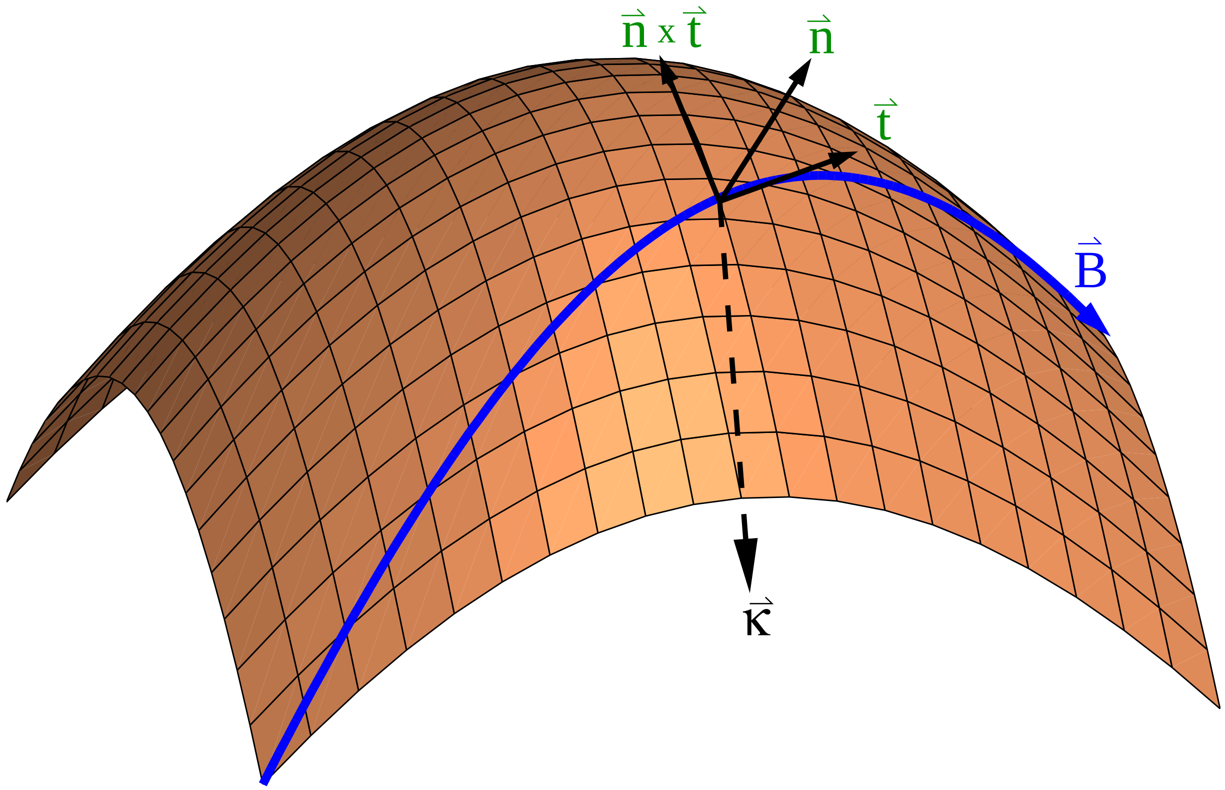
$j_\|$ **produces vertical field** $B_{z,0}$

$\Rightarrow$ **Plasma shifted outwards**

$\Delta \approx \frac{R}{\iota B_0} B_{z,0}$



Curvature



Curvature ($\vec{b} = \frac{1}{B}\vec{B}$):

$$\vec{\kappa} := \left(\vec{b}\cdot\nabla\right)\vec{b}$$

$$\vec{\kappa} = \kappa_{\mathrm{n}}\,\vec{n} + \kappa_{\mathrm{g}}\,\vec{n} \times \vec{t}$$

κ_{n}: **normal curvature**

κ_{g}: **geodesic curvature (intrinsic geometry)**

Equilibrium β–limit: $\Delta \approx a \;\Rightarrow\; \beta_{\rm eq} \approx \frac{\iota^2}{A}$ $\qquad$ A: aspect ratio

- **Tokamak:** $A \approx 3,\ \iota \lesssim 1 \;\Rightarrow$ reasonable $\beta_{\rm eq}$
- **Stellarator:** $A \approx 10,\ \iota < 1 \;\Rightarrow$ $\beta_{\rm eq}$ too low

$\Rightarrow$ Increase $\beta_{\rm eq}$ by reducing $j_{\|} \Rightarrow |\vec{B}(\vartheta, \varphi)|$ essential
Use 3D freedom in stellarator to design $|\vec{B}|$ appropriately

Example: Helical symmetry of $|\vec{B}|$ (not of $\vec{B}$)

$B = B(r, h)$ with $h = \vartheta - M\varphi$ and $M \gg \iota$

using $\nabla_{\|}\, j_{\|} = -\dfrac{2\,p'|\nabla r|}{B}\kappa_{\rm g}$ and simple model $B = B_0(1 - \dfrac{r}{R}\cos h)$

$$\Rightarrow j_{\|} = -\frac{2\,p'}{B_0}\,\frac{1}{\iota - M}\,\cos h$$

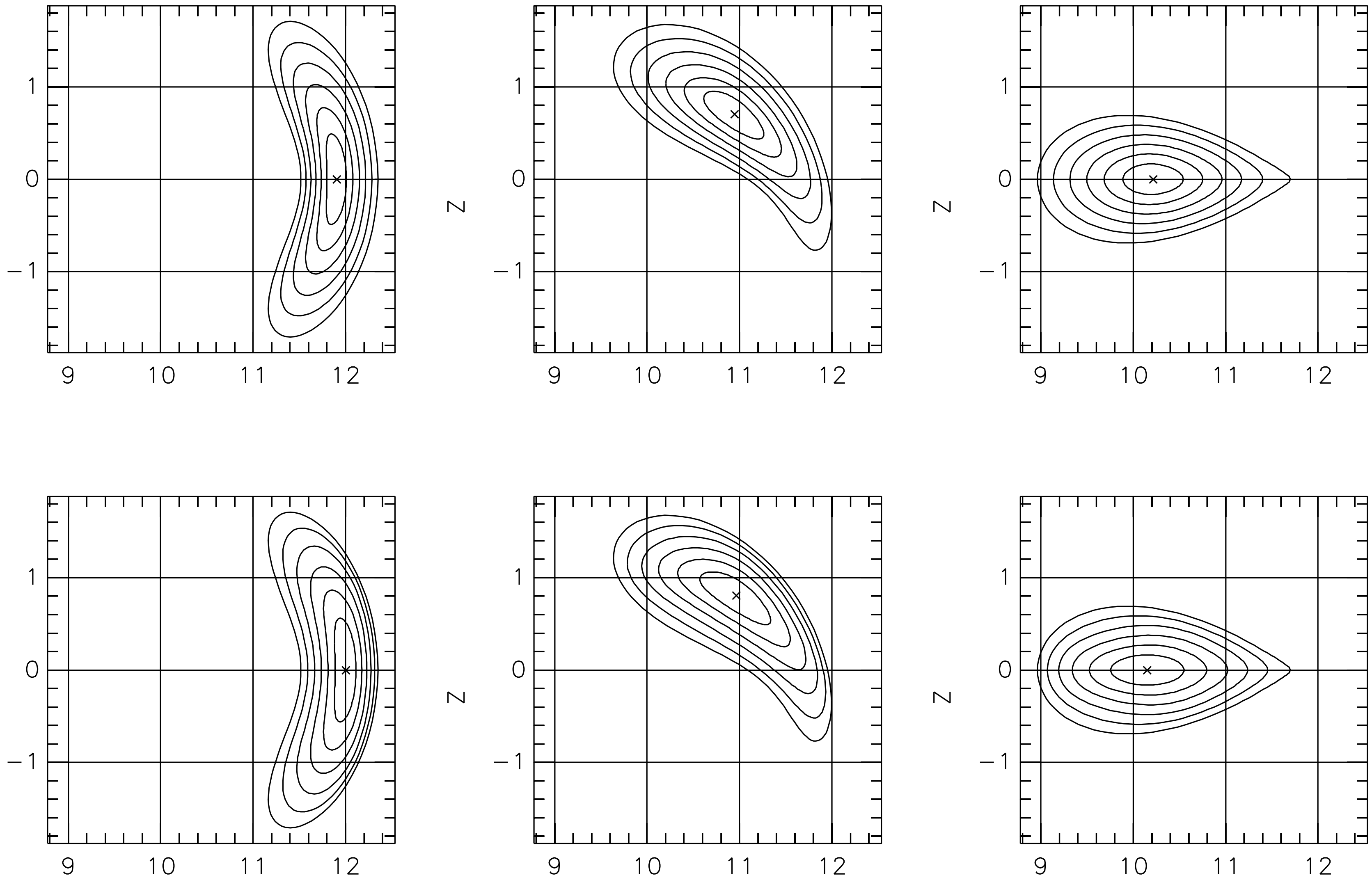
$\Rightarrow$ Shafranov shift not a problem in helically symmetric systems

