MS-C1081 – Abstract Algebra **Problem set 1**
2022–2023 (Period III)
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During the exercise sessions you may work on the Warm-up problems, and the TAs will present solutions to those. Submit your solutions to Homework 1,2,3 and to the Fill-in-the-blanks problem on MyCourses before the deadline. Remember that the Homeworks and the Fill-in-the-blanks go to separate return boxes, and each return box accepts a single pdf file.

Warm-ups

Warm-up 1. Given a set X, denote $X^X := \{\text{functions } X \to X\}$ and define the operation

$$\begin{aligned} X^X \times X^X &\longrightarrow X^X \\ (f,g) &\longmapsto g \circ f. \end{aligned}$$

1. Show that this operation is associative.
2. In the case of $X = \{1,2\}$, write down the Cayley table.
3. Show that if the cardinality of X is at least 2, this operation is not commutative.

Warm-up 2. Recall that $S_{\mathbb{N}}$ is the group of *bijective* functions $\mathbb{N} \to \mathbb{N}$, equipped with composition. For all $n \in \mathbb{N}$, find an element of $S_{\mathbb{N}}$ of order n. Find an element of $S_{\mathbb{N}}$ of infinite order.

Warm-up 3. The dihedral group D_n is the group of symmetries of a regular n-gon.

1. Consider D_3. Use the same notation as in the lecture notes for the rotation R and the flips F_1, F_2, F_3. Write down the Cayley table for D_3.
2. Find the order of all the elements of D_3 and D_4.
3. List all the subgroups of D_3.

Homework

Homework 1. Given two *subsets* A and B of a group $(G,\cdot)$, we denote

$$AB := \{a \cdot b \mid a \in A, b \in B\}.$$

Prove the following:

1. Given three subsets A, B and C of a group G, we have $(AB)C = A(BC)$. [3 points]
2. For a *subgroup* $H \subseteq G$, we have $HH = H$. [3 points]

Homework 2. Consider $\mathrm{GL}_2(\mathbb{Z}_2)$, the group of invertible 2×2 matrices with coefficients in $\mathbb{Z}_2$, equipped with matrix multiplication.

1. What is the identity of this group? [1 points]
2. Find all the elements of $\mathrm{GL}_2(\mathbb{Z}_2)$ and their order. [3 points]
3. Is $\mathrm{GL}_2(\mathbb{Z}_2)$ abelian? [2 points]

Homework 3. Recall that S_n is the group of permutations of $\{1,\dots,n\}$, equipped with function composition.

1. Write down the Cayley table of S_3. [1.5 points]
2. Show that S_n is not abelian for $n \geq 3$. [2.5 points]
3. Compute the order of all the elements of S_3. [2 points]

Fill-in-the-blanks. Complete the proof of the following claim:

Claim. Let $(G,\cdot)$ be a group with identity e. If $a \cdot a = e$ for all $a \in G$, then G is abelian.

Proof. By definition of abelian group, we have to show that, for all $x, y \in G$, ____________.
Indeed, for all $x, y \in G$,

$$x \cdot y \overset{(*)}{=} \big(y \cdot \underline{\quad\quad}\big) \cdot \underline{\quad\quad\quad\quad} \cdot \big(\underline{\quad\quad} \cdot x\big) \overset{(**)}{=} y \cdot \big(\underline{\quad\quad\quad\quad\quad\quad}\big) \cdot x \overset{(***)}{=} y \cdot e \cdot x = y \cdot x,$$

where in equality $(*)$ we used the given hypothesis both with $a = x$ and with $a = y$, in equality $(**)$ we used the associativity of $\cdot$, and in equality $(***)$ we used the hypothesis with $a =$ ________. □

[3 points]