Microeconomic Theory 1
Helsinki GSE – Fall 2019
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Problem Set 1 - Solutions

Problem 1

- **Completeness.** Take $x, y \in X$. If $x \succ y$, then the asymmetry of $\succ$ implies $\neg[y \succ x]$, which is equivalent to $x \succeq y$. If $\neg[x \succ y]$, then this is equivalent to $y \succeq x$ by definition.
- **Transitivity.** Transitivity of $\succeq$ is the contrapositive of negative transitivity of $\succ$, so the two conditions are equivalent. More specifically, the contrapositive of negative transitivity is
$$\neg[x \succ z] \text{ and } \neg[z \succ y] \implies \neg[x \succ y].$$
By definition of $\succeq$, the latter is equivalent to
$$[z \succeq x] \text{ and } [y \succeq z] \implies [y \succeq x],$$
which establishes transitivity.

Problem 2

I use WA to indicate the main definition of the Weak Axiom, and WA* for the alternative condition.

- **WA $\implies$ WA*.** Assume that WA holds. Take $x, y \in B \cap B'$ such that $x \in C(B)$ and $y \notin C(B)$. By way of contradiction, suppose that $y \in C(B')$. By WA, we have that $x, y \in B \cap B'$, $y \in C(B')$, and $x \in C(B)$, imply $y \in C(B)$, so leading to a contradiction.
- **WA* $\implies$ WA.** Assume that WA* holds. Take $x, y \in B \cap B'$ such that $x \in C(B)$ and $y \in C(B')$. By way of contradiction, suppose that $x \notin C(B')$. By WA*, we have that $x, y \in B \cap B'$, $y \in C(B')$, and $x \notin C(B')$, imply $x \notin C(B)$, so leading to a contradiction.

Problem 3

Before proving the equivalence with the WA, notice that property i) is known as Sen's α condition, and ii) is Sen's β condition.

- **i) & ii) $\implies$ WA.** Take $x, y \in B \cap B'$ such that $x \in C(B)$ and $y \in C(B')$. Since $B \cap B' \subset B$, it follows from property i) that $x \in C(B \cap B')$. Now, since $y \in B \cap B' \subset B'$ and $y \in C(B')$, we have that $C(B \cap B') \subset C(B')$ by property ii). Combining the latter with $x \in C(B \cap B')$ we finally obtain $x \in C(B')$.

- **WA $\implies$ i) & ii).** Suppose $x \in B \subset A$ and $x \in C(A)$. By non-emptiness, there exists $z \in C(B)$. By WA, it follows that $x \in C(B)$.

  Now suppose $y \in B \subset A$ and $y \in C(A)$. By non-emptiness, there exists $z \in C(B)$. For any such $z \in C(B)$, we have $z, y \in B \cap A$, $z \in C(B)$, and $y \in C(A)$. By WA, this implies $z \in C(A)$, so proving that $C(B) \subset C(A)$.

In the table below, you can find the values of two choice rules C and C' that violate either i) or ii). Notice that the choice set is $X = \{x, y, z\}$, and the first column of the table indicates the feasible sets. You can easily verify that C satisfies i) but not ii), whereas C' satisfies ii) but not i).

	C	C'
$\{x\}$	$\{x\}$	$\{x\}$
$\{y\}$	$\{y\}$	$\{y\}$
$\{z\}$	$\{z\}$	$\{z\}$
$\{x, y\}$	$\{x, y\}$	$\{x, y\}$
$\{x, z\}$	$\{x\}$	$\{x\}$
$\{y, z\}$	$\{y, z\}$	$\{y, z\}$
$\{x, y, z\}$	$\{x, y\}$	$\{x, y, z\}$

Problem 4

(a) We need to show that, for every $x, y \in X$, $x \succeq y$ if and only if $u(x) \geq u(y)$.

- $\boldsymbol{x \succeq y \implies u(x) \geq u(y)}$**.** Take $x, y \in X$ and suppose $x \succeq y$. By transitivity, we have

$$\{z : x \succeq z\} \supset \{z : y \succeq z\},$$

which implies

$$\#\{z : x \succeq z\} \geq \#\{z : y \succeq z\}.$$

The latter is equivalent to $u(x) \geq u(y)$.

- $\boldsymbol{u(x) \geq u(y) \implies x \succeq y}$**.** Take $x, y \in X$ and suppose $u(x) \geq u(y)$, which is equivalent to

$$\#\{z : x \succeq z\} \geq \#\{z : y \succeq z\}. \tag{1}$$

By completeness, we must have either $x \succeq y$ or $y \succeq x$. By transitivity, the latter disjunction implies

$$\{z : x \succeq z\} \supset \{z : y \succeq z\} \text{ or } \{z : x \succeq z\} \subset \{z : y \succeq z\}. \tag{2}$$

Combining (1) and (2), we obtain

$$\{z : x \succeq z\} \supset \{z : y \succeq z\},$$

from which it follows that $x \succeq y$.

