Microeconomic Theory I: Lecture 4

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Consumer Theory and Utility Maximization

Classical Consumer Theory

Objectives::

- Preference-based approach to consumer theory
- The additional positive implications implied by the preference maximization
- Again: endogenous variables change in response to exogenous variables.
- Endogenous: Consumption choices.
- Exogenous: Prices and income.

Recall the Framework:

- X the set of all possible consumptions.
- Normally we take $X = \mathbb{R}_+^L$, where $L \in \mathbb{N}$.
- $x = (x_1, ..., x_L)$, where each $x_l \in \mathbb{R}_+$ for each $l \in \{1, ..., L\}$.

- $B(p,w) = \{x \in X \,|p \cdot x \leq w\}$. (homogeneity of degree 0)
- $p \gg 0$.
- Local non-satiation: $p \cdot x(p,w) = w$. (Walras' Law).
- Strict convexity of preferences implies that $x(p,w)$ is single valued.
- We prove next the existence of a continuous utility representation for strongly monotone rational preferences.

Existence of Utility Representation

Theorem

Suppose that $X = \mathbb{R}^n_+$ and that preferences are complete, transitive, continuous and strongly monotonic. There there exists a continuous function $u : X \to \mathbb{R}$ such that, for any $\boldsymbol{x}, \boldsymbol{y} \in X$, $u(\boldsymbol{x}) \geq u(\boldsymbol{y})$ if and only if $\boldsymbol{x} \succeq \boldsymbol{y}$

proof

- let $\boldsymbol{e}$ be the vector in $\mathbb{R}^n_+$ consisting of all ones
- given any $\boldsymbol{x} \in X$, let $u(\boldsymbol{x})$ be the number such that $\boldsymbol{x} \sim u(\boldsymbol{x})\boldsymbol{e}$
- have to show that such a number exists
- let $BS \equiv \{t \in \mathbb{R} | t\boldsymbol{e} \succeq \boldsymbol{x}\}$ and $WS \equiv \{t \in \mathbb{R} | \boldsymbol{x} \succeq t\boldsymbol{e}\}$
- strong monotonicity: BS is non-empty
- WS is definitely non-empty (contains zero, at least)
- continuity: both BS and WS are closed
- since real line is connected, it is not the union of two disjoint closed sets
- hence there is some t_x such that $t_x\boldsymbol{e} \sim \boldsymbol{x}$

Existence of Utility Representation

- let

$$u(\boldsymbol{x}) = t_x, \quad u(\boldsymbol{y}) = t_y$$

- if $t_x < t_y$, then
  - strong monotonicity $\Rightarrow t_x\boldsymbol{e} \prec t_y\boldsymbol{e}$
  - transitivity $\Rightarrow \boldsymbol{x} \sim t_x\boldsymbol{e} \prec t_y\boldsymbol{e} \sim \boldsymbol{y}$
- if $\boldsymbol{x} \succ \boldsymbol{y}$, then $t_x\boldsymbol{e} \succ t_y\boldsymbol{e}$, and so $t_x > t_y$
- continuity follows since the preimage $u^{-1}(A)$ of all open sets $A \subset \mathbb{R}_+$ is open

Consumer's problem (UMP):

Version 1:

$$\max_{x \in B(p,w)} u(x). \tag{1}$$

Version 2:

$$\max_{x \geq 0} u(x) \tag{2}$$

$$\text{s.t. } p \cdot x \leq w.$$

When does a solution to this problem exist?

Proposition

(Weierstrass). Let $f : X \to \mathbb{R}$ be a continuous function and X a compact set. Then f attains its maximum on x, i.e. there is a point $x^* \in X$ such that for all $x \in X$,

$$f(x) \leq f(x^*).$$

- Recall: if $X \subset \mathbb{R}^n$, then X is compact if and only if it is closed and bounded.
- Since $p \gg 0$, the budget set $B(p, w)$ is bounded.
- Since the budget set is defined by an inequality, it is closed.
- It is easy to see through counterexamples that continuity, boundedness and closedness are all required for the result.
- How to characterize x^*?

Lagrangean for the problem:

$$\mathcal{L}(x, \lambda, \mu) = u(x) - \lambda(p \cdot x - w) + \mu \cdot x.$$

- Note: $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}_+^L$.
- Constraint qualification always holds for UMP (why?)

