Microeconomic Theory I: Lecture 3

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WA, Walras' law and homogeneity for demand functions

Recall WA from Lecture 1:

Axiom

If $x, y \in B$ and $x \in C(B)$, then for all B' such that $x, y \in B'$ and $y \in C(B')$, we have $x \in C(B')$.

In the context of Walrasian budget sets and demand functions, WA takes the form:

Axiom

$x(p, w)$ satisfies WA if for any two budget situations $B(p, w)$ and $B(p', w')$, we have

$$\begin{aligned} [p \cdot x(p', w') &\leq w \text{ and } x(p, w) \neq x(p', w')] \\ &\Rightarrow \\ [p' \cdot x(p, w) &> w'] \end{aligned}$$

Implications of WA for $x(p, w)$

- Recall the Law of Demand from intermediate microeconomics: $x_l(p, w)$ is decreasing in p_l.
- But an increase in p_l changes relative prices (slope of the budget line) and effective wealth (i.e. is not feasible with new prices).
- To isolate the substitution effect we consider compensated price changes
- Idea: Look at the effects of relative price changes by forcing the original consumption point to lie on the new budget line.

Definition

$x(p, w)$ satisfies compensated law of demand (CLD) if
$\{(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0\}$
for all (p', w') such that $p' \cdot x(p, w) = w'$, and the inequality is strict if $x(p, w) \neq x(p', w')$.

WA and Compensated Law of Demand

Proposition

Suppose the demand function $x(p, w)$ is homogenous of degree 0 and satisfies Walras' Law. Then we have:

$$x(p, w) \text{ satisfies WA} \iff x(p, w) \text{ satisfies CLD.}$$

WA and Compensated Law of Demand

proof

We need to show the implication in both directions. Notice first that if $x(p, w) = x(p', w')$, there is nothing to prove so suppose $x(p, w) \neq x(p', w')$

i) WA implies CLD.

Let $x(p, w)$ be the demand at budget situation (p, w). Consider a compensated budget situation (p', w') where $p' \cdot x\,(p, w) = w'$. Since $x(p, w)$ is feasible at (p', w'), and $x(p, w) \neq x(p', w')$, WA implies that $p \cdot x(p', w') > w$.

Hence

$$(p' - p) \cdot (x(p', w') - x\,(p, w))$$

$$= (p' \cdot (x(p', w') - x(p, w)) - p \cdot (x(p', w') - x(p, w)))$$

$$= (w' - w') - (p \cdot x(p', w') - w) < 0.$$

WA and Compensated Law of Demand

ii) CLD implies WA.
We prove this by contrapositive: if $x(p, w)$ does not satisfy WA, then it does not satisfy CLD. Not satisfying WA means that there are budget situations (p, w) and (p', w') and associated demands $x(p, w) \neq x(p', w')$ such that

$$p' \cdot x(p, w) \leq w' \text{ and } p \cdot x(p', w') \leq w.$$

The key step in the proof is to show that an arbitrary violation of WA implies a violation of WA for some compensated price change.

WA and Compensated Law of Demand

If one of the inequalities above is satisfied as an equality, then the budget change is a compensated price change and there is nothing to prove. Hence assume that

$$p' \cdot x(p, w) < w' \text{ and } p \cdot x(p', w') < w.$$

Choose $\alpha \in (0, 1)$ so that

$$\alpha p \cdot (x(p, w) - x(p', w')) = (1 - \alpha) p' \cdot (x(p', w') - x(p, w)).$$

Define a new budget situation (p^c, w^c) by

$$p^c = \alpha p + (1 - \alpha) p',$$

$$w^c = (\alpha p + (1 - \alpha) p') \cdot x(p, w).$$

WA and Compensated Law of Demand

Notice that (p^c, w^c) is a compensated price change from both (p, w) and (p', w'). By WA,

$$\begin{aligned} \alpha w + (1-\alpha)w' &> \alpha p \cdot x(p', w') + (1-\alpha)p' \cdot x(p, w) \\ &= w^c = p^c \cdot x(p^c, w^c) \\ &= \alpha p \cdot x(p^c, w^c) + (1-\alpha)p' \cdot x(p^c, w^c). \end{aligned}$$

Hence either $p \cdot x(p^c, w^c) < w$ or $p' \cdot x(p^c, w^c) < w'$. But either of these possibilities gives a violation of WA for a compensated price change (from (p, w) to (p^c, w^c) in the first and from (p', w') to (p^c, w^c) in the second.

