ELEC-E8101 Digital and optimal control
Exam and solutions
Autumn 2022 - 2nd part
December 7, 2022

Instructions

1. Write the name of the course, your name and student number to each answer sheet.
2. There are three (3) problems and each one must be answered. Each problem is worth maximum 6 points.
3. No other literature except the table of formulas prepared by the course staff is allowed.
4. A calculator can be used. Only basic, non-symbolic calculations are allowed with the calculator (e.g. no symbolic solving of equations, taking Laplace- or z-transformations).
5. The table of formulas must be returned, if you have received it from the exam supervisor.

1 (6 points total)
Choose the correct alternative for each of 1.1-1.6.
1.1 Consider different ways to approximate discretization. Which of the following statements is correct?

a) Under Tustin approximation, if the discretized system is stable, there may exist a corresponding continuous system that is unstable.
b) Under forward approximation, if the discretized system is stable, there may exist a corresponding continuous system that is unstable.
c) Under forward approximation, if the continuous system is stable, the discretized system may be unstable. (Correct answer. See Lect 7)
d) Under backward approximation, if the continuous system is stable, the discretized system may be unstable.

1.2 The simplified dynamics of a single link robot arm can be written as

$$J\ddot{\theta} + mgl\sin\theta = \tau$$

The system is converted to state-space form and linearized at $\theta = \pi$. Which of the following represents the linearized state-space form $\dot{x} = Ax + Bu$?

a)
$$x = \theta,\ u = \tau,\ A = 1,\ B = 1/J$$

b)
$$x = \left(\theta,\, \dot{\theta}\right),\ u = \tau,\ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ B = 1/J$$

c)
$$x = \left(\theta,\, \dot{\theta}\right),\ u = \tau,\ A = \begin{pmatrix} 0 & 1 \\ mgl & 0 \end{pmatrix},\ B = 1/J$$

d) (correct answer)
$$x = \left(\theta,\, \dot{\theta}\right),\ u = \tau,\ A = \begin{pmatrix} 0 & 1 \\ mgl/J & 0 \end{pmatrix},\ B = 1/J$$

Explanation: begin by solving the equation for $\ddot{\theta}$, substitute $\sin x\,|_{x=-\pi} \approx -x$ (this comes from Taylor expansion), convert to state space form

Solving $\ddot{\theta}$ first

$$J\ddot{\theta} + mglsin(\theta) = \tau$$
$$J\ddot{\theta} = -mglsin(\theta) + \tau$$
$$\ddot{\theta} = -\frac{1}{J}mglsin(\theta) + \frac{1}{J}\tau$$

When $\theta = \pi$ -> $sin(\theta = \pi) = -\theta$. Hence we get

$\ddot{\theta} = -\frac{1}{J}mgl(-\theta) + \frac{1}{J}\tau = \frac{1}{J}mgl\theta + \frac{1}{J}\tau$. This corresponds to **option d**.

1.3 Consider a feedback system with plant G(z) and controller C(z). An additive input-disturbance D(z) perturbs the system. Which of the following represents the transfer function of the system from reference R(z) to output Y(z)?

a) **(Correct answer. See Lect 9)**
$$\frac{G(z)C(z)}{1+G(z)C(z)}$$
b)
$$\frac{G(z)}{1+G(z)C(z)}$$
c)
$$\frac{C(z)}{1+G(z)C(z)}$$
d)
$$\frac{1}{1+G(z)C(z)}$$

If the input disturbance D(z) is added before the controller, the input-output relation is
$$Y(z) = \frac{G(z)C(z)}{1+G(z)C(z)}R(z) + \frac{G(z)C(z)}{1+G(z)C(z)}D(z)$$

If the input disturbance D(z) is added between the controller and the plant, the input-output relation is
$$Y(z) = \frac{G(z)C(z)}{1+G(z)C(z)}R(z) + \frac{G(z)}{1+G(z)C(z)}D(z)$$

This corresponds to **option a**.

1.4 Consider the discrete time system

$$\vec{x}_{t+1} = A\vec{x}_t + \vec{n}$$

where $\vec{x} = (u, v)$ and $\vec{n}$ is Gaussian (white) noise

$$n \sim N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right).$$

Let the initial covariance of the system be zero, that is,

$$cov(\vec{x}_0) = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

What is the covariance after two time steps?

a)
$$cov(\vec{x}_2) = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

b)
$$cov(\vec{x}_2) = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

c)
$$cov(\vec{x}_2) = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$$

d) The task does not provide sufficient information to determine it. (Correct answer. cov(x_2)=A A^T + I)

1.5 Consider a discrete-time system

$$x[k+1] = \Phi x[k] + \Gamma u[k]$$

is to be controlled to minimize the cost

$$J = 0.5x[N]^T Q_N x[N] + 0.5 \sum_{k=0}^{N-1} x[k]^T Q x[k] + u[k]^T R u[k]$$

for a given $x[0] \neq 0$ and $N = 100$.

