Microeconomic Theory I: Lecture 2

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Choice-Based Approach to Consumer Theory

- use choice theory to derive positive implications for consumer theory
- derive the Walrasian demand without utility maximization
- how complete is the description of the consumer behavior that can be obtained by applying WA?
- endogenous variables change in response to exogenous variables
- endogenous: consumption choices.
- exogenous: prices and income.

An Interpretation for the Framework:

- X the choice set, the consumption set
- we take $X = \mathbb{R}^L_+$, where $L \in \mathbb{N}$.
- $x = (x_1, ..., x_L)$, where each $x_l \in \mathbb{R}_+$ for each $l \in \{1, ..., L\}$.
- goods are divisible
- the choice set is convex
- recall conventions on matrices and derivatives from math camp.

Feasible Set B

- nature of opportunity sets in general
- $B \in \mathcal{B}$ gives the budget set of a consumer.
- $\mathcal{B}$ gives the set of all possible budget situations.
- The feasible budget is defined by prices p and income or wealth w.
- A budget feasible consumption is one that can be purchased with the disposable income.

In classical consumer theory (and in this lecture), we assume that prices are linear:

- The price of an additional unit of good l is independent of the amount of good l purchased
- The price of an additional unit of good k is independent of consumptions of goods $l \neq k$
- Rules out choice between phone plans with different fixed costs and different cost per unit of data transmission
- Rules out progressive taxes, exemptions etc.
- $p \in \mathbb{R}^L_+,\ w \in \mathbb{R}_+$.

We define the Walrasian budget set:

$$B = \{x \in X \,|p \cdot x \leq w\} \text{ or}$$

$$B = \left\{x \in X \,\middle|\Sigma_{l=1}^{L} p_l x_l \leq w\right\}.$$

Notice that B is then determined by p and w and we write $B(p, w)$.

Note

$B(p, w)$ rules out nonlinearities, indivisibilities, uncertainties, and interdependencies between individuals

The choice rule $C(B)$

The Walrasian demand correspondence $x(p, w)$
We assume that $x(p, w)$ is single valued. $x : \mathbb{R}_{+}^{L+1} \rightarrow \mathbb{R}_{+}^{L}$.
Two assumptions regarding $x(p, w)$
Assumption 1: the Walras' Law:

$$p \cdot x(p, w) = w \text{ for } \forall p, w.$$

$$\Sigma_{l=1}^{L} p_l x_l (p, w) = w.$$

- sometimes called the adding-up restriction
- wealth/income vs. expenditure

Assumption 2: Homogeneity restriction

$$x(\lambda p, \lambda w) = x(p, w) \text{ for all } \lambda > 0 \text{ and all } p, w.$$

- that is, $x(p, w)$ is homegenous of degree zero in (p, w)
- no money illusion
- the effect of units on the consumer's perception of opportunities

Notice that

$$p \cdot x\,(p, w) : \mathbb{R}_+^{L+1} \to \mathbb{R}_+.$$

Denote by $D_p x\,(p, w)$ the derivative of $x\,(p, w)$ with respect to p and by $D_w x\,(p, w)$ the derivative w.r.t. w.
Then

$$D_p x(p, w) : \mathbb{R}_+^L \to \mathbb{R}_+^L,$$

$$D_w x(p, w) : \mathbb{R}_+ \to \mathbb{R}_+^L.$$

Implications of the Walras' Law:

Engel aggregation:

$$p \cdot D_w x(p, w) = 1. \tag{1}$$

Cournot aggregation

$$p \cdot D_p x(p, w) + x\left(p, w\right)^T = 0 \tag{2}$$

Homogeneity of Degree 0

$$D_p x\left(p, w\right) p + D_w x\left(p, w\right) w = 0. \tag{3}$$

Elasticities and Budget Shares

It is convenient to find expressions in terms of more easily understood concepts

Let $b_l(p, w) = p_l x_l(p, w)/w$ denote the budget share of l.

Let $\varepsilon_{lk}(p, w) = \frac{\partial x_l(p,w)}{\partial p_k}\frac{p_k}{x_l(p,w)}$ denote price elasticity between of good l with respect to price p_k.

Let $\varepsilon_{lw}(p, w) = \frac{\partial x_l(p,w)}{\partial w}\frac{w}{x_l(p,w)}$ denote the income elasticity of demand for good l.

- Engel and Cournot aggregation imply:

$$\Sigma_{l=1}^{L} b_l\left(p,w\right)\varepsilon_{lk}\left(p,w\right)+b_k\left(p,w\right)=0$$

$$\Sigma_{l=1}^{L} b_l\left(p,w\right)\varepsilon_{lw}\left(p,w\right)=1$$

- Homogeneity restriction implies:

$$\Sigma_{k=1}^{L}\varepsilon_{lk}\left(p,w\right)+\varepsilon_{lw}\left(p,w\right)=0$$

WA and Walras' law

Recall WA:

Axiom

If $x, y \in B$ *and* $x \in C(B)$, *then for all* B' *such that* $x, y \in B'$ *and* $y \in C(B')$, *we have* $x \in C(B')$.

In the context of Walrasian budget sets WA takes the form:

Axiom

$x(p, w)$ *satisfies WA if for any two budget situations* $B(p, w)$ *and* $B(p', w')$, *we have*

$$\begin{array}{rcl} [p \cdot x(p', w') & \leq & w \text{ and } x(p, w) \neq x(p', w')] \\ & \Rightarrow & \\ [p' \cdot x(p, w) & > & w'] \end{array}$$

Why is it that the two axioms are equivalent?