$\Rightarrow$ reduce toroidal part and increase helical part of B

Quasi–helically symmetric stellarator

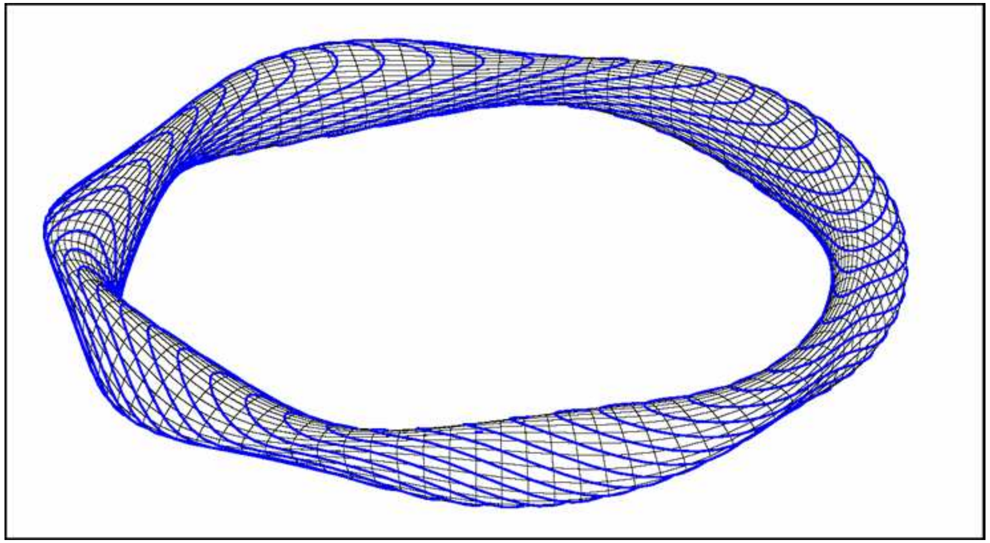
$\beta = 0$

$\beta = 40\%$

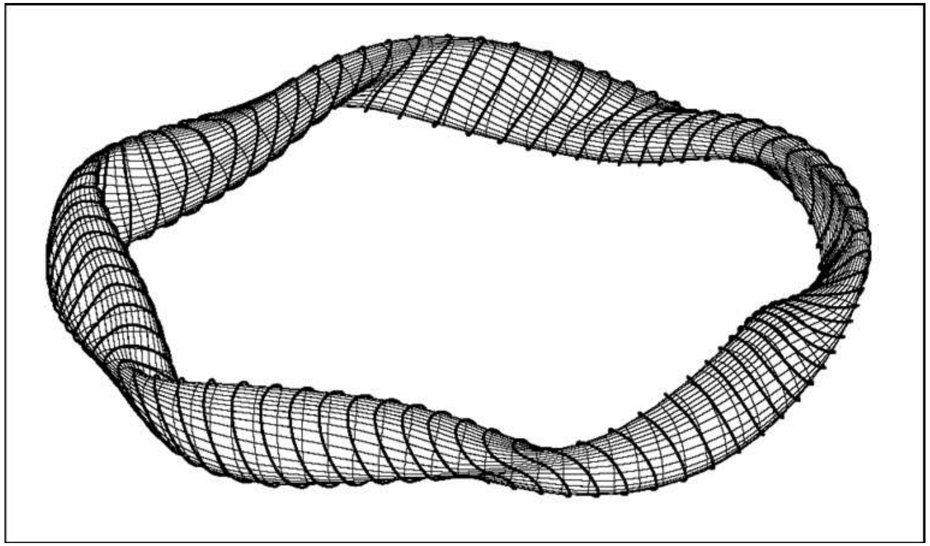


Plasma currents

$\ell = 2$ stellarator (W7-A)



Helias configuration (W7-X)



	W7-A (non opt.)	W7-AS (partly opt.)	W7-X (fully opt.)
$\left\langle \frac{j_\parallel^2}{j_\perp^2} \right\rangle$	12	3	0.5

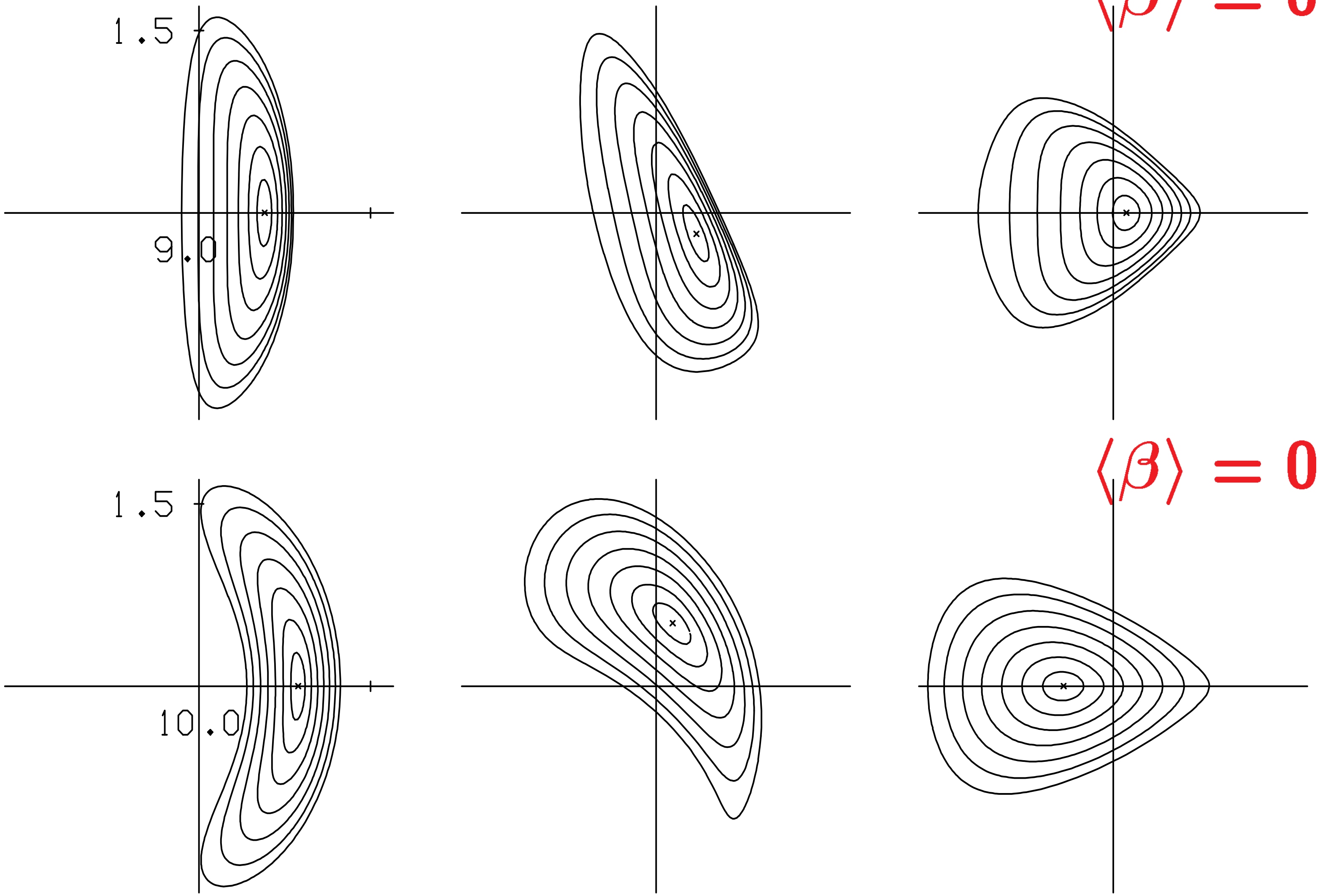
Shafranov shift for W7-AS and W7-X

$\langle\beta\rangle = 0.02$

$\langle\beta\rangle = 0.05$

W7–AS

W7–X



Neoclassical Transport

Neoclassical transport:

Radial diffusion of particles/heat by collisions in toroidal geometry
$\Rightarrow$ Loss of plasma

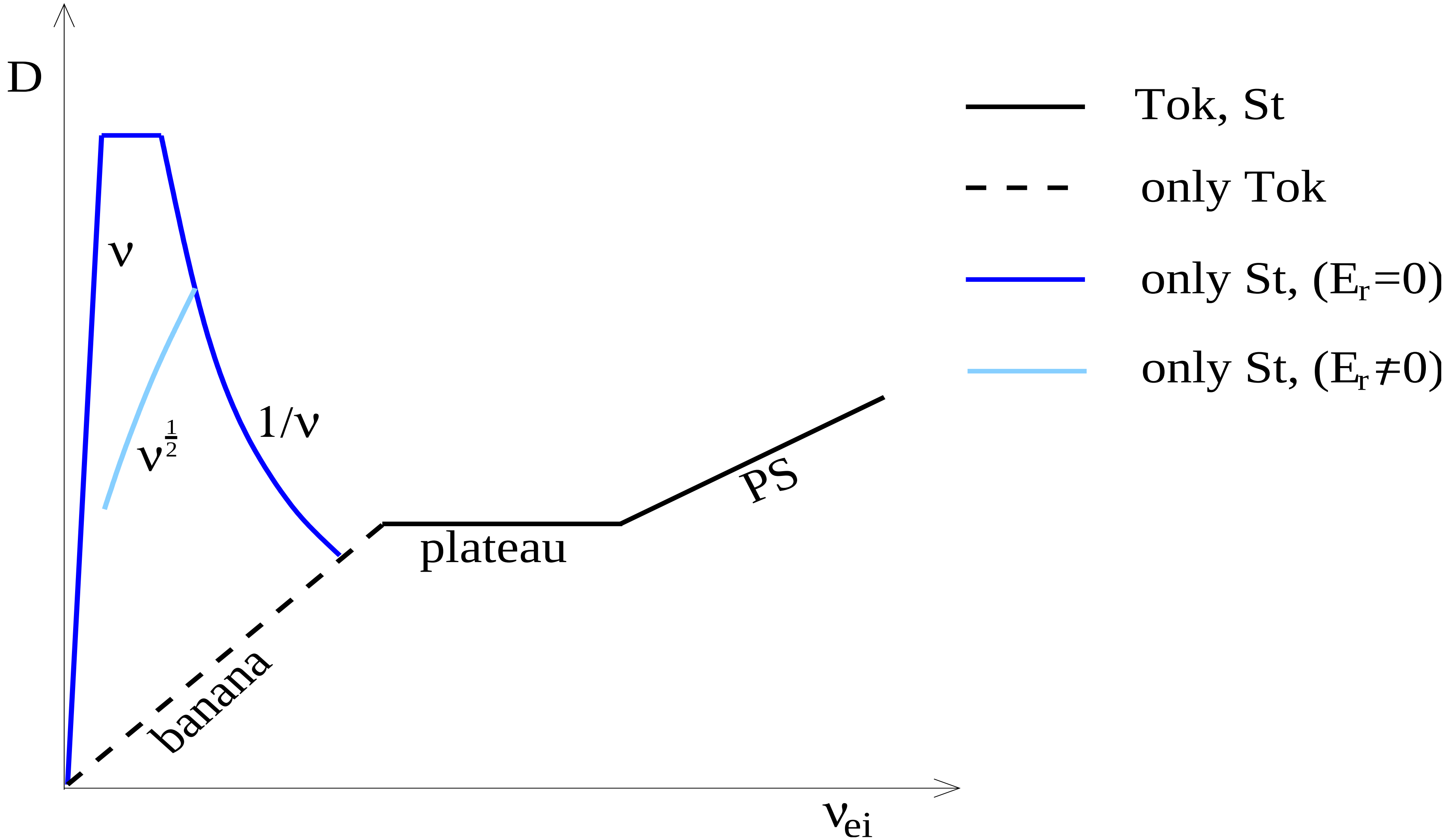
- High collision frequency: Fluid treatment possible
- Low collision frequency: Kinetic theory necessary
  Detailed particle motion important
  (orbit types, trapping of particles)