WA and Compensated Law of Demand

Hence if WA does not hold there exists a compensated price from some (p, w) to (p', w') with $x(p, w) \neq x(p', w')$, $p \cdot x(p', w') = w$ and $p' \cdot x(p, w) \leq w'$. By Walras' law,

$$p \cdot [x(p', w') - x(p, w)] = 0,$$
$$p' \cdot [x(p', w') - x(p, w)] \geq 0.$$

But this gives

$$(p' - p) \cdot [x(p', w') - x(p, w)] \geq 0,$$

with $x(p, w) \neq x(p', w')$ violating CLD. **QED**

CLD and Slutsky

The compensated law of demand, $\Delta p \cdot \Delta x \leq 0$, has implications for the substitution matrix also called the Slutsky matrix of $x(p, w)$. Consider the change induced in $x(p, w)$ by changing the budget to $(x(p + dp, w + dw)$. For small dp, dw, we have:

$$dx = D_p x(p, w) dp + D_w x(p, w) dw.$$

For compensated changes from (p, w) to $(p + dp, w + dw)$, CLD states $dp \cdot dx \leq 0$. Hence we have:

$$dp \cdot (D_p x(p, w) dp + D_w x(p, w) dw) \leq 0,$$

for compensated price changes. The price change is compensated if $dw = dp \cdot x(p, w) = x(p, w) \cdot dp$.

CLD and Slutsky

Substituting for *dw* gives:

$$dp \cdot [D_p x(p,w) + D_w x(p,w)\, x(p,w)^T] dp \leq 0.$$

This says that $L \times L$ matrix in the brackets is negative semidefinite. Call this matrix the Slutsky matrix and denote it by $S(p,w)$.

Proposition

Suppose $x(p,w)$ satisfies Assumptions 1-2 and the WA. Then, at any (p,w) the Slutsky matrix by $S(p,w)$ is negative semidefinite.

CLD and Slutsky

- The implications
- substitution effects w.r.t. own price
- Giffen goods
- Does negative semidefinite $S(p, w)$ generated by $x(p, w)$ satisfying the Walras Law and Homogeneity restiction imply that WA is also satisfied?
- $S(p, w)$ is not symmetric in general but is symmetric if L=2.
- $S(p, w)$ has rank less than L

Consumer Theory based on Preference Maximization

From lecture 1:

- If $\succeq$ is rational, then $C^*(\mathcal{B}, \succeq)$ satisfies WA.
- Let $x(p, w) = C^*(B(p, w), \succeq)$. We call $x(p, w)$ the Walrasian demand of the consumer with preferences $\succeq$.
- If $x(p, w)$ is Homogenous of degree 0 in (p, w) and satisfies Walras' Law, then $x(p, w)$ also satisfies compensated law of demand and hence the Slutsky matrix is negative semidefinite.
- When does a preference maximizing choice exist? When is such a choice unique so that the Walrasian demand is a demand function?
- When will Homogeneity of degree 0 and Walras' Law be satisfied?

Existence of Preference Maximizing Choice

With an infinite choice set X, it is not obvious that a choice $x \in X$ exists such that $x \succeq y$ for all $y \in X$. (Just think of the largest real number in $(0, 1)$).

It is easy to see that similar problems can arise whenever the choice set is not closed and bounded, i.e. compact.

Preferences must also be regular enough to avoid problems with existence. To see this, consider $X = [0, 1]$ and $x \succ y$ if $x > y$ and $x < 1/2$, and $x \sim 0$ if $x \geq 1/2$.

Note that the existence problem arises from discontinuity in $\succeq$.