Implications of WA for $x(p, w)$

- Recall the Law of Demand: $x_l(p, w)$ is decreasing in p_l.
- This result is surprisingly hard to get in microeconomics.
- reason for the difficulty: An increase in p_l changes relative prices (slope of the budget line) and effective wealth (i.e. is not feasible with new prices).
- to isolate the substitution effect we consider compensated price changes
- we obtain the compensated law of demand
- Idea: Look at the effects of relative price changes by forcing the original consumption point to lie on the new budget line.
- Formally (p', w') is a compensated price change if $p' \cdot x(p, w) = w'$

Proposition

Suppose $x(p, w)$ satisfies Assumptions 1-2 and (p', w') is a compensated price change.
i) If $x(p, w)$ satisfies WA, then

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0,$$

where the inequality is strict whenever $x(p', w') \neq x(p, w)$.
ii) Conversely if for all compensated price changes from (p, w) to (p', w')

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0,$$

then $x(p, w)$ satisfies WA.

The compensated law of demand, $\Delta p \cdot \Delta x \leq 0$, has implications for the substitution matrix called the Slutsky matrix of $x(p, w)$.

The differential analog $dp \cdot dx \leq 0$ implies

$$dp \cdot [D_p x(p,w) + D_w x(p,w)\, x(p,w)^T] dp \leq 0.$$

This says that $L \times L$ matrix in brackets is negative semidefinite. Denote the Slutsky matrix by $S(p,w)$.

Proposition

Suppose $x(p,w)$ satisfies Assumptions 1-2 and the WA. Then, at any (p,w) the Slutsky matrix by $S(p,w)$ is negative semidefinite.

- The implications
- substitution effects w.r.t. own price
- Giffen goods
- Does negative semidefinite $S(p,w)$ generated by $x(p,w)$ satisfying the Walras Law and Homogeneity restiction imply that WA is also satisfied?
- $S(p,w)$ is not symmetric in general
- has rank less than L

Preliminaries for the Preference-based approach to consumer demand

From lecture 1:

- If $\succeq$ is rational, then $C^*(\mathcal{B}, \succeq)$ satisfies WA.
- Let $x(p, w) = C^*(B(p, w), \succeq)$. We call $x(p, w)$ the Walrasian demand of the consumer with preferences $\succeq$.
- If $x(p, w)$ is Homogenous of degree 0 in (p, w) and satisfies Walras' Law, then $x(p, w)$ also satisfies compensated law of demand and hence the Slutsky matrix is negative semidefinite.
- When will Homogeneity of degree 0 and Walras' Law be satisfied?

- Homogeneity of Degree 0
  This follows from the fact that for all $\lambda > 0$,

$$B(p, w) = B(\lambda p, \lambda w)$$

- Walras' Law

  We need new assumptions on consumer's "tastes".

Definition

$\succeq$ is:

(i) strongly monotonic if $\forall x, y \in X$

$$(x \geq y \text{ and } x \neq y) \Rightarrow x \succ y;$$

(ii) monotonic if $\forall x, y \in X$

$$(x_i \geq y_i, \forall i) \Rightarrow x \geq y;$$

(iii) locally nonsatiated if $\forall x$ and scalars $\delta > 0$, $\exists y \in X$ such that

$$1.\ \|y - x\| < \delta \text{ and} \tag{4}$$

$$2.\quad y \succ x. \tag{5}$$

Local non-satiation of $\succeq$ implies Walras' law.

When is $x(p, w)$ single valued?

Definition

$\succeq$ is said to be

i) Convex if for all $x, y, \in X$ and for all $t \in [0, 1]$, we have

$$x \succeq y \Rightarrow (tx + (1-t)y) \succeq y$$

ii) Strictly Convex if for all $x, y, \in X$ and for all $t \in (0, 1)$, we have

$$x \succeq y \Rightarrow (tx + (1-t)y) \succ y$$

Assume next that a utility function u representing $\succeq$ exists. Recall the definition of quasi-concave functions:

Definition

Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is a quasiconcave function if and only if

$$f(tx_1 + (1-t)x_2) \geq \min[f(x_1), f(x_2)]$$

for every $x_1, x_2 \in X$,and $0 \leq t \leq 1$.

It is a straightforward exercise to show the following alternative characterization for quasiconcave functions.

Theorem

Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is said to be **quasiconcave** *if its upper-level sets*

$$U(f, \alpha) = \{x : x \in X, f(x) \geq \alpha\}$$

are convex sets for every real α.

- Notice from here the connection between quasiconcavity of a representation and the convexity of the underlying preferences.
- It is clear that a concave function is also quasiconcave, since

$$\begin{aligned} f(tx_1 + (1-t)x_2) &\geq \\ tf(x_1) + (1-t)f(x_2) &\geq \min\left[f(x_1), f(x_2)\right]. \end{aligned}$$

The most important property of quasiconcave functions for consumer theory theory is, however, the following:

Theorem

If $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconcave and if $g : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $g\left(f\left(x\right)\right)$ is also quasiconcave.

These definitions can be extended to the strict case in a straightforward manner:

Definition

Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is said to be strictly quasiconcave if

$$f(tx_1 + (1-t)x_2) > \min[f(x_1), f(x_2)]$$

for every $x_1, x_2 \in X, x_1 \neq x_2$, and $0 < t < 1$.

Hence strict quasiconcavity of a representation is equivalent to strict convexity of preferences.

Proposition

i) If $\succeq$ *is convex, then* $x(p, w)$ *is a convex set for all* p, w. *ii) If* $\succeq$ *is strictly convex, then* $x(p, w)$ *is a singleton for all* p, w.