Which one of the following cost designs is the best to achieve the objective of "*getting the state variables close to zero as soon as possible while expending relatively little control input*"?

a) $Q_N = 10\,I$, $Q = 10\,I$ and $R = 0$
b) $Q_N = 10\,I$**,** $Q = 10\,I$ **and** $R = 1$ **[Correct answer: b]**
c) $Q_N = 10\,I$, $Q = 0\,I$ and $R = 0$
d) $Q_N = 10\,I$, $Q = 0\,I$ and $R = 1$

Note: I in the options above refers to the identity matrix.

Explanation: Q_N adds a control penalty only to the final state, while Q controls the intermediate states. To reach zero state as soon as possible, the intermediate states also need to be included in the cost function, hence Q > 0. Additionally, to expend as little control input as possible, the control signal also needs to be included in the cost (so R > 0).

1.6 Consider that we combine the state-feedback with the observer to deal with non-measurable but observable states (output feedback) in discrete-time linear time-invariant systems. Which of the following statements is correct?

a) the state feedback controller is the same as the state estimator.
b) the state feedback controller and state estimator should be designed jointly.
c) the state feedback controller and state estimator can be designed separately.
d) the problem cannot be solved.

[Correct answer: **c. See Lecture 11, slides 88–99 *(separation principle)***]

2

A continuous-time model of a DC motor can be modeled with the transfer function (from input voltage to rotational velocity)

$$G(s) = \frac{\omega(s)}{V(s)} = \frac{K}{(R+Ls)(Js+b)+K^2}$$

where ω is the rotational velocity, and *V* the input voltage. *K* is the electromotive force constant, *R* the electric resistance, and *L* the electric inductance of the armature. *J* is the moment of inertia of the rotor and *b* the (damping) friction of the mechanical system.

a. Use backward approximation to determine the corresponding discrete-time model of the motor. Let discretization time (sampling time) be denoted by T_S. (2 points)
Answer (first 2 points):
Substitute

$$s = \frac{1-z^{-1}}{T_S}$$

gives

$$G(z) = \frac{K}{\left(R+\frac{L}{T_S}(1-z^{-1})\right)\left(\frac{J}{T_S}(1-z^{-1})+b\right)+K^2}$$

b. The motor is controlled by a discrete time controller with the transfer function

$$C(z) = K_P + K_D\frac{1-z^{-1}}{T_S} + K_I\frac{T_S}{z-1}$$

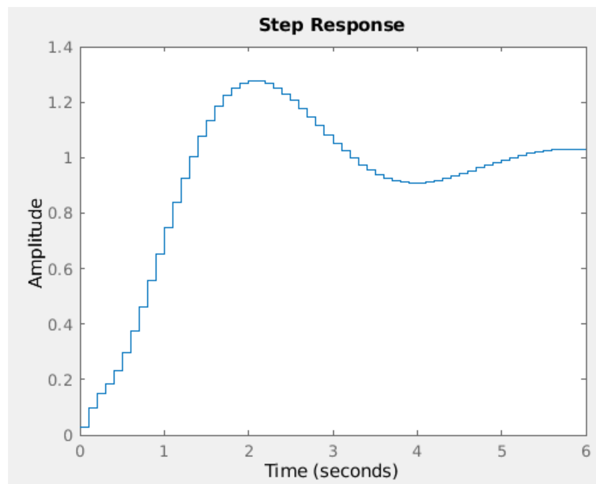
Write the corresponding time-domain equation of the controller in terms of $u[k]$ (controller input) and $y[k]$ (controller output). Which approximations (forward, backward, Tustin) are used for the derivative and integral terms? (2 points)

Answer:

$$y[k] = K_P u[k] + \frac{K_D}{T_S}(u[k] - u[k-1]) + K_I Ts \sum_{i=0}^{k-1} u[i]$$

The derivative uses backward approximation, the integral the forward approximation. Note: according to the task, u[k] is the controller input, that is, the error signal.

c. A particular motor is controlled by the controller, with the resulting pulse response shown here. This response is inadequate, because it has too much overshoot and too slow rise time. How would you adjust the controller parameters to account for those? That is, which parameters would need to be increased/decreased? (1 point)



Answer:
Raise time can be decreased by increasing the proportional gain. Overshoot can be decreased by increasing the derivative gain (alternatively, decreasing integral term).

d. A physical system typically has actuator limits, for example, here the input voltage would have limits. Explain briefly (maximum 5 sentences) how the limits would affect the system behavior and controller design. (1 point)

Answer:
Actuator limits (actuator saturation) causes integrator wind-up. This is because the integrator keeps integrating the tracking error even when the system input is saturated. (0.5 points on the phenomenon)

The controller needs to have an anti-windup mechanism. For example, negative feedback from saturation error to integral term can be used. (0.5 points on the solution (controller design))

3 Consider the discrete-time system given by the following difference equation

$$x[k+1] = x[k] + 0.1v[k]$$
$$v[k+1] = v[k] + u[k]$$

where $x[k]$ is the position, $v[k]$ is the velocity and $u[k]$ is the control input.