Neoclassical losses:

- Small for a tokamak (smaller than anomalous transport)
  (due to symmetry and associated constant of motion)
- Can be extremely large for a classical (non–optimized) stellarator
  (larger than anomalous transport)
  $\Rightarrow$ Find ways to minimize losses

Overview: Neoclassical regimes

Particle diffusion coefficient versus collision frequency



⇒ **Different physics and approximations in each regime**

Diffusion (general)

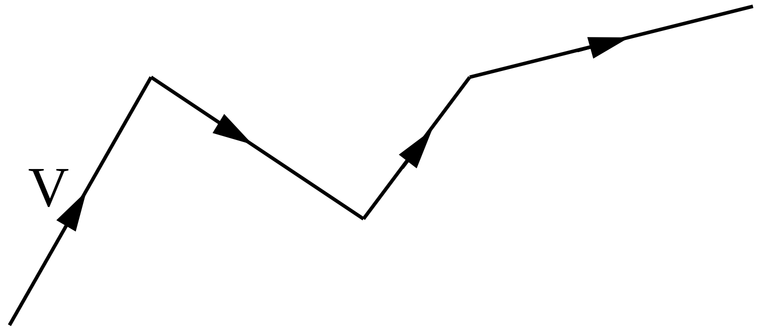
- **Radial particle flux:** $\Gamma = D\dfrac{\partial n}{\partial r}$ $\qquad [D] = \dfrac{m^2}{s}$

- **Estimate diffusion coefficient as** $D \sim \dfrac{\lambda^2}{\tau}$

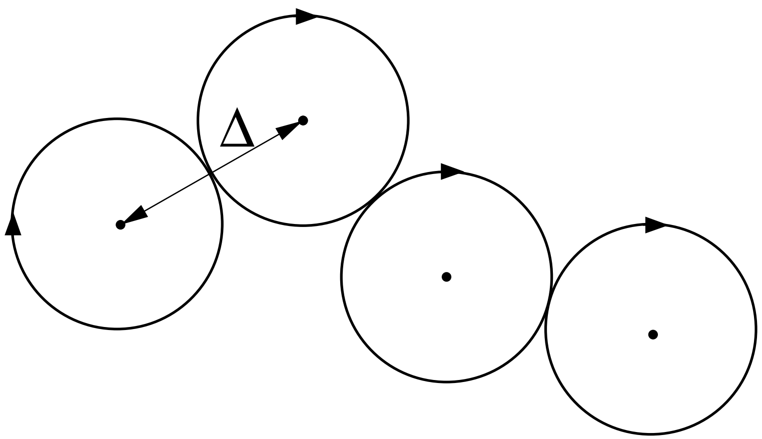
  λ: **displacement during collisions**

  collision frequency $\nu = 1/\tau \sim nT^{-\frac{3}{2}}$:

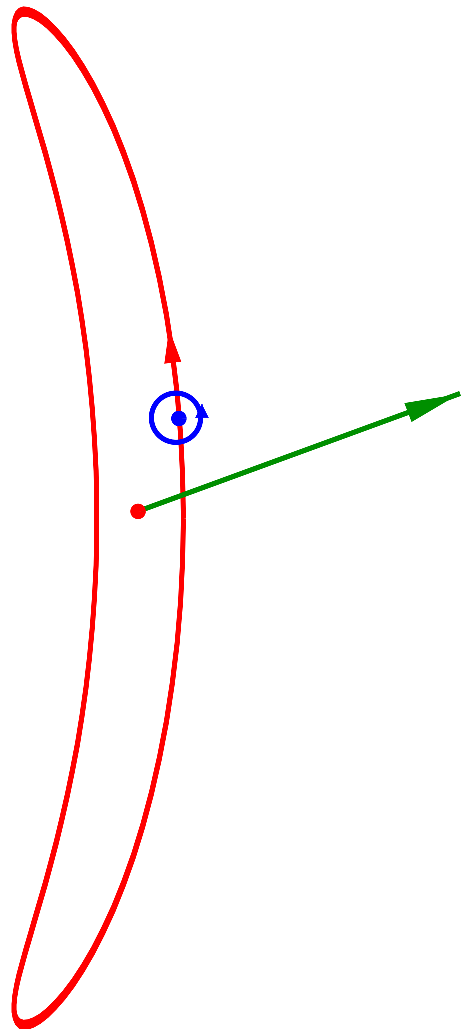
1) **Particles stream with velocity** v
(ordinary diffusion)
$\Rightarrow \lambda = v\tau$
$\Rightarrow D \sim v^2/\nu$



2) **Particles move on orbit** $(\omega \gg \nu)$
(e.g. gyro orbit)
collisions: Particle jumps to different orbit
$\Rightarrow \lambda = \Delta$
$\Rightarrow D \sim \Delta^2\nu$



Different scales of motion



motion on different time scales:

- **gyro motion**
- **bounce motion (of guiding centre)**
- **bounce centre motion**

$\Downarrow$ comparison with ν $\Downarrow$

it is possible to average over the corresponding fast scale

$\Rightarrow$ simplified equations for slow scale motion

Pfirsch-Schlüter regime

- **High collision rate: Fluid treatment possible**

Equilibrium: $\vec{j} \times \vec{B} = \nabla p$
Ohm's law: $\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j}$

$$\Rightarrow \qquad \vec{v}_\perp = \frac{\vec{E} \times \vec{B}}{B^2} - \frac{1}{B^2}\eta \nabla p$$

Particle flux: $\vec{\Gamma} = n\vec{v}_\perp$
Fick's law: $\vec{\Gamma} = -D\nabla n$

$$\Rightarrow \qquad D_{\rm cl} = \left(1 + \frac{T_{\rm i}}{T_{\rm e}}\right) \rho_{\rm e}^2 \, \nu_{\rm ei}$$

- **Particle picture:**

Electron on gyro orbit suffers collision $\Rightarrow$ Jumps to different gyro orbit
Diffusion step size: $\lambda = \rho_{\rm e}$

$$\Rightarrow D_{\rm cl} \sim \rho_{\rm e}^2 \, \nu_{\rm ei}$$

- **Including parallel dynamics in a torus $\Rightarrow$ neoclassical contribution:**
Passing particles can move distance $\lambda = \dfrac{\rho_{\rm e}}{\iota}$ **away from flux surface**
gives additional contribution (PS factor): $D_{\rm nc} = D_{\rm cl}\left(1 + \dfrac{1}{\iota^2}\right)$

Elimination of gyro motion: Guiding centre equations

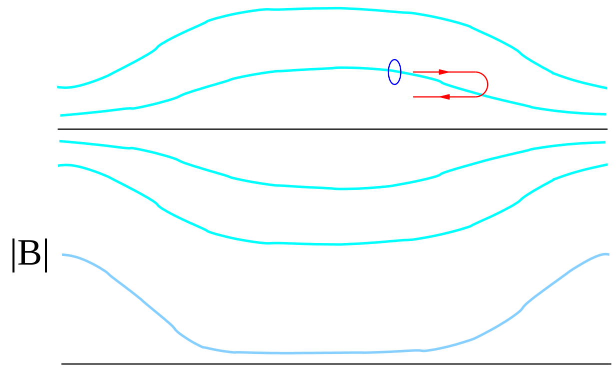
Small gyroradius $\left(\rho\frac{|\nabla B|}{B} \ll 1\right)$**: Particle orbit replaced by orbit of guiding centre**

Six-dim. phase space $\{\vec{r}, \vec{v}\}$ **reduced to five-dim.** $\{\vec{R}, v_{\|}, \mu\}$

$$\begin{aligned}
\dot{\vec{R}} &= v_{\|}\vec{b} + \vec{v}_{\mathrm{d}} & \vec{b} &= \vec{B}/|\vec{B}| \\
\dot{v}_{\|} &= -\mu\,\vec{b}\cdot\nabla|\vec{B}| + \frac{q}{m}\vec{b}\cdot\vec{E} & &\text{(energy conservation)} \\
\dot{\mu} &= 0
\end{aligned}$$

Perpendicular drift velocity: $\vec{v}_{\mathrm{d}} = \frac{1}{B}\left(\frac{\mu B + v_{\|}^2}{\Omega}\vec{b}\times\nabla|\vec{B}| + \vec{E}\times\vec{b}\right)$