Definition

A rational preference relation $\succeq$ is continuous if the set $\{y | y \succeq x\}$ and $\{y | x \succeq y\}$ are closed.

Existence of Preference Maximizing Choice

We will also call $B(x) := \{y | y \succeq x\}$ the better than set for x and $W(x) := \{y | x \succeq y\}$ the worse than x set.

A rational preference relation is then continuous if $\{z_n \in B(x)\}$ and $z_n \to z$ implies that $\{z \in B(x)\}$ (and similarly for $W(x)$).

A relatively deep theorem due to Debreu shows that any continuous rational preference relation has a continuous utility representation. The existence of maximizers to continuous functions on compact sets implies that continuous rational preference relations on compact choice sets also has a preference maximizing choice.

We show the existence of a continuous utility representation for strictly monotone preferences in the next section.

Walras' law and Homogeneity

Walras' Law

Definition

A rational preference relation $\succsim$ is:
(i) strongly monotonic if $\forall x, y \in X$

$$(x \geq y \text{ and } x \neq y) \Rightarrow x \succ y;$$

(ii) monotonic if $\forall x, y \in X$

$$(x_i \geq y_i, \forall i) \Rightarrow x \geq y;$$

(iii) locally nonsatiated if $\forall x$ and scalars $\delta > 0$, $\exists y \in X$ such that

$$1. \; \|y - x\| < \delta \text{ and}$$

$$2. \qquad y \succ x.$$

Walras' law and Homogeneity

Local non-satiation of $\succeq$ implies Walras' law.

Homogeneity of Degree 0

This follows from the fact that for all $\lambda > 0$,

$$B(p, w) = B(\lambda p, \lambda w)$$

Single valuedness

When do we have a demand *function*? When is $x(p, w)$ single valued?

Definition

$\succeq$ is said to be
i) Convex if for all $x, y, \in X$ and for all $t \in [0, 1]$, we have

$$x \succeq y \Rightarrow (tx + (1-t)y) \succeq y$$

ii) Strictly Convex if for all $x, y, \in X$ and for all $t \in (0, 1)$, we have

$$x \succeq y \Rightarrow (tx + (1-t)y) \succ y$$

Assume next that a utility function u representing $\succeq$ exists. Recall the definition of quasi-concave functions:

Definition

Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is a quasiconcave function if and only if

$$f(tx_1 + (1-t)x_2) \geq \min[f(x_1), f(x_2)]$$

for every $x_1, x_2 \in X$,and $0 \leq t \leq 1$.

It is a straightforward exercise to show the following alternative characterization for quasiconcave functions.

Theorem

Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is said to be **quasiconcave** *if its upper-level sets*

$$U(f, \alpha) = \{x : x \in X, f(x) \geq \alpha\}$$

are convex sets for every real α.

- Notice from here the connection between quasiconcavity of a representation and the convexity of the underlying preferences.
- It is clear that a concave function is also quasiconcave, since

$$\begin{aligned} f(tx_1 + (1-t)x_2) &\geq \\ tf(x_1) + (1-t)f(x_2) &\geq \min\left[f(x_1), f(x_2)\right]. \end{aligned}$$

The most important property of quasiconcave functions for consumer theory theory is, however, the following:

Theorem

If $f : \mathbb{R}^n \to \mathbb{R}$ *is quasiconcave and if* $g : \mathbb{R} \to \mathbb{R}$ *is strictly increasing, then* $g(f(x))$ *is also quasiconcave.*

These definitions can be extended to the strict case in a straightforward manner:

Definition

Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is said to be strictly quasiconcave if

$$f(tx_1 + (1-t)x_2) > \min[f(x_1), f(x_2)]$$

for every $x_1, x_2 \in X, x_1 \neq x_2$, and $0 < t < 1$.

Hence strict quasiconcavity of a representation is equivalent to strict convexity of preferences.

Proposition

i) If $\succeq$ is convex, then $x(p, w)$ is a convex set for all p, w. ii) If $\succeq$ is strictly convex, then $x(p, w)$ is a singleton for all p, w.