(b) Also in this part we need to show that, for every $x, y \in X$, $x \succeq y$ if and only if $u(x) \geq u(y)$. Unlike what we did in part (a), now we are going to prove the statement by induction. More specifically, the argument is by induction on $\#X$, i.e. the cardinality of X. If $\#X = 1$, the statement that we need to prove is trivially true. For the induction step, suppose that u is a valid utility representation when $\#X = n$. All we need to prove is that this implies that u is a valid utility representation also when $\#X = n + 1$. To do this, we consider three cases. In the first case, x_{n+1} is indifferent to some x_k with $k \leq n$. As per the procedure with which we compute u, this is equivalent to $u(x_k) = u(x_{n+1})$. Since u is a valid utility representation for $\{x_1, \ldots, x_n\}$ by the induction hypothesis, the result follows immediately. In the second case, x_{n+1} is strictly better than x_k for all $k \leq n$. This is equivalent to $u(x_{n+1}) > u(x_k)$ for all $k \leq n$. Combining this with the hypothesis that u is a valid utility representation for $\{x_1, \ldots, x_n\}$, we can conclude that u is a valid utility representation over $\{x_1, \ldots, x_n, x_{n+1}\}$. An analogous reasoning applies to when $x_{n+1} \prec x_k$ for all $k \leq n$. In the third and final case, x_{n+1} is strictly worse than some element in $\{x_1, \ldots, x_n\}$ and strictly better than some other element in the same set. Let $A := \{x_k : k \leq n \text{ and } x_k \succ x_{n+1}\}$ be the set of elements that are strictly better than x_{n+1}. By definition, we have that x^o is a minimal element in this set. By the induction hypothesis, the latter is equivalent to $u(x^o) \leq u(x_k)$ for all x_k in A. Similarly, let $B := \{x_k : k \leq n \text{ and } x_k \prec x_{n+1}\}$ be the set of elements that are strictly worse than x_{n+1}. By definition again, x_o is a maximal element in this set, which is equivalent to $u(x^o) \geq u(x_k)$ for all x_k in B. By transitivity and the induction hypothesis, $u(x^o) > u(x_o)$. Since $u(x_{n+1}) = \frac{u(x^o)+u(x_o)}{2}$, we have $u(x^o) > u(x_{n+1}) > u(x_o)$, from which we can conclude that u is a valid utility representation over $\{x_1, \ldots, x_n, x_{n+1}\}$. And this ends our proof.

It is apparent that the utility function in this part is not the same as the utility function in part (a), i.e. they take on different values. In addition, the utility function in part (b) is not independent of how we enumerate the elements in X. To see this, consider the case where $X = \{a, b, c\}$ and the preference relation is such that $a \succ b \succ c$. Now set $x_1 = a, x_2 = b, x_3 = c$. Thus the utility function u is such that $u(a) = \frac{1}{2}$, $u(b) = \frac{1}{4}$, and $u(c) = \frac{1}{8}$. But if we enumerate elements in the reverse order, i.e. $x_1 = c, x_2 = b, x_3 = a$, we would obtain a utility function u' such that $u'(c) = \frac{1}{2}$, $u'(b) = \frac{3}{4}$, and $u'(a) = \frac{7}{8}$. Clearly, u and u' are different utility functions but they do represent the same preference relation.

(c) The utility representation in part (a) cannot be applied directly to infinite sets because the cardinality of sets like $\{y : x \succeq y\}$ could be infinite. To circumvent this issue, we are going to modify our utility representation in such a way that it works not just for finite sets but also for countably infinite ones[1]. Let X be countably infinite, and fix an enumeration of its elements $\{x_1, x_2, \dots\}$. Define a function $f : X \longrightarrow \mathbb{R}$ such that $f(x_n) = \left(\frac{1}{2}\right)^n$. For each x, the utility function u is defined as

$$u(x) = \sum_{z' \in \{z : x \succeq z\}} f(z').$$

Since the series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent, $u(x)$ is always well-defined and finite. To show that u is indeed a utility representation, suppose first that $x \succeq y$. By transitivity, we have $\{z : x \succeq z\} \supset \{z : y \succeq z\}$. Therefore, the sum that defines $u(x)$ contains all the terms in the sum that defines $u(y)$. Since all the terms in both sums are strictly positive, we have $u(x) \geq u(y)$. For the other direction, suppose $u(x) \geq u(y)$. By completeness and transitivity, we already know that either $\{z : x \succeq z\} \supset \{z : y \succeq z\}$ or $\{z : x \succeq z\} \subset \{z : y \succeq z\}$. By the definition of u, we have that $u(x) \geq u(y)$ only if $\{z : x \succeq z\} \supset \{z : y \succeq z\}$, from which it follows that $x \succeq y$. Therefore, u is a valid utility representation.