$\dot{\mu} = 0 \Rightarrow$ **Particles can be reflected in B field (trapped particles)**

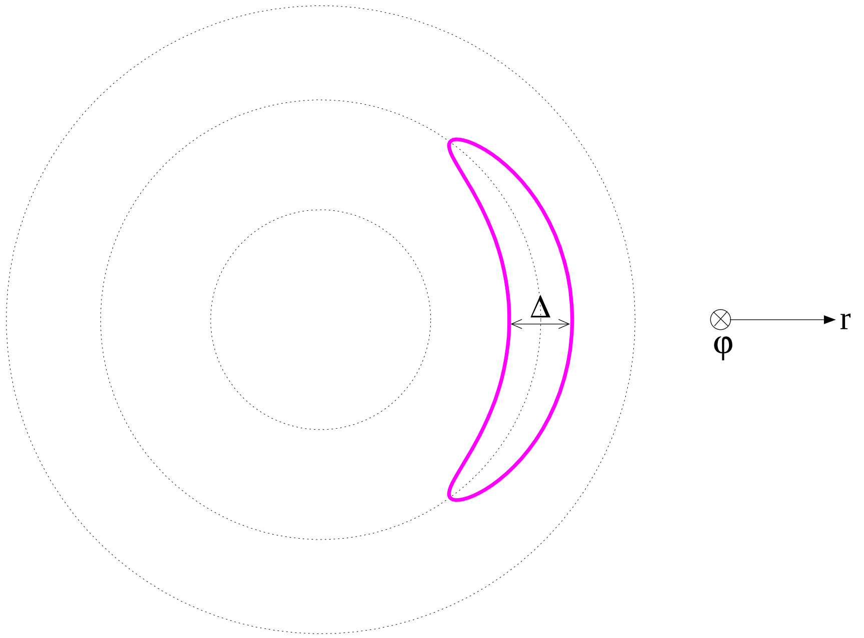


- **Tokamak: Axisymmetry** $\Rightarrow p_\varphi$ **conserved** $\Rightarrow$ **Particles stay near flux surface**
- **Stellarator: In general no conserved quantity** $\Rightarrow$ **Particles can leave quickly**

Movie: Reflected particle in a tokamak

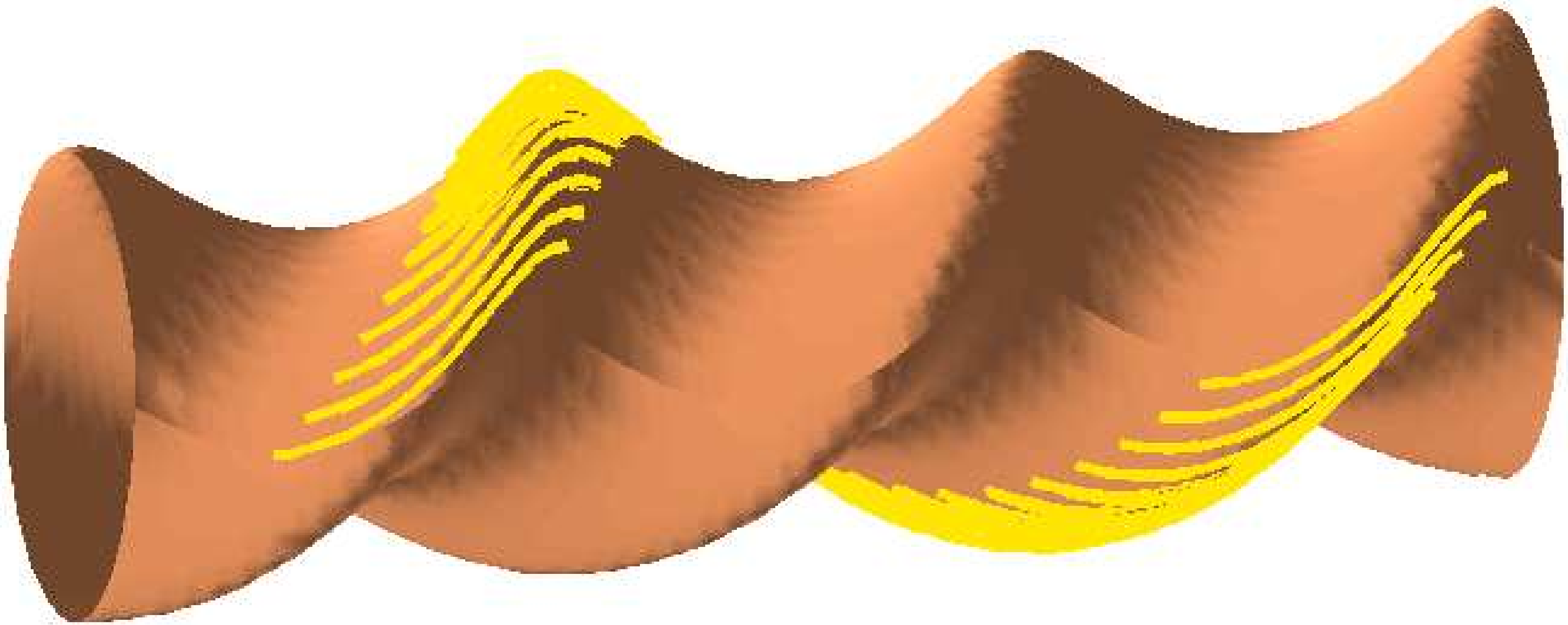
Banana regime in a tokamak

ν **small enough: Trapped particles become important**
since full banana orbit can be completed
$\Rightarrow$ **Diffusion step size λ given by banana width**
$\Delta \sim \rho_\mathrm{e}/(\iota\sqrt{\epsilon})$ **(inverse aspect ratio: $\epsilon = A^{-1}$)**



$$\Rightarrow D_\mathrm{b} \sim \sqrt{\epsilon} \cdot \Delta^2 \cdot \frac{\nu_\mathrm{ei}}{\epsilon} = \epsilon^{-\frac{3}{2}}\, \iota^{-2}\, \rho_\mathrm{e}^2\, \nu_\mathrm{ei}$$

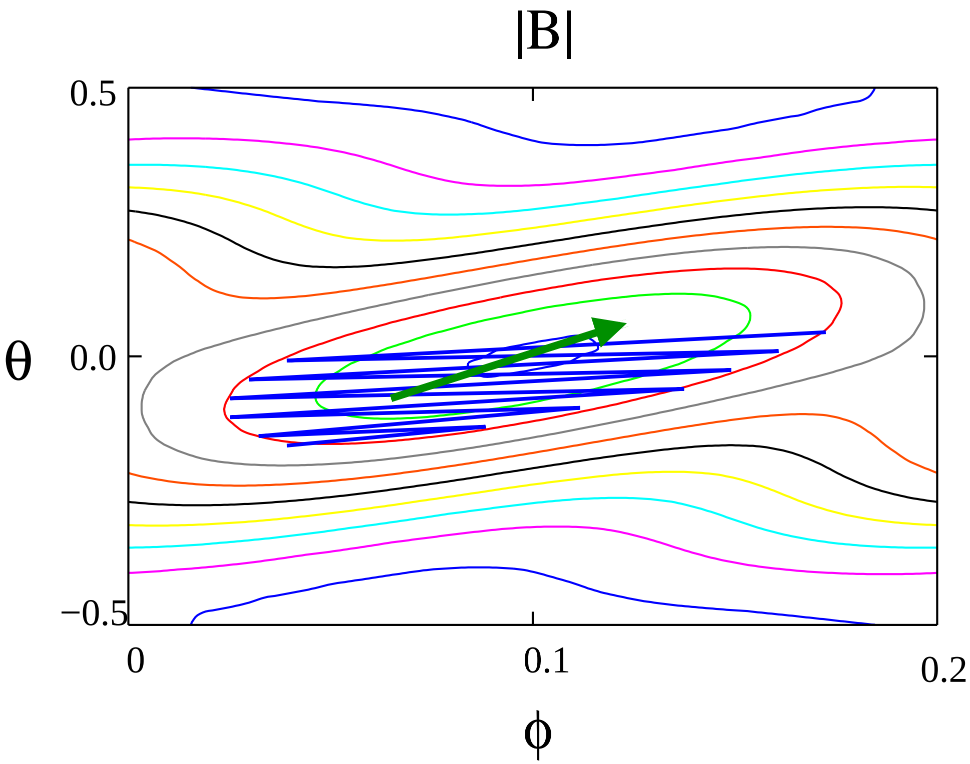
Trapped particle in a straight $\ell = 2$ stellarator



$\Rightarrow$ **Bounce centre rotates helically with frequency ω_h**

Bounce average

Particles can get trapped in local minima of $|B|$



further reduction of motion possible:

Guiding centre motion

$\Downarrow$

Bounce centre motion

Average over fast motion of trapped particle $\langle\cdot\rangle_{\mathrm{b}} = \dfrac{1}{\tau_{\mathrm{b}}}\oint \cdot\, dt$

$\Rightarrow$ **Motion of bounce centre determined by 2nd adiabatic invariant:** $J = \oint v_{\|} dl$

