Analysis, Random Walks and Groups

Exercise sheet 1

Homework exercises: Return these for marking to Kai Hippi in the tutorial on Week 2. Contact Kai by email if you cannot return these in-person, and you can arrange an alternative way to return your solutions. Remember to be clear in your solutions, if the solution is unclear and difficult to read, you can lose marks. Also, if you do not know how to solve the exercise, attempt something, you can get awarded partial marks.

1. (5pts) In the **weak Borel shuffle** one lifts the top card from a deck of 52 cards and inserts it into the deck in a random position.

(a) For $0 \leq j \leq 51$ determine the permutation $\sigma_j \in S_{52}$ corresponding to the outcome of placing the top card in the jth position amongst the remaining cards. (The 0th position is on top of the remaining cards and the 51st position is on the bottom of the remaining cards.)

(b) Supposing that one performs consecutive weak Borel shuffles with $0 \leq j \leq 51$ chosen uniformly at random (e.g. by rolling a 52-sided die each time) what is the probability that the card on the top of the deck before shuffling is on top of the deck after 2 shuffles?

2. (5pts) Prove the *variational formula* for the total variation distance between two probability distributions μ, ν in $\mathbb{Z}_p$:

$$d(\mu,\nu) = \frac{1}{2}\max\{|\mu(f) - \nu(f)| : \|f\|_\infty \leq 1, f : \mathbb{Z}_p \to \mathbb{R}\}.$$

Hint: For the more difficult upper bound, first define a suitable function $g : \mathbb{Z}_p \to \mathbb{R}$ using the set $B = \{t \in \mathbb{Z}_p : \mu(t) \geq \nu(t)\}$ such that you can ensure $\mu(g) - \nu(g) = \sum_{t\in\mathbb{Z}_p} |\mu(t) - \nu(t)|$ (try to think something quite simple for g), and then use the L^1 identity from the lecture notes.

Further exercises: Attempt these before the tutorial, they are not marked and will be discussed in the tutorial. If you cannot attend the tutorial, but want to do the attendance marks, you can return your attempts to these before the tutorial to Kai. Here Kai will not mark the further exercises, but will look if an attempt has been made and awards the attendance mark for that week's tutorial.

Let $0 < \alpha < 1$, integer $p \geq 2$ and define the following probability distribution on $\mathbb{Z}_p$:

$$\mu_\alpha = \alpha\delta_1 + (1-\alpha)\delta_{-1}.$$

3. Find the probabilities of the events:

(a) *"a randomly chosen $t \in \mathbb{Z}_p$ with respect to μ_α is even"*

(b) *"a randomly chosen $t \in \mathbb{Z}_p$ with respect to μ_α is prime"*

4. Define a function $f : \mathbb{Z}_p \to \mathbb{C}$ by

$$f(t) = \begin{cases} +1; & t \text{ is even;} \\ -1; & t \text{ is odd.} \end{cases}$$

Find the integral (i.e. expectation) $\mu_\alpha(f)$ of f.

5. Define a function $f : \mathbb{Z}_p \to \mathbb{C}$ by

$$f(t) = \begin{cases} +1; & t \text{ is even;} \\ -1; & t \text{ is odd.} \end{cases}$$

Find the integral (i.e. expectation) $\lambda(f)$ of f with respect to the uniform measure.

6. The **distance** between two points $t, s \in \mathbb{Z}_p$ is

$$\mathrm{dist}(t, s) = \min\{t \oplus (-s), (-t) \oplus s\},$$

which measures the shortest distance between t and s at the dinner table. A function $f : \mathbb{Z}_p \to \mathbb{R}$ is **Lipschitz** if there exists $L \geq 0$ such that

$$|f(t) - f(s)| \leq L\mathrm{dist}(t, s)$$

for all $t, s \in Z_p$. If $f : \mathbb{Z}_p \to \mathbb{R}$, then the smallest such $L \geq 0$ (it exists for any f, see below), is denoted as $\mathrm{Lip}(f)$. The **earth mover's distance** or also known as the first **Wasserstein distance** $W_1(\mu, \nu)$ of two probability distributions μ, ν in $\mathbb{Z}_p$ is given by

$$W_1(\mu, \nu) = \max\Big\{|\mu(f) - \nu(f)| : \mathrm{Lip}(f) \leq 1, f : \mathbb{Z}_p \to \mathbb{R}\Big\}.$$

Wasserstein distance appears commonly in a field called **mass transportation theory**, which has many applications throughout economics, physics and mathematics of PDEs. We can state it in our course's language as follows. Given a probability distribution μ on $\mathbb{Z}_p$ and a map $T : \mathbb{Z}_p \to \mathbb{Z}_p$, define the **push-forward** $T_*\mu$ by the formula

$$T_*\mu(t) = \mu(T^{-1}\{t\}), \quad t \in \mathbb{Z}_p,$$

where $T^{-1}\{t\} = \{s \in \mathbb{Z}_p : T(t) = s\}$ is the pre-image of the singleton $\{t\}$. Given two probability distributions μ, ν in $\mathbb{Z}_p$, a mapping $T : \mathbb{Z}_p \to \mathbb{Z}_p$ that maps μ onto ν, $T_*\mu = \nu$, is called an **optimal transportation** if it minimises the "cost":

$$\int \mathrm{dist}(t, T(t))\, d\mu(t). \qquad \Big(= \sum_{t \in \mathbb{Z}_p} \mathrm{dist}(t, T(t))\, \mu(t) \Big)$$

The **Monge-Kantorovich duality theorem** says the minimal cost is the first Wasserstein distance:

$$\min\Big\{ \int \mathrm{dist}(t, T(t))\, d\mu(t) : T_*\mu = \nu \Big\} = W_1(\mu, \nu),$$

the proof can be found from literature on optimal transportation theory.

(a) Prove that any $f : \mathbb{Z}_p \to \mathbb{R}$ is Lipschitz. Which functions $f : \mathbb{Z}_p \to \mathbb{R}$ satisfy $\mathrm{Lip}(f) = 0$?

(b) Prove that

$$d(\mu, \nu) \leq W_1(\mu, \nu),$$

where $d(\mu, \nu)$ is the total variation distance.

(c) Fix $s \in \mathbb{Z}_p$. Define a transportation map $T : \mathbb{Z}_p \to \mathbb{Z}_p$ by $T(t) = s$, $t \in \mathbb{Z}_p$. Verify that the uniform distribution $\lambda(t) = 1/p$, $t \in \mathbb{Z}_p$, satisfies $T_*\lambda = \delta_s$, where δ_s is the singular distribution at s. What is the cost of transporting the uniform mass λ to a point δ_s? That is, find the cost

$$\int \mathrm{dist}(t, T(t))\, d\lambda(t).$$

Can you transport δ_s to λ?