Up to now, we have seen utility representations that work for the finite or countably infinite case and that require only rational, i.e. complete and transitive, preferences. But what if the consumption space X is uncountable? Under the rationality assumption alone, one cannot find a utility representation that works for any given uncountable consumption space. The classic counterexample is given by lexicographic preferences. To find utility representations in these cases, one needs to make stronger assumptions, e.g. preference continuity.

[1]The utility representation I am using is the same as in the proof of Proposition 1.11 in Kreps (2013).

Problem 5

(a) If we subscribet to plan (f_i, c_i), the set of allocations that we can afford is

$$B_i = \{(x, y) : f_i + c_i y + x \leq w\},$$

where w is our income. If we choose not to subscribe to any plan, we can afford any $x \leq w$. Formally,

$$B_0 = \{(x, y) : y = 0 \text{ and } x \leq w\}.$$

Summing this up, our overall budget constraint is $\mathcal{B} = \cup_{i=0}^{K} B_i$.

(b) Assume that $r_b \geq r_\ell$. If we lend money, we are willing to consume less in the first period in order to consume more later on. More specifically, our consumption in the second period is such that

$$c_2 \leq w_2 + (1 + r_\ell)(w_1 - c_1).$$

If we borrow money, we can increase our consumption c_1 and we pay this back by consuming less in the second period. Our consumption in the second period must be

$$c_2 \leq w_2 - (1 + r_b)(c_1 - w_1).$$

In sum, the inter-temporal budget set is

$$\mathcal{B} = \{(c_1, c_2) \in \mathbb{R}^2_+ : c_2 \leq w_2 + (1 + r_\ell)(w_1 - c_1) \text{ and } c_2 \leq w_2 - (1 + r_b)(c_1 - w_1)\}.$$

If you draw it, you can easily see that, when $r_b > r_\ell$, the budget line has a kink at (w_1, w_2).

(c) Let T be our daily time endowment and ℓ the amount of time we choose to work. Leisure is $T - \ell$. If we write the budget constraint in terms of consumption and labor we have

$$c \leq \bar{w} + \omega\ell, \tag{3}$$

where $\bar{w}$ is some fixed nonlabor income and ω is the wage rate. It is clear that $\ell \leq T$.

To rewrite the budget constraint in terms of consumption and leisure, we can add ωT to each side of (3) and rearrange so as to get

$$c + \omega(T - \ell) \leq \bar{w} + \omega T.$$

In addition to this baseline case, an important source of nonlinearity is given by income taxation, which is progressive in most countries. Suppose for simplicity that $\bar{w} = 0$, and that labor income is taxed at a proportional tax rate τ if $\omega\ell$ is less than or equal to a certain threshold A. If the income is strictly larger than A, then the tax rate is $\tau' > \tau$. Suppose that your labor income is bigger than A. In the (c, ℓ)-space, the set affordable allocations is given by all points that simultaneously satify $c \leq (1-\tau)\omega\ell$ and $c \leq (1-\tau')\omega\ell + (\tau' - \tau)A$. If you draw it, you can easily verify that the budget line has a kink at $\left(\frac{A}{\omega}, (1-\tau)A\right)$. In economic terms, a kink means that the marginal tax rate is changing at that income level. Since these kinks typically take place at lower income levels, this form of non-linearity is more likely to affect the consumption-leisure decisions of those with lower wages.

Problem 6

The Weak Axiom requires that choices are independent and that preferences are stable across choices. An example where such assumptions are justified is given in part (a) of the previous exercise: mobile phone subscriptions. It is indeed fairly reasonable to assume that in this particular market consumers choices are independent across time. Since product quality is more or less the same across mobile service providers, and since switching costs are negligible, my choice to subscribe to a certain plan today is unlikely to affect my future preferences over subscription plans.

From a more general perspective, the independence assumption is easier to justify when one considers composite goods instead of a particular good, e.g. households food expenditures instead of a particular food item. Importantly, this kind of aggregate data is also what economists typically work with.

An example where the independence assumption is much harder to justify is given by experience goods, i.e. goods whose quality can be fully assessed only after consumption. For example, restaurants, hairdressers, books, etc. In this case, your choices today are more likely to affect your future preferences. While the Weak Axiom postulates that the consumer has a clear preference ordering in her mind before choosing, this is not necessarily true when it comes to experience goods. Loosely put, when you choose A over B, this does not imply that you like A better than B. If you are uncertain about A's quality, it may well be the case that you choose it just because you want to find out. In other words, your choice is not derived from a clear and pre-formulated preference ordering. Instead, it is part of an experimentation process through which you are trying to formulate such a complete and consistent ordering.