Bounce centre motion satisfies $J = \mathrm{const.}$

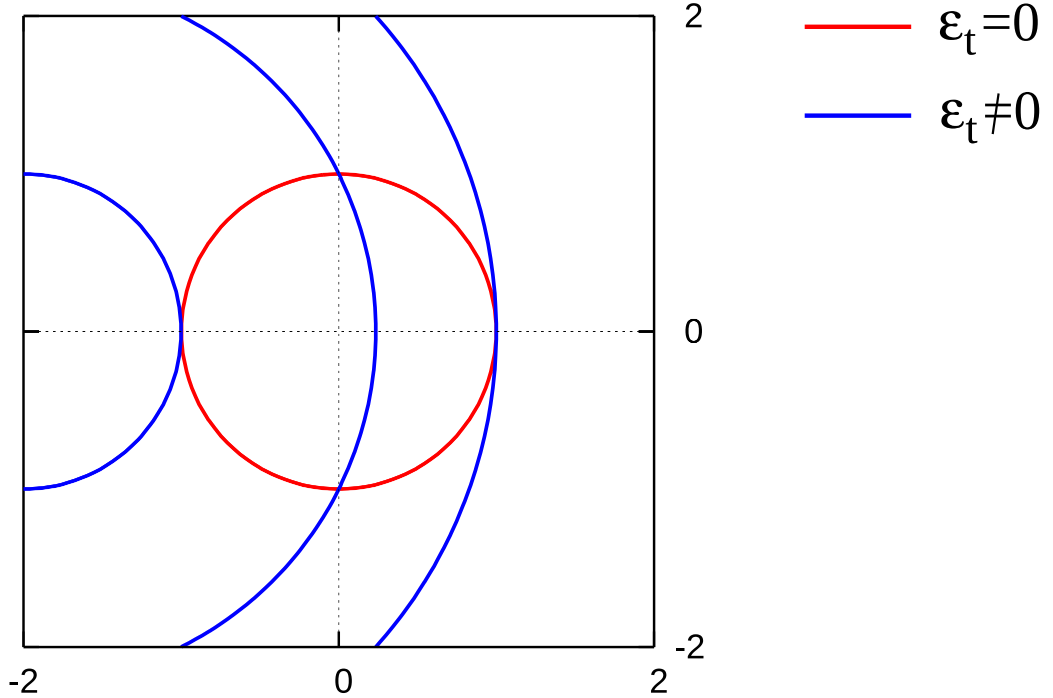
Bounce centre motion (for model fields)

- **Axisymmetry:** $\dot{r} = 0, \dot{\phi} = \omega$
  $\Rightarrow$ **No radial motion of banana centre**
- **Stellarator (simple model): Straight stellarator + toroidal variation of** B

$$B = B_0(1 - \epsilon_h \cos l(\theta - k\phi) - \epsilon_t \cos\theta)$$

Bounce centre motion:

$$\dot{r} = \epsilon_t v_\perp \sin\theta$$
$$\dot{\theta} = \epsilon_h \omega_h + \epsilon_t \frac{v_\perp}{r} \cos\theta$$



$\Longrightarrow$ $\epsilon_t \neq 0$: **Trapped particles move away from flux surface with velocity** $v_\perp$

Movie: Particle in $\ell = 2$ stellarator

$1/\nu$ regime

- **Helically trapped particles move out radially**

- **Radial velocity:**

$$v^{\mathrm{r}} = \vec{v}_{\mathrm{d}} \frac{\nabla r}{|\nabla r|} = \frac{\mu B + v_{\parallel}^2}{B\Omega} \left(\frac{\nabla r}{|\nabla r|} \times \vec{b} \right) \cdot \nabla |\vec{B}| = \frac{\mu B + v_{\parallel}^2}{\Omega} \kappa_{\mathrm{g}}$$

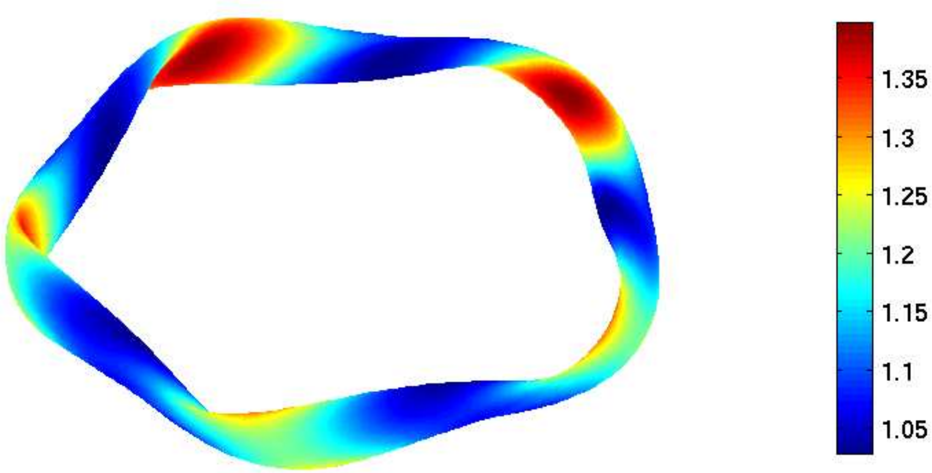
- **Diffusion coefficient: Use $\lambda = \langle v^{\mathrm{r}} \rangle_{\mathrm{b}} \tau$ ($\langle v^{\mathrm{r}} \rangle_{\mathrm{b}}$: Radial velocity of bounce centre)**

$$D_{1/\nu} \sim f_{\mathrm{t}} \frac{\lambda^2}{\tau} \sim f_{\mathrm{t}} \frac{1}{\nu} \langle v^{\mathrm{r}} \rangle_{\mathrm{b}}^2 \sim f_{\mathrm{t}} \frac{T^{\frac{7}{2}}}{nB^2} \kappa_{\mathrm{g}}^2$$

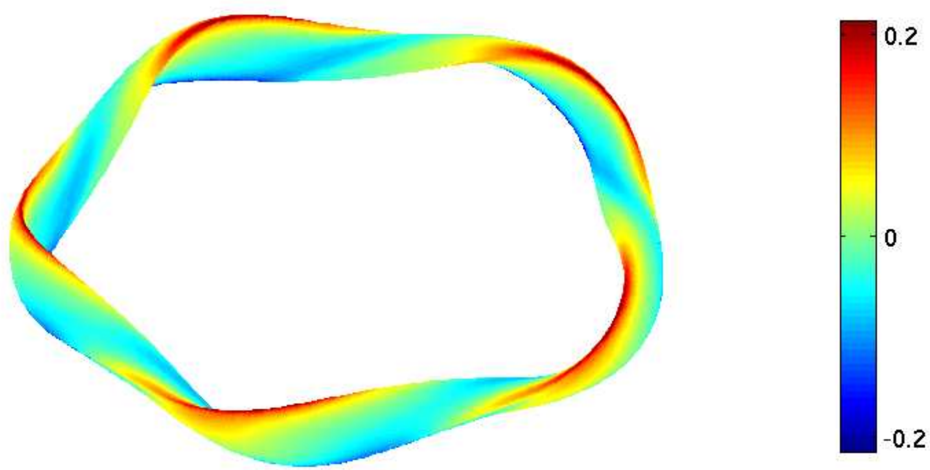
Problem: Strong T dependence ($\chi_{\mathrm{heat}} \sim T^{\frac{9}{2}}$) can lead to large losses for reactor
Possible solution: - Reduce $D_{1/\nu}$ by trapping particles in region where κ_{g} is small
- Reduce number of trapped particles f_{t}

Comment: for $E_{\mathrm{r}} \neq 0$: $1/\nu$ regime $\Rightarrow \sqrt{\nu}$ regime

B



$\boldsymbol{\kappa}_{\mathrm{g}}$



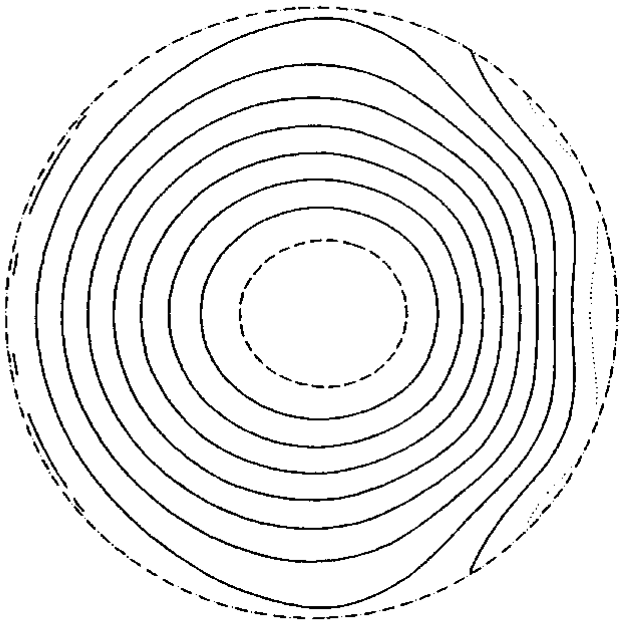
Optimisation: Reduce neoclassical losses by trapping particles in a region with small κ_g

W7-X: Quasi isodynamic stellarator

- **isodynamic:** $B = B(s) \rightarrow$ **guiding centre follows flux surface**
  $\Rightarrow$ **not possible in a torus**

- **quasi isodynamic: relax requirement: bounce centre follows flux surface**
  $\Rightarrow J \approx J(s)$

$J = \mathrm{const.}$ **contours in W7-X** ($\beta = 4.8\%$)



Movie: Reflected particle in W7-X

Ambipolarity

(monoenergetic) diffusion coefficients of species p: $D_p(K, \nu, E_\mathrm{r}, r)$

$$\Downarrow$$

particle flux: $\Gamma_p = -\int D_p \frac{\partial f_\mathrm{M}}{\partial r}\, d^3v$

$$\Gamma_p = -\frac{2n}{\sqrt{\pi}} \int D_\mathrm{p} \left[\frac{\partial \ln n}{\partial r} + \left(K - \frac{3}{2} \right) \frac{\partial \ln T_\mathrm{p}}{\partial r} + \frac{q_\mathrm{p}}{T} E_\mathrm{r} \right] e^{-K} \sqrt{K}\, dK$$

ambipolarity condition: prevent charge accumulation

$\Rightarrow \Gamma_\mathrm{i}(\nu, E_\mathrm{r}, r) = \Gamma_\mathrm{e}(\nu, E_\mathrm{r}, r)$ (on each flux surface)

tokamak: axisymmetry $\rightarrow$ ambipolarity intrinsically fulfilled, $E_\mathrm{r} \approx 0$