Kuhn-Tucker necessary conditions for the UMP:

If x^* solves UMP, then there exist $\lambda \in \mathbb{R}, \mu \in \mathbb{R}^L_+$ such that:

$$\begin{aligned}
\frac{\partial u(x^*)}{\partial x_l} - \lambda p_l + \mu_l &= 0 \text{ for all } l = 1, ...L, \\
p \cdot x^* &= w, \\
\mu_l x_l^* &= 0 \text{ for all } l = 1, ...L, \\
x^* &\geq 0,
\end{aligned}$$

Alternatively, there exists a scalar $\lambda \in \mathbb{R}$ such that:

$$\begin{aligned}
\frac{\partial u(x^*)}{\partial x_l} - \lambda p_l &\leq 0 \text{ for all } l, \\
\frac{\partial u(x^*)}{\partial x_l} - \lambda p_l &= 0 \text{ if } x_l^* > 0, \\
p \cdot x^* &= w, \\
x^* &\geq 0.
\end{aligned}$$

When can we say that these conditions are also sufficient?

Proposition

Suppose that $u(x)$ is quasiconcave and

$$\Delta u\left(x^{*}\right) \neq 0,\ \forall x \in B\left(p, w\right).$$

Then, if x^{*} satisfies the Kuhn-Tucker first-order conditions, x^{*} solves UMP.

Note:

- We have assumed $p \gg 0$, so that $B\left(p, w\right)$ is a compact convex set.

Interpretation

Marginal rate of substitution (MRS) between goods l and k.
For all l, k such that $x_l^*, x_k^* > 0$, we have:

$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_l}{p_k}.$$

Lagrange multiplier λ: the marginal utility value of wealth

$$\lambda = D_w u\left(x\left(p, w\right)\right).$$

To see this, use chain rule to get

$$D_w u\left(x\left(p, w\right)\right) = \nabla u\left(x\right) \cdot D_w x\left(p, w\right).$$

FOC's give:

$$\nabla u\left(x\right) = \lambda p,$$

and recall Engel aggregation:

$$p \cdot D_w x\left(p, w\right) = 1.$$

Alternative proof by the Envelope Theorem.
Let $z(\bar{q})$ denote value function depending on a parameter vector $\bar{q}$
Objective is $f(x;\bar{q})$ and $g_1(x(\bar{q});\bar{q}), ...g_m(x(\bar{q});\bar{q})$ are constraints.
The Envelope Theorem for the constrained optimization problem says that

$$\nabla z(\bar{q}) = \nabla_q f(x(\bar{q});\bar{q}) - \sum_{m=1}^{M} \lambda_m \nabla_q g_m(x(\bar{q});\bar{q})$$

For the UMP, this equation gives the interpretation for λ discussed above.

The value function or the indirect utility function of UMP:

$$v(p, w) \equiv u(x(p, w)).$$

What are the properties of $v(p, w)$ implied by the utility maximization problem?
Conversely, what properties of $v(p, w)$ guarantee the existence of a utility function $u(x)$?

Proposition

(Properties of $v(p,w)$)
i) Homogenous of degree 0 in (p,w).
ii) Strictly increasing in w, non-decreasing in p_l for all l.
iii) Continuous in (p,w).
iv) Quasiconvex in (p,w).

$u(x)$ **can be recovered if** $v(p,w)$ **is known:**

Proposition

If $v(p,w)$ satisfies i)-iv), then there exists a locally non-satiated and quasiconcave $u(x)$ such that $v(p,w) = u(x(p,w))$. Furthermore, we can solve for such a $u(x)$ from the problem

$$\min_{p \in \mathbb{R}^L_{++}} v(p,w)$$

$$s.t. p \cdot x = w.$$

When $v(p,w)$ is known, we can recover the demands:

Proposition

(Roy's Identity)
Given an indirect utility function $v(p,w)$, the Walrasian demands $x(p,w)$ can be recovered from

$$x_l(p,w) = -\frac{\frac{\partial v(p,1)}{\partial p_l}}{\frac{\partial v(p,1)}{\partial w}}.$$

This follows from the fact that
$\frac{\partial v(p,w)}{\partial p_l} = \nabla u(x) \cdot D_{p_l} x(p,w) = \lambda p_l \cdot D_{p_l} x(p,w) = -\lambda x_l(p,w)$
where the last equality comes from Cournot aggregation.