Given $x[0]$ = 3 and $v[0]$ = -10, the system is to be controlled to minimize the following cost

$$J = \frac{1}{2}\left\{100x[2]^2 + v[2]^2 + u[0]^2 + u[1]^2\right\}$$

The objective is to get the final position and velocity ($x[2]$ and $v[2]$) close to zero, while expending relatively little control input.

a) Calculate the optimal control gains for $k = 1$ (2 points)
b) Given $u[0]$ = -2, calculate
    i) Optimal control input for $k = 1$ (1 point)
    ii) Optimal cost for $k = 1$ (2 points)
    iii) Final/terminal cost assuming $u[1]$ is optimal control input (1 point)

Q3 Solution is on the following pages (2 pages)

$$\begin{bmatrix} x[k+1] \\ v[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}}_{\Phi} \begin{bmatrix} x[k] \\ v[k] \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\Gamma} u[k] \implies \mathbf{x}_{k+1} = \Phi \mathbf{x}_k + \Gamma u_k$$

$$S_2 = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q_0 = Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad R_0 = R_1 = R = 1$$

a) Calculate the optimal control gains for $k=1$

$\Rightarrow$ Calculate L_1

$$L_k = (\Gamma^T S_{k+1} \Gamma + R)^{-1} \Gamma^T S_{k+1} \Phi$$

$$L_1 = (\Gamma^T S_2 \Gamma + R)^{-1} \Gamma^T S_2 \Phi$$

$$= \underbrace{\left(\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \right)^{-1}}_{\left(\underbrace{[0 \ 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1}_{2}\right)^{-1}} \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}}_{[0 \ 1]} \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2} [0 \ \ 1] \implies \boxed{L_1 = [0 \ \ 0.5]}$$

b) Given $u[0] = -2$, calculate $\Rightarrow u_0 = -2$

i) Optimal control input for $k=1$ $\Rightarrow u_1^* = ?$

$$\underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\mathbf{x}_1} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -10 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} -2 \implies \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \end{bmatrix}$$

$$u_1^* = -L_1 \mathbf{x}_1$$

$$= -[0 \ \ 0.5] \begin{bmatrix} 2 \\ -12 \end{bmatrix} \implies \boxed{u_1^* = 6}$$

ii) Optimal cost for $k=1$ $\Rightarrow J_1^* = ?$

$$S_k = (\Phi - \Gamma L_k)^T S_{k+1} (\Phi - \Gamma L_k) + \overset{0}{Q} + L_k^T \overset{1}{R} L_k$$

$$J_k^* = \frac{1}{2} x_k^T S_k x_k$$

$$S_1 = (\Phi - \Gamma L_1)^T S_2 (\Phi - \Gamma L_1) + L_1^T L_1$$

$$S_1 = \left(\begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0.5 \end{bmatrix}\right)^T \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0.5 \end{bmatrix}\right) + \overbrace{\begin{bmatrix} 0 \\ 0.5 \end{bmatrix}\begin{bmatrix} 0 & 0.5 \end{bmatrix}}^{\begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0.1 \\ 0 & 0.5 \end{bmatrix}^T \qquad \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0.1 & 0.5 \end{bmatrix}\begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0.1 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 100 & 0 \\ 10 & 0.5 \end{bmatrix}\begin{bmatrix} 1 & 0.1 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 100 & 10 \\ 10 & 1.25 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.25 \end{bmatrix} \implies S_1 = \begin{bmatrix} 100 & 10 \\ 10 & 1.5 \end{bmatrix}$$

$$J_1^* = \frac{1}{2} x_1^T S_1 x_1 = \frac{1}{2}\begin{bmatrix} 2 & -12 \end{bmatrix} \cdot \begin{bmatrix} 100 & 10 \\ 10 & 1.5 \end{bmatrix}\begin{bmatrix} 2 \\ -12 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 80 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -12 \end{bmatrix} = \frac{1}{2} \cdot 136 = 68 \qquad \boxed{J_1^* = 68}$$

iii) Final / terminal cost assuming u[1] is optimal control input $\Rightarrow J_2 = ?$ $u_1 = u_1^* = 6$ (found in i)

$$J_2 = \frac{1}{2} \mathbf{x}_2^T S_2 \mathbf{x}_2$$

$$\mathbf{x}_2 = \Phi \mathbf{x}_1 + \Gamma u_1^* \qquad \mathbf{x}_2 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -12 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 6 = \begin{bmatrix} 2 - 1.2 \\ -12 + 6 \end{bmatrix} \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 0.8 \\ -6 \end{bmatrix}$$

$$J_2 = \frac{1}{2}\begin{bmatrix} 0.8 & -6 \end{bmatrix}\begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.8 \\ -6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 80 & -6 \end{bmatrix}\begin{bmatrix} 0.8 \\ -6 \end{bmatrix} = \frac{100}{2} = 50 \qquad \boxed{J_2 = 50}$$