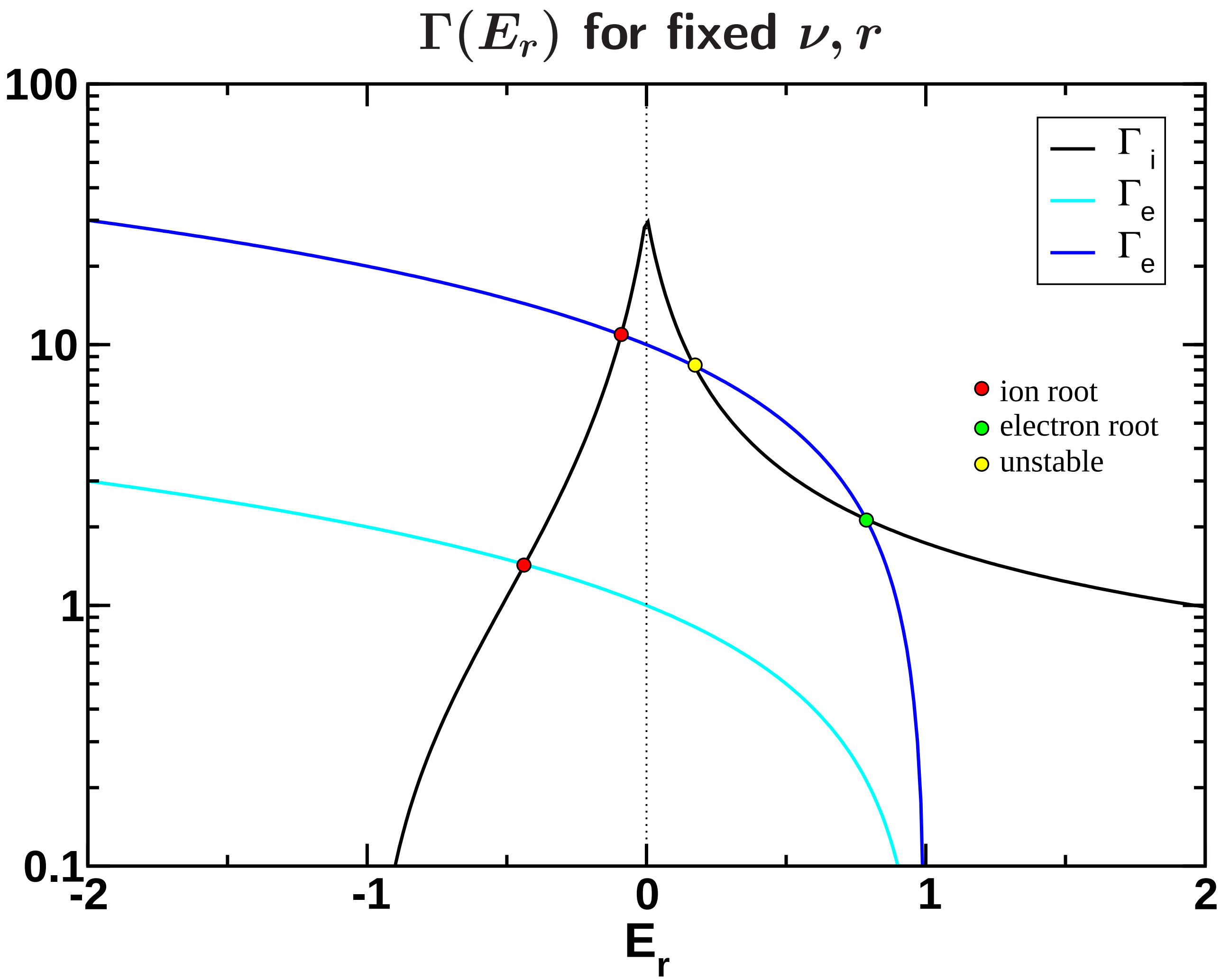
stellarator: E_r determined by ambipolarity condition

Electron/ion root

Example: $D_\mathrm{i} \sim K^2 \nu^{\frac{1}{2}} |E_\mathrm{r}|^{-\frac{3}{2}}$

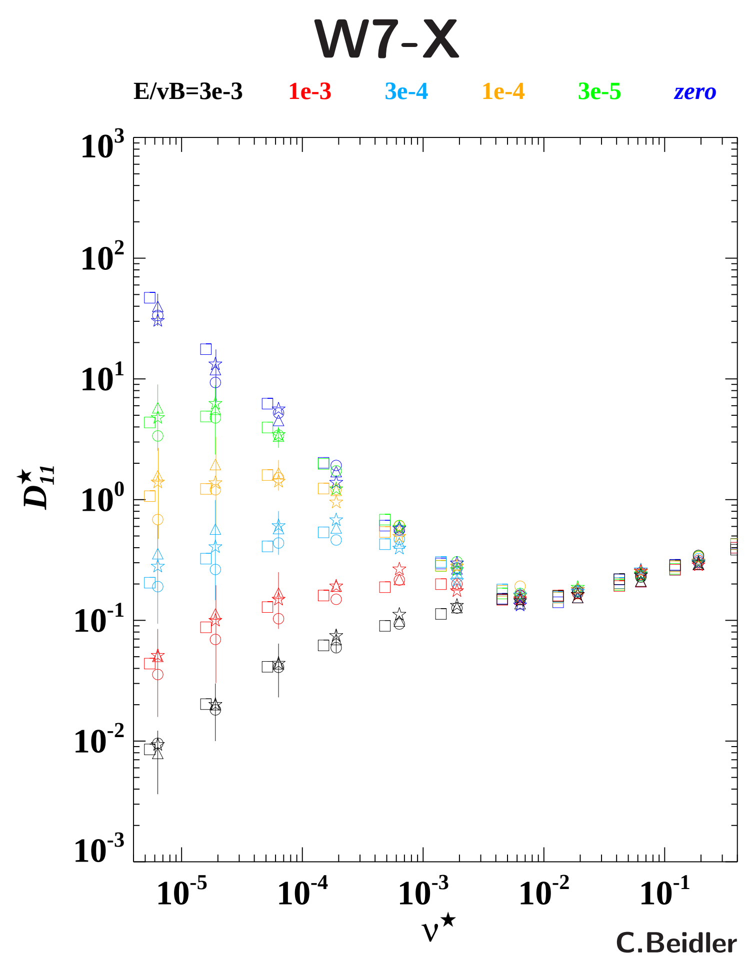
$$D_\mathrm{e} \sim K^2 \nu^{-1}$$

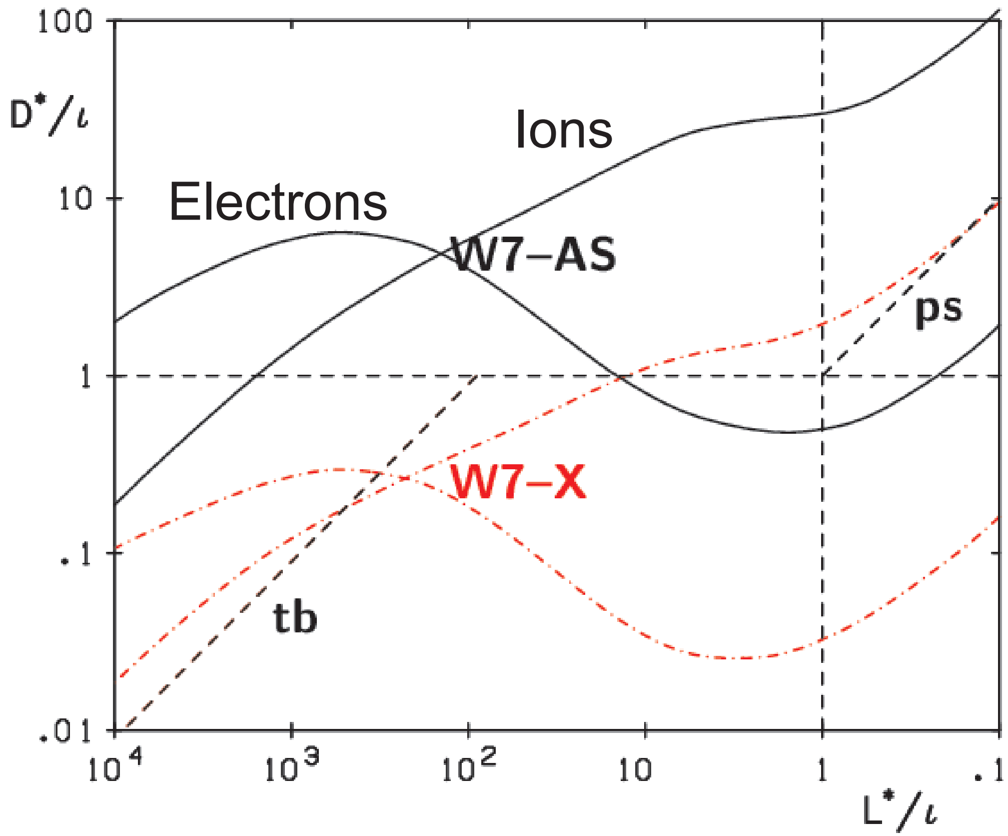
- **solve** $\Gamma_\mathrm{i} = \Gamma_\mathrm{e}$ **for each flux surface** $\Rightarrow E_\mathrm{r}(r)$



$\Rightarrow$ **stellarators have a large E_r which (usually) points inwards (ion root)**

Transport coefficients





⇒ **Neoclassical losses in W7-X much smaller than in W7-AS due to optimisation**

Quasisymmetry

Lagrangian of GC motion

$$L = \frac{m}{2}(\vec{b}\cdot\dot{\vec{R}})^2 + q\vec{A}\cdot\dot{\vec{R}} - \mu B$$

especially simple in Boozer coordinates

$$L = \frac{m}{2B}(J\dot{\vartheta}+I\dot{\varphi})^2+q(F_{\mathrm{T}}'\,\dot{\vartheta}+F_{\mathrm{P}}'\,\dot{\varphi})-\mu B$$

GC motion determined only by $B(s,\vartheta,\varphi)$ but not by the spatial structure of the flux surface (in magnetic coordinates particles do not "see" the geometry)

Noether's theorem: Symmetry (invariance of L) $\vec{v}\cdot\nabla L = 0$ for a vector field $\vec{v} = v^i\vec{e}_i$ leads to a conserved quantity $P = v^i\frac{\partial L}{\partial \dot{R}^i}$

For L in Boozer coordinates: $\vec{v}\cdot\nabla L = 0 \Leftrightarrow \vec{v}\cdot\nabla B = 0$

A symmetry of B (quasisymmetry) leads to a conserved quantity.

Assume $\vec{v} = M\vec{e}_\vartheta + N\vec{e}_\varphi$ with $M, N \in \mathbb{Z}$ (closed lines on a torus)
If $\vec{v}$ is a symmetry of B then there exists a conserved quantity

$$P = \frac{m}{B}v_\parallel(MI + NJ) - q(NF_{\mathrm{P}} - MF_{\mathrm{T}}) \quad \text{with} \quad v_\parallel = \vec{b}\cdot\dot{\vec{R}}$$

For three-dimensional systems quasisymmetry can only be achieved approximately

Quasisymmetry

$B(s, \vartheta, \varphi)$ (magnetic topography) is the only important quantity for gc motion

- $B = B(s, \vartheta, \varphi)$: **Trajectories of reflected particles in general not confined**

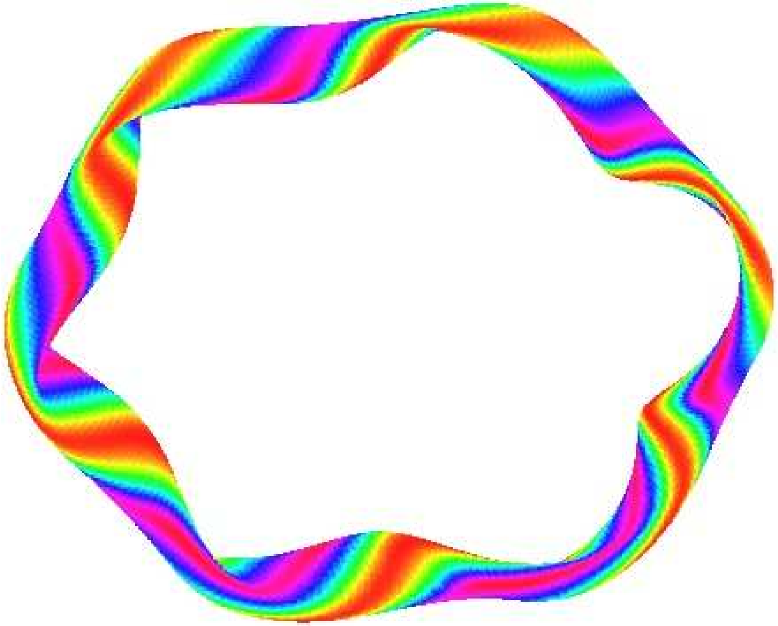
To get better confined particles try to make B symmetric
(symmetry in B, not in spatial structure)
Quasisymmetry
$\Downarrow$
Conserved quantity
$\Downarrow$
Better particle confinement

- **Quasi–axial symmetry** ($M = 0$): $B = B(s, \vartheta) \Rightarrow$ **Similar to a tokamak**
- **Quasi–poloidal symmetry** ($N = 0$): $B = B(s, \varphi)$
- **Quasi–helical symmetry:** $B = B(s, N\vartheta - M\varphi) \Rightarrow$ **Similar to a straight stellarator**

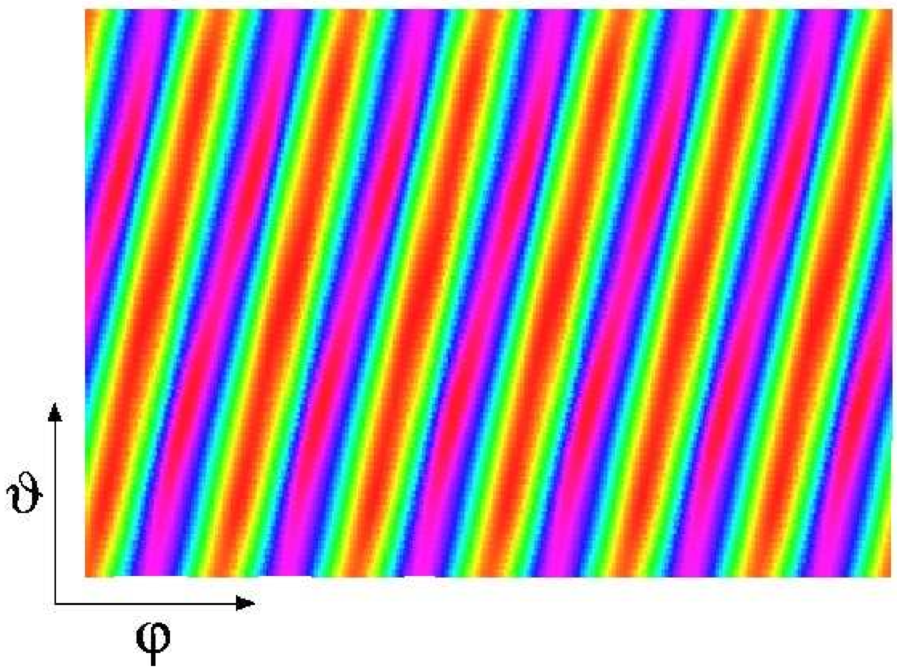
$\Longrightarrow$ **No $1/\nu$ regime**

Quasi helically symmetric stellarator

B, real space

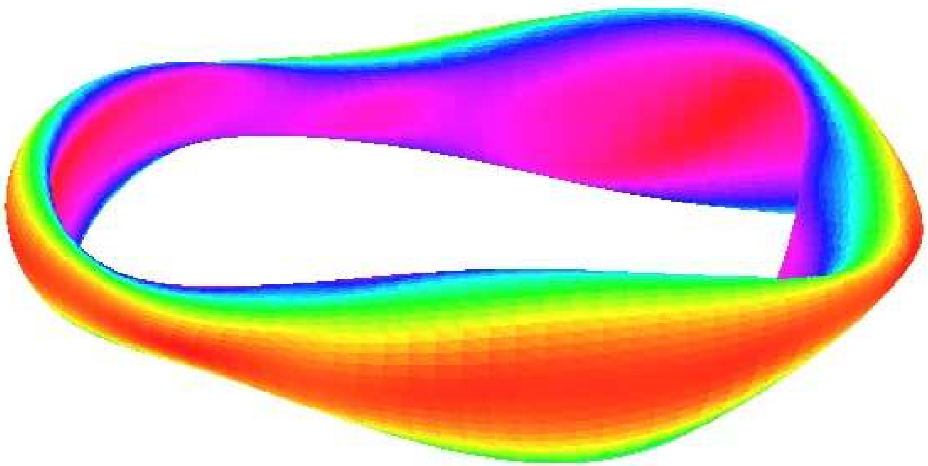


B, magnetic coordinates

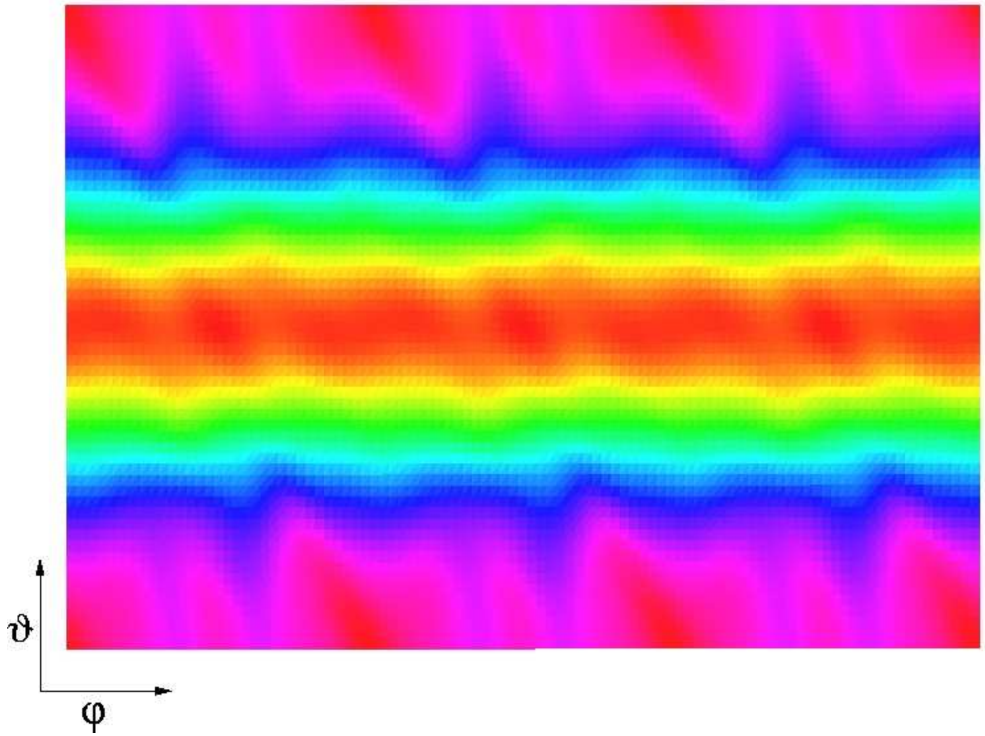


Quasi axially symmetric stellarator

B, real space



B, magnetic coordinates



Quasi symmetries can only be achived approximately

Stellarator Optimisation

Stellarator optimisation

3D geometry offers high flexibility to shape the equilibrium
Needed: Unified concept to design equilibrium with good confinement properties

$\Downarrow$

Stellarator optimisation

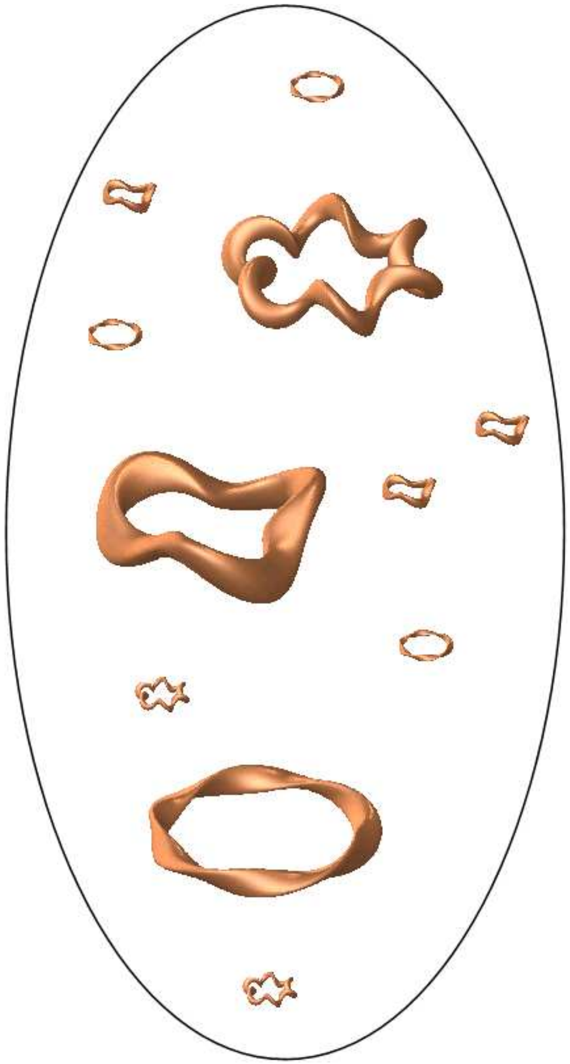
Define what "good" properties mean by constructing a quality measure $\mathcal{Q} : \mathrm{Con} \mapsto \mathbb{R}$ over the configuration space

$\mathcal{Q}$ **includes (e.g. W7–X):**

- **Good magnetic surfaces**
- **Good equilibrium (small Shafranov shift for $\bar{\beta} \approx 5\%$)**
- **Small neoclassical transport**
- **Good α–particle confinement**
- **MHD stability (stable for $\bar{\beta} \approx 5\%$)**
- **Small bootstrap current**
  (ι does not change much with β)
- **Feasible modular coils (engineering constraints)**

Stellarator optimisation

Numerical optimisation procedure:



Specify outer flux surface

$c \in \mathrm{Con}$: **Fourier coefficients of boundary**

$\Downarrow$

Calculate equilibrium (VMEC)

$\Downarrow$

Evaluate $\mathcal{Q}(c)$

maximum ?

$\Downarrow$

Optimized stellarator equilibrium

e.g. W7–X

$(\Downarrow)$

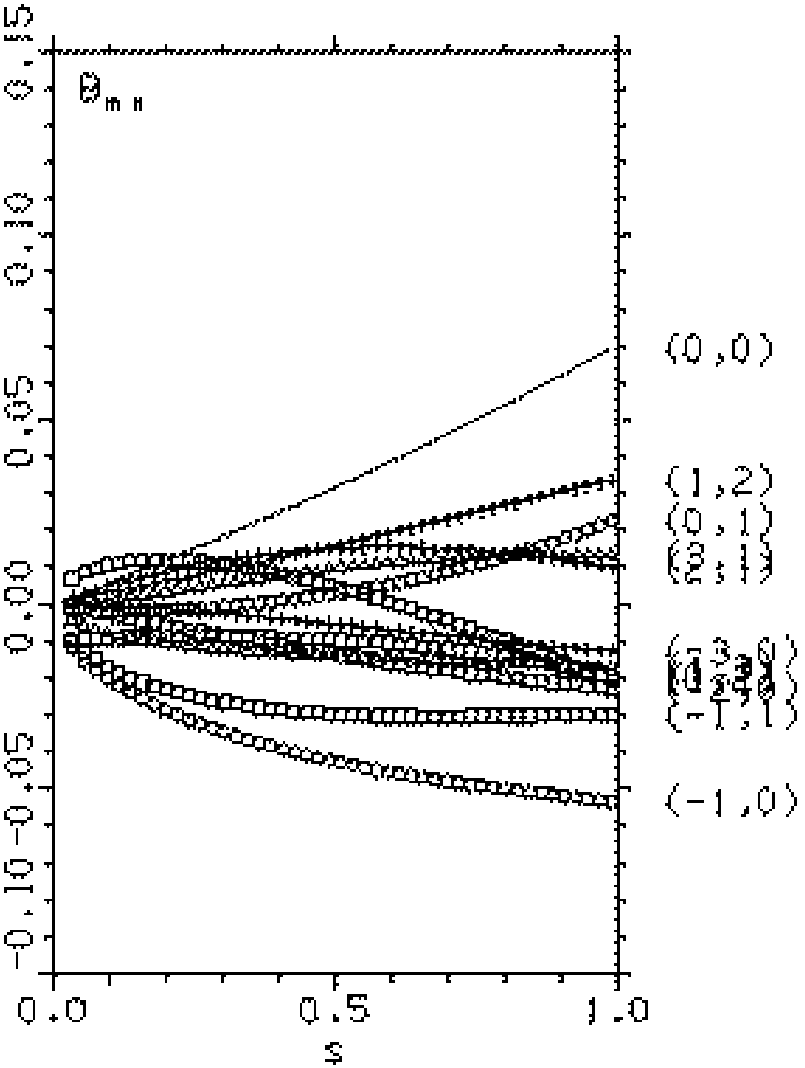
Calculate coil system

WENDELSTEIN 7-X

IPP

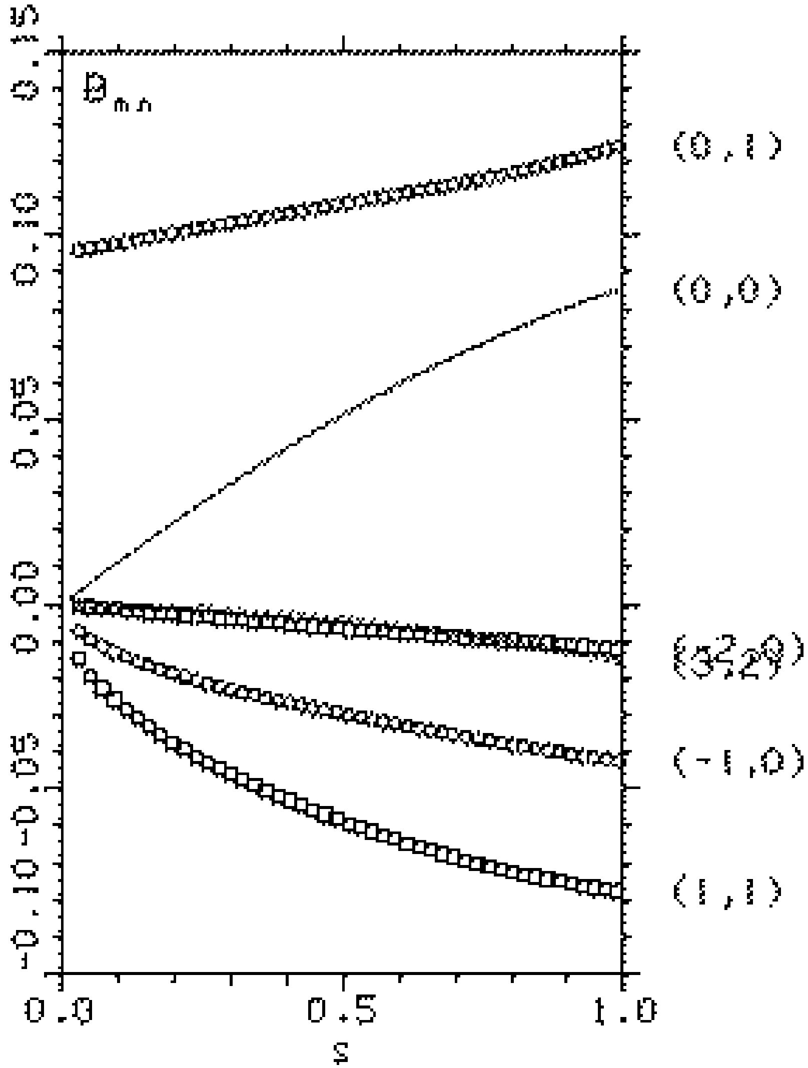
Fourier components (m, n) of B

W7-AS



Dominant toroidal component

W7-X



Mirror and helical component

Conclusions

Due to their three-dimensional structure stellarators offer great flexibility for their design, but some subtle points must be considered:

- **Magnetic surfaces**
- **Shafranov shift**
- **Neoclassical/α–particle confinement**
- **MHD stability**

Connection via geodesic curvature

Unification by

- **Using optimisation for designing equilibria**
- **Concept of quasi symmetries**

Stellarator theory (as many other branches of modern theoretical physics) needs computational physics